RIVISTA DEL NUOVO CIMENTO VOL. 9, N. 10 1986

Operator Quantization and Abelization of Dynamical Systems Subject to First-Class Constraints.

I. A. BATALIN and E. S. FRADKIN

Physical Lebedev Institute - Moscow

(ricevuto il 4 Marzo 1985)

Introduction.

The development of the general theory of dynamical constrained systems is traced back to the famous works by DIRAC [1-3] and BERGMANN with collaborators [4-6]. Apart from Dirac's lecture notes, which have become classical, there exists a number of excellent reviews [7-9] which reflect the present status and methods for solving the quantization problems of constrained dynamical systems. These reviews may be recommended to a reader who is interested in the state of the problem as a whole.

The purpose of the present paper is to consistently present, on an unique ground, original author's results which solve the problem of operatorial canonical quantization of relativistic dynamical systems subject to first-class constraints generating a gauge algebra of the most general kind.

It is conventional to use the path integral formalism when quantizing constrained dynamical systems. Our view, however, is that the operatorial quantization is most appropriate, since quantum mechanics is to be taken as an algebraic entity. As for the path integral it should follow as a closed functional solution to operatorial equations of motion.

When developing operatorial quantization, the question about equal-time commutation relations becomes crucial. If dynamical systems with first-class constraints are dealt with, and gauge conditions are chosen as relativistic, their relativistic nature shows itself in two ways. First, in such gauges the reduction of the original phase space onto the hypersurface specialized by constraints is not possible. Second, the Lagrange multipliers to the first-class constraints and to the corresponding gauge conditions become dynamically active pairs of canonically conjugate variables. Consequently, when quantizing, one becomes able to impose canonical equal-time commutation relations on the operators belonging to the extended (relativistic) phase space without encountering any contradictions. Then the problem of operatorial quantization reduces to that of finding the unitarizing Hamiltonian.

The unitarizing Hamiltonian is constructed using generating operators of the gauge algebra of constraints in such a way that there exists the BRS-algebra formed by the conserved BRS-charge and ghost number operators. Then, according to the remarkable theorem proved by KUGO and OJIMA [10, 11], the S-matrix proves to be unitary in the physical subspace of BRS singlets (see also the work by NISHIJIMA [12]).

Such is the general outline of the operatorial method for canonical quantization of dynamical systems subject to first-class constraints and relativistic gauges, presented in this paper.

Unfortunately, the situation in operatorial quantization of systems subject to second-class constraints is far less satisfactory. Apart from rather trivial cases, the crucial problem of equal-time commutation relations remains here unsolved. The main difficulty for the second-class constraints is in that the original phase space is actually reduced onto the hypersurface specialized by the constraints. Therefore, canonical commutation relations cannot be imposed, as being definitely incompatible with the operatorial constraint equations. The heart of the as yet unsolved problem of operatorial quantization of second-class constrained systems is just in finding the general form of equal-time commutation relations compatible with the operatorical constraint equations and besides self-consistent under the Jacobi identities.

It is a widespread opinion that the quantization problem of second-class constrained systems may be supposedly (at least partially) solved by equating equal-time commutation relations among the phase space variables with the corresponding Dirac brackets. The following remark is in order at this point. It is sure that the Dirac brackets allow us to achieve a completely consistent treatment of classical dynamics of systems with second-class constraints. In the quantum domain, however, the use of the Dirac brackets for treating general second-class constraints does not lead to a consistent solution of the equal-time commutation relation problem, as not, generally, meeting the above two requirements. Only a very trivial case of linear constraints, when one can explicitly separate physical degrees of freedom, admits the operatorial analogue of the Dirac brackets to be uniquely defined and self-consistent.

Since the general operatorial quantization problem of second-class constrained system is yet unsolved (*), one should treat with some precautions the results concerning their euristic quantization in terms of the path integral. We must be aware that, generally, the euristic path integral drastically depends on the choice of a finite-dimensional approximation [14], as not being derived from an accurate operatorial formalism. This circumstance may mask the inconsistency in the equal-time commutation relations.

It is well known that the combination of first-class constraints with admissible unitary gauges makes a set of second-class constraints. Everything said above about the operatorial quantization of a general second-class constrained system holds true for this special case as well. Accordingly, the operatorial quantization of first-class constrained systems can be as a matter of fact carried out accurately in a unitary gauge only for simplest cases, when one manages to explicitly separate physical degrees of freedom. Electrodynamics and Yang-Mills field theory in the Coulomb or axial gauges belong to these cases.

It is remarkable that the method presented in this paper for operatorial canonical quantization of first-class constrained systems in relativistic gauges has a quite general applicability and is not restricted by the technical requirement that the constraints and gauge conditions be explicitly solvable. The operatorial gauge algebra generated by the original constraints may also be of the most general nature, (any rank) open and (any stage) reducible. In this respect the proposed method is, seemingly, most advanced towards solving the problem of operatorial quantization of constrained systems.

After these preliminary remarks we shall dwell in more detail only on those works which lie in the channel of the general method developed and are, thereby, most closely related to the purpose of the present work. Unless the opposite is indicated, we mean that the papers quoted below deal with the quantization understood in terms of the formal (euristic) path integral.

The canonical S-matrix for dynamical Bose systems was obtained for unitary (nonrelativistic) gauges in ref. [15]. The solution for the S-matrix of dynamical Bose-Fermi systems subjected to second-class constraints was obtained in ref. [16]. As a consequence of this result, the canonical S-matrix was obtained for Fermi-Bose systems with first- and second-class constraints in unitary gauges.

In ref. [17] the canonical formalism was extended to relativistic gauges. It was shown that constructing the S-matrix in a relativistic gauge reduces

(*) A new consistent approach to the solution of the operator quantization problem in second-class constraint case was developed recently by the present authors [13] in the framework of the *generalized* canonical quantization method.

to finding a unitarizing Hamiltonian in the relativistic phase space. The explicit form of this Hamiltonian was obtained in [17] for the case of dynamical Bose systems subject to first-class constraints that generate the rank-1 gauge algebra. Simultaneously, an important observation was made in [17] that the gauge part of the unitarizing Hamiltonian may be represented as the Poisson bracket between a fermion function, determined by the choice of the gauge, and another fermion function, formed by the constraints and the structural coefficients of their involutions. That was the first step towards understanding a universal relation between the structure of the unitarizing Hamiltonian and the process of the gauge algebra generation.

The basic formula for the classical unitarizing Hamiltonian, as well as the classical generating equations for the Hamiltonian gauge algebra, were formulated in ref. [18]. Their explicit solution was obtained in that reference for Bose-Fermi systems subject to first-class constraints which generate a rank-1 gauge algebra.

The authors of ref. [19] extended the definition of the unitarizing Hamiltonian and the form of the generating equations to the case when the second-class constraints are also present, and obtained the solution for the canonical S-matrix in relativistic gauges for Bose-Fermi systems subjected to constraints of both classes in the general case of any rank, irreducible gauge algebra.

The generalized canonical formalism developed in ref. [17-19] gave the possibility to achieve essential results when applied to special dynamical systems. The most important is the consistent canonical quantization and construction of the S-matrix for the Einstein gravity [20] and supergravity [21].

The generalized canonical formalism for systems subject to linearly dependent first-class constraints was first developed in ref. [22]. This allowed one to obtain the solution for the canonical S-matrix of dynamical Bose-Fermi systems subject to first-class constraints generating an open (any rank) and reducible (of any stage) Hamiltonian gauge algebra when (linearly independent) second-class constraints are also present. In the subsequent paper [23] this result was extended to the most general case when the second-class constraints may be also linearly dependent and of any stage of reducibility. All these papers dealt with quantization of constrained systems using the formal path integral in the phase space.

The clue to operatorial quantization of relativistic gauge systems was pointed out in ref. [24]. It is to impose canonical commutation relations on operators of the relativistic phase space that include, in the irreducible situation, the original dynamical variables, Lagrange multipliers for relativistic gauges and first-class constraints and the corresponding ghosts. Using this basic idea, the operatorial quantization of the Yang-Mills theory with relativistic gauge was considered in ref. [25, 26].

In ref. [27], which is a further development of the same idea, the operatorial canonical quantization was carried out for Bose-Fermi systems subject to first-class constraints which generate an open (any rank) irreducible gauge algebra. In that paper the operatorial version of the universal definition of the unitarizing Hamiltonian and of the gauge-algebra–generating equations was first formulated. In more explicit form, these results are presented in ref. [28], where the corresponding solution for the canonical S-matrix is also given in the form of accurate (nonformal) path integral in the phase space. In other words, it was first shown in that paper that on the virtual phase trajectories there acts a quantum gauge algebra, generated by the operator symbols of the first-class constraints.

In the next paper [29], dealing with the case of irreducibility, the closure and abelization procedures are formulated for the operatorial gauge algebra. The classical counterpart of this result is well known: first-class constraints may always be converted into Abelian ones, at least locally, by means of a rotation (*i.e.* by finding a linear superposition) performed with the use of a reversible matrix, depending, generally, on phase variables. In [29] a ghost-operator–dependent unitary transformation is defined that reduces the fermion generating operator to the form that corresponds to new constraint operators which commute among themselves. The same transformation, but accompanied with an effective form variation of the gauge fermion operator when applied to the bosonic generating operator, reduces the latter to the form which corresponds to a new Hamiltonian, commuting with the new constraints. Of course, as far as the field theory is concerned, the new operator-valued constraints and Hamiltonian are, generally, nonlocal and do not possess correct properties under the Lorentz group. Nevertheless, the dynamics of the new operator-valued phase variables governed by them is physically equivalent to the original one.

The goal of the present paper is to synthesize the results of [22, 27, 29]. Namely we are going to present 1) the consistent solution to the problem of canonical operatorial quantization of dynamical Bose-Fermi systems subject to first-class constraints which generate an open (any rank) and any-stage–reducible gauge algebra, 2) the formulation, for the general case of any-stage reducibility, of a regular procedure for closing and abelizing the operatorial gauge algebra. The paper is organized as follows.

Section **1** is of auxiliary nature and is intended to recall some elementary facts and relations concerning the correspondence between the symbol and its operator.

In sect. **2** we consider in every detail how the gauge algebra of any stage of reducibility is generated. Here the generating equations are formulated for the fermionic and bosonic generating operators of the gauge algebra which depend upon canonical pairs of operators belonging to the so-called minimal sector. Solution to these equations is searched for in the form of normally ordered power series in the ghost operators of the algebraic sector with all the ghost canonical momenta placed to the left of their canonical co-ordinates.

The coefficients of these series depend only on canonical pairs of operators of the original dynamical variables and are the structural operators of the gauge algebra.

In sect. 3 we give the operatorial version of the unitarizing Hamiltonian which is made up of the three basic components: fermionic and bosonic generating operators of the gauge algebra and the operator-valued gauge fermion. This unitarizing Hamiltonian makes a basis for the operatorial dynamical description of systems subject to first-class constraints.

Section 4 contains an explicit formulation of the operatorial dynamics. The time evolution of the operators belonging to the relativistic phase space is governed by the Heisenberg equations of motion depending on the unitarizing Hamiltonian of the theory. By introducing external sources in an usual way, one defines the generating functional of T-products of the Heisenberg dynamical variables. Simultaneously this generating operator-valued functional defines a canonical transformation that puts an operator in the new external-source–dependent representation into correspondence with each Heisenberg operator. An important consequence of the equations of motion in the new representation are the operatorial Ward identities.

In sect. 5 we derive, following the standard scheme [30-33], variational derivative equations for the generating functional of the quantum Green's functions, using the operatorial equations of motion dependent on the external sources. The solution to these equations has the form of a functional path integral in relativistic phase space. The effective action in this path integral contains the symbol of the unitarizing Hamiltonian with its time arguments moved apart. Thereby, the ordering of the operatorial factors is completely allowed for [33].

The procedure of closing and abelizing the operator-valued gauge algebra is formulated in sect. 6. We find the equation for the unitary operator of the canonical transformation in the minimal sector that reduces the fermionic generating operator of the gauge algebra to the form linear in the ghost canonical momenta. This means that the commutation relations of the new operator-valued first-class constraints have been closed among themselves. In order to close the commutation relations of the new constraints also with the new Hamiltonian, one must remove from the bosonic generating operator of the gauge algebra the terms nonlinear in the ghost momenta which have remained after the canonical transformation. This goal is achieved by a special form variation of the gauge fermion operator. This results in a new unitarizing Hamiltonian operator, which describes an equivalent dynamical system characterized by a closed, or even Abelian, gauge algebra.

Some conventions. Each quantity we shall be dealing with belongs to either of the two sets of uniform elements: bosons orfer mions. If n is the number of some objects, then n_+ (n_-) is the number of bosons (fermions) among them,

$n = n_+ + n_-$. Each quantity A is characterized by a function of the statistics —the so-called Grassmann parity $\mathscr{E}(A)$:

$$\mathscr{E}(A) \equiv \begin{cases} \bar{0} & \text{if } A \text{ is a boson,} \\ \bar{1} & \text{if } A \text{ is a fermion,} \end{cases}$$

where $(\bar{0}, \bar{1})$ are elements of the field of residues modulo 2. Evidently, $\mathscr{E}(AB) = \mathscr{E}(A) + \mathscr{E}(B)$. Each quantity A bears, besides, an internal label called a « ghost number » $\mathrm{gh}\,(A)$. After appropriate values of the ghost number have been attached to the elementary dynamical variables, we have, by definition, $\mathrm{gh}\,(AB) = \mathrm{gh}\,(A) + \mathrm{gh}\,(B)$ for any two quantities A and B. In what follows $\mathrm{rank}_\pm \|M\|$ denotes the set of ranks of the Bose-Bose (+) and Fermi-Fermi (—) boxes of the even rectangular matrix $\|M_{ab}\|$, $\mathscr{E}(M_{ab}) = \mathscr{E}_a + \mathscr{E}_b$. Right and left derivatives are, as usual, denoted as ∂_r and ∂_l respectively.

1. – Operators and symbols.

The quantum operator which corresponds to a classical *observable* A is denoted as $\hat{A}$. Their (common) Grassmann parity is denoted as $\mathscr{E}(\hat{A}) = \mathscr{E}(A)$, and the ghost number as $\mathrm{gh}\,(\hat{A}) = \mathrm{gh}\,(A)$. The supercommutation of any two operators $\hat{A}$ and $\hat{B}$ is defined as

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}(-1)^{\mathscr{E}(\hat{A})\mathscr{E}(\hat{B})}, \tag{1.1}$$

and possesses the following standard algebraic properties:

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}](-1)^{\mathscr{E}(\hat{A})\mathscr{E}(\hat{B})}, \tag{1.2}$$

$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}](-1)^{\mathscr{E}(\hat{A})\mathscr{E}(\hat{B})}, \tag{1.3}$$

$$[\hat{A}, [\hat{B}, \hat{C}]](-1)^{\mathscr{E}(\hat{A})\mathscr{E}(\hat{C})} + \text{cycl. perm.}\ (\hat{C}, \hat{A}, \hat{B}) = 0\,. \tag{1.4}$$

The symbol of an operator $\hat{A}$ is designated as $\tilde{A}$ and

$$\mathscr{E}(\tilde{A}) = \mathscr{E}(\hat{A})\,, \quad \mathrm{gh}\,(\tilde{A}) = \mathrm{gh}\,(\hat{A})\,, \quad [\tilde{A}, \tilde{B}] = [A, B] = 0\,. \tag{1.5}$$

The correspondence « symbol $\leftrightarrow$ operator » is one to one:

$$\hat{A} \leftrightarrow \tilde{A} \tag{1.6}$$

and defines an associative $*$-multiplication law of symbols

$$\hat{A}\hat{B} \leftrightarrow \tilde{A} * \tilde{B}\,, \tag{1.7}$$

so that commutation (1.1) is mapped onto the $*$-commutation of the symbols

$$[\hat{A}, \hat{B}] \leftrightarrow [\tilde{A}, \tilde{B}]_* , \tag{1.8}$$

where

$$[\tilde{A}, \tilde{B}]_* \equiv \tilde{A} * \tilde{B} - \tilde{B} * \tilde{A}(-1)^{\varepsilon(\hat{A})\varepsilon(\hat{B})} . \tag{1.9}$$

The $*$-commutation (1.8) has the same algebraic properties (1.2)-(1.4) under the $*$-multiplication as commutation (1.1) has under the operator one.

An explicit form of the $*$-multiplication law of symbols depends on how the normal form has been chosen, *i.e.* on how the elementary operators $\hat{P}_M$, $\hat{Q}^M$, $\hat{1}$ of the Heisenberg superalgebra

$$[\hat{P}_M, \hat{P}_{M'}] = 0 , \quad [\hat{Q}^M, \hat{P}_{M'}] = i\hbar\delta^M_{M'}\hat{1} , \quad [\hat{Q}^M, \hat{Q}^{M'}] = 0 \tag{1.10}$$

have been ordered. We shall be using the $\hat{P}\hat{Q}$-symbols corresponding to the $\hat{P}\hat{Q}$-normal form, *i.e.* the one when all $\hat{P}$ are placed to the left of all $\hat{Q}$:

$$\hat{A}(\hat{P}, \hat{Q}) = \exp\left[\hat{P}_M \frac{\partial_l}{\partial P_M}\right] \exp\left[\hat{Q}^M \frac{\partial_l}{\partial Q^M}\right] \tilde{A}(P, Q)\Big|_{P=0,Q=0} . \tag{1.11}$$

For the $\hat{P}\hat{Q}$-symbols the $*$-multiplication law has the form

$$\tilde{A} * \tilde{B} \equiv \tilde{A}\left(P, Q + i\hbar \frac{\partial_l}{\partial P'}\right)\tilde{B}(P + P', Q)\Big|_{P'=0} . \tag{1.12}$$

If the $\hat{P}\hat{Q}$-symbols of operators $\hat{A}$ and $\hat{B}$ are expandable into power series in

$$\tilde{A} \equiv A + \sum_{n=1}^{\infty} \hbar^n \tilde{A}_n , \quad \tilde{B} \equiv B + \sum_{n=1}^{\infty} \hbar^n \tilde{B}_n , \tag{1.13}$$

$$A = \lim_{\hbar\to 0} \tilde{A} , \quad B = \lim_{\hbar\to 0} \tilde{B} , \tag{1.14}$$

one has for their $*$-product (1.12) the expansion

$$\tilde{A} * \tilde{B} = AB + \sum_{n=1}^{\infty} \hbar^n \frac{i^n}{n!} (A, B)_n + \tag{1.15}$$
$$+ \sum_{n=2}^{\infty} \hbar^n \sum_{m=1}^{n-1} \frac{i^m}{m!} [(A, \tilde{B}_{n-m})_m + (\tilde{A}_{n-m}, B)_m] + \sum_{n=3}^{\infty} \hbar^n \sum_{m=2}^{n-1} \sum_{l=1}^{m-1} \frac{i^l}{l!} (\tilde{A}_{m-l}, \tilde{B}_{n-m})_l ,$$

wherein the concise notation

$$(A, B)_n \equiv \frac{\partial_r^n A}{\partial Q^{M_1} \dots \partial Q^{M_n}} \frac{\partial_l^n B}{\partial P_{M_1} \dots \partial P_{M_n}} \tag{1.16}$$

is used. The corresponding $\hbar$-power expansion of the $*$-commutation (1.9) has the form

$$(1.17)\qquad [\tilde{A}, \tilde{B}]_* = \sum_{n=1}^{\infty} \hbar^n \frac{i^n}{n!}\{A, B\}_n + \sum_{n=2}^{\infty} \hbar^n \sum_{m=1}^{n-1} \frac{i^m}{m!}(\{A, \tilde{B}_{n-m}\}_m + \{\tilde{A}_{n-m}, B\}_m) + \sum_{n=3}^{\infty} \hbar^n \sum_{m=2}^{n-1} \sum_{l=1}^{m-1} \frac{i^l}{2l!}(\{\tilde{A}_{m-l}, \tilde{B}_{n-m}\}_l + \{\tilde{A}_{n-m}, \tilde{B}_{m-l}\}_l),$$

where we used the notation

$$(1.18)\qquad \{A, B\}_n \equiv (A, B)_n - (B, A)_n(-1)^{\varepsilon(A)\varepsilon(B)}.$$

Thereby, for the classical limit (1.14) one has

$$(1.19)\qquad \lim_{\hbar\to 0}(i\hbar)^{-1}[\tilde{A}, \tilde{B}]_* = \{A, B\},$$

where

$$(1.20)\qquad \{A, B\} \equiv \{A, B\}_1 = \frac{\partial_r A}{\partial Q^M}\frac{\partial_l B}{\partial P_M} - \frac{\partial_r B}{\partial Q^M}\frac{\partial_l A}{\partial P_M}(-1)^{\varepsilon(A)\varepsilon(B)}$$

is the classical Poisson bracket.

Let now $\mathscr{T}_{\mathscr{D}}$ denote the Dyson time-ordering. Then, for every operator $\hat{A}(\hat{P}(t), \hat{Q}(t))$ one has

$$(1.21)\qquad \hat{A} = \lim_{\varepsilon\to+0}\mathscr{T}_{\mathscr{D}}\tilde{A}(\hat{P}(t+\varepsilon), \hat{Q}(t)),$$

$$(1.22)\qquad [\hat{Q}^M, \hat{A}] = i\hbar \lim_{\varepsilon\to+0}\mathscr{T}_{\mathscr{D}}\frac{\partial_l\tilde{A}(\hat{P}(t+\varepsilon), \hat{Q}(t))}{\partial\hat{P}_M},$$

$$(1.23)\qquad [\hat{A}, \hat{P}_M] = i\hbar \lim_{\varepsilon\to+0}\mathscr{T}_{\mathscr{D}}\frac{\partial_r\tilde{A}(\hat{P}(t+\varepsilon), \hat{Q}(t))}{\partial\hat{Q}^M}.$$

2. – Generating operatorial gauge algebra.

Let

$$(2.1)\qquad (\hat{p}_i, \hat{q}^i), \qquad i = 1, \dots, n,$$

be operators of original dynamical variables, there being n_+ bosonic and n_- fermionic pairs among them:

$$(2.2)\qquad n = n_+ + n_-,$$

and

$$\mathscr{E}(\hat{p}_i) = \mathscr{E}(\hat{q}^i) . \tag{2.3}$$

The zero value of the ghost number is attached to the initial operators (2.1):

$$\mathrm{gh}\,(\hat{p}_i) = -\,\mathrm{gh}\,(\hat{q}^i) = 0 . \tag{2.4}$$

The equal-time commutation relations for operators (2.1) are meant to be canonical, like eqs. (1.10), so that the commutator

$$[\hat{q}^j, \hat{p}_k] = i\hbar\delta^j_k \hat{1} \tag{2.5}$$

is the only nonzero one. Let, further, for each $s = 0, \ldots, L$ the distribution of the Grassmann parity

$$\mathscr{E}_{s\alpha_s} , \qquad \alpha_s = 1, \ldots, m_s , \tag{2.6}$$

be given, and $(m_s)_+$ and $(m_s)_-$ be the numbers of bosonic and fermionic coordinates respectively, among those distributed according to (2.6):

$$m_s = (m_s)_+ + (m_s)_- . \tag{2.7}$$

Define the sequence

$$\gamma_s(L)_\pm \equiv \sum_{s'=s}^{L} (m_{s'})_\pm (-1)^{(s-s')} , \tag{2.8}$$

and impose the following conditions on the numbers $(m_s)_\pm$:

$$n_\pm \geqslant \gamma_0(L)_\pm , \qquad \gamma_s(L)_\pm \geqslant 0 , \qquad s = 0, \ldots, L. \tag{2.9}$$

Consider now the following operators that form the ghost algebraic sector

$$(\hat{\bar{\mathscr{P}}}_{s\alpha_s}, \hat{C}^{\alpha_s}_s) , \qquad \alpha_s = 1, \ldots, m_s ;\ s = 0, \ldots, L. \tag{2.10}$$

Their statistics and the ghost number are, respectively,

$$\mathscr{E}(\hat{\bar{\mathscr{P}}}_{s\alpha_s}) = \mathscr{E}(\hat{C}^{\alpha_s}_s) = \mathscr{E}_{s\alpha_s} + s + 1 , \tag{2.11}$$

$$\mathrm{gh}\,(\hat{\bar{\mathscr{P}}}_{s\alpha_s}) = -\,\mathrm{gh}\,(\hat{C}^{\alpha_s}_s) = -\,(s+1) . \tag{2.12}$$

The equal-time commutation relations for operators (2.10) are meant to be canonical, in the sense of eqs. (1.10), too. The only nonzero commutator among them is

$$[\hat{C}^{\alpha_s}_s, \hat{\bar{\mathcal{P}}}_{r\beta_r}] = i\hbar\delta_{sr}\delta^{\alpha_s}_{\beta_r}\hat{1}\,. \tag{2.13}$$

We postulate, besides, the following Hermite-conjugation properties of (2.10), compatible with (2.13):

$$(\hat{\bar{\mathcal{P}}}_{s\alpha_s})^\dagger = \hat{\bar{\mathcal{P}}}_{s\alpha_s}(-1)^{\varepsilon(\bar{\mathcal{P}}_{s\alpha_s})}, \qquad (\hat{C}^{\alpha_s}_s)^\dagger = \hat{C}^{\alpha_s}_s\,. \tag{2.14}$$

The original operators (2.1) and the ghost operators (2.10) of the algebraic sector form together the so-called minimal sector, henceforth designated as $\hat{\Gamma}_{\rm min}$. For the operators (2.10) of the algebraic sector we shall be using the following collective notation:

$$(\hat{\bar{\mathcal{P}}}_A, \hat{C}^A) = (\hat{\bar{\mathcal{P}}}_{s\alpha_s}, \hat{C}^{\alpha_s}_s)\,, \tag{2.15}$$

where the collective index $A \equiv (s, \alpha_s)$ runs the set of values $s = 0, \dots, L$; $\alpha_s = 1, \dots, m_s$.

Thus

$$\hat{\Gamma}_{\rm min} \equiv (\hat{p}_i, \hat{\bar{\mathcal{P}}}_A; \hat{q}^i, \hat{C}^A)\,. \tag{2.16}$$

Define the fermionic ($\hat{\Omega}_{\rm min}$) and bosonic ($\hat{H}_{\rm min}$) generating operators of the gauge algebra to be solutions of the equations

$$[\hat{\Omega}_{\rm min}, \hat{\Omega}_{\rm min}] = 0\,, \qquad {\rm gh}\,(\hat{\Omega}_{\rm min}) = 1\,, \tag{2.17}$$

$$[\hat{H}_{\rm min}, \hat{\Omega}_{\rm min}] = 0\,, \qquad {\rm gh}\,(\hat{H}_{\rm min}) = 0\,, \tag{2.18}$$

in the minimal sector (2.16). We require also that the generating operators satisfy the formal Hermiticity condition

$$(\hat{\Omega}_{\rm min})^\dagger = \hat{\Omega}_{\rm min}\,, \tag{2.19}$$

$$(\hat{H}_{\rm min})^\dagger = \hat{H}_{\rm min}\,. \tag{2.20}$$

Due to the Jacobi identities (1.4) one has

$$[[\hat{\Omega}_{\rm min}, \hat{\Omega}_{\rm min}], \hat{\Omega}_{\rm min}] \equiv 0\,, \tag{2.21}$$

$$[[\hat{H}_{\rm min}, \hat{\Omega}_{\rm min}], \hat{\Omega}_{\rm min}] \equiv \tfrac{1}{2}[\hat{H}_{\rm min}, [\hat{\Omega}_{\rm min}, \hat{\Omega}_{\rm min}]] = 0\,, \tag{2.22}$$

the cyclic operator identity (2.21) providing the necessary solvability conditions for eq. (2.17). On the other hand, the fact that the right-hand side of (2.22) vanishes due to (2.17) guarantees that the necessary conditions for the solvability of eq. (2.18) are fulfilled.

Solution to eqs. (2.17), (2.18) is looked for in the form of a power series (*) in operators (2.15) taken in the $\hat{\bar{\mathscr{P}}}\hat{C}$-normal form (*i.e.* where all $\hat{\bar{\mathscr{P}}}$ are on the left of all $\hat{C}$)

$$\hat{\Omega}_{\min} = \hat{U}_0 + \sum_{n=1}^{\infty} \hat{\bar{\mathscr{P}}}_{A_n} \dots \hat{\bar{\mathscr{P}}}_{A_1} \hat{U}^{A_1 \dots A_n}, \tag{2.23}$$

$$\hat{H}_{\min} = \hat{V}_0 + \sum_{n=1}^{\infty} \hat{\bar{\mathscr{P}}}_{A_n} \dots \hat{\bar{\mathscr{P}}}_{A_1} \hat{V}^{A_1 \dots A_n}. \tag{2.24}$$

The coefficients of these series

$$\hat{U}_0, \quad \hat{U}^{A_1 \dots A_n}, \qquad n = 1, \dots, \tag{2.25}$$

$$\hat{V}_0, \quad \hat{V}^{A_1 \dots A_n}, \qquad n = 1, \dots, \tag{2.26}$$

possess the following statistics and ghost number

$$\mathscr{E}(\hat{U}_0) = 1, \qquad \mathscr{E}(\hat{U}^{A_1 \dots A_n}) = \sum_{j=1}^{n} \mathscr{E}(\hat{C}^{A_j}) + 1, \tag{2.27}$$

$$\mathscr{E}(\hat{V}_0) = 0, \qquad \mathscr{E}(\hat{V}^{A_1 \dots A_n}) = \sum_{j=1}^{n} \mathscr{E}(\hat{C}^{A_j}), \tag{2.28}$$

$$\mathrm{gh}\,(\hat{U}_0) = 1, \quad \mathrm{gh}\,(\hat{U}^{A_1 \dots A_n}) = \sum_{j=1}^{n} \mathrm{gh}\,(\hat{C}^{A_j}) + 1, \tag{2.29}$$

$$\mathrm{gh}\,(\hat{V}_0) = 0, \quad \mathrm{gh}\,(\hat{V}^{A_1 \dots A_n}) = \sum_{j=1}^{n} \mathrm{gh}\,(\hat{C}^{A_j}), \tag{2.30}$$

and the following property, called generalized symmetry:

$$\hat{U}^{A_1 \dots A_n} = (\hat{U}_{\mathrm{sym}})^{A_1 \dots A_n}, \tag{2.31}$$

$$\hat{V}^{A_1 \dots A_n} = (\hat{V}_{\mathrm{sym}})^{A_1 \dots A_n}, \tag{2.32}$$

where for every $\hat{K}^{A_1 \dots A_n}$ the generalized symmetrization is defined as

$$(\hat{K}_{\mathrm{sym}})^{A_1 \dots A_n} \equiv S^{A_1 \dots A_n}_{B_1 \dots B_n} K^{B_1 \dots B_n}, \tag{2.33}$$

$$n!\, S^{A_1 \dots A_n}_{B_1 \dots B_n} \equiv \left(\frac{\partial_l}{\partial \bar{\mathscr{P}}_{A_1}} \dots \frac{\partial_l}{\partial \bar{\mathscr{P}}_{A_n}} \bar{\mathscr{P}}_{B_n} \dots \bar{\mathscr{P}}_{B_1} \right). \tag{2.34}$$

(*) If series (2.23), (2.24) contain a finite number of terms, the highest power of the ghost momenta $\hat{\bar{\mathscr{P}}}_A$ being (r, r') we refer to the rank-(r, r') gauge algebra. Any closed algebra is of rank $- r = r' = 1$.

By substituting expansion (2.23) into eq. (2.17) and transforming the l.h.s. to the $\hat{\hat{\mathscr{P}}}\hat{\mathcal{C}}$-normal form, we obtain the following relations for the operator-valued coefficients (2.31):

$$[\hat{U}_0, \hat{U}_0] + 2i\hbar \frac{\partial_r \hat{U}_0}{\partial \hat{\mathcal{C}}^A} \hat{U}^A = 0 \,, \tag{2.35}$$

$$[\hat{U}^{A_1 \ldots A_n}, \hat{U}_0] + \sum_{k=1}^{\infty} (i\hbar)^k \frac{\partial_r^k \hat{U}^{A_1 \ldots A_n}}{\partial \hat{\mathcal{C}}^{B_1} \ldots \partial \hat{\mathcal{C}}^{B_k}} \hat{U}^{B_1 \ldots B_k} + (\hat{X}_{\mathrm{sym}})^{A_1 \ldots A_n} = -i\hbar(n+1) \frac{\partial_r \hat{U}_0}{\partial \hat{\mathcal{C}}^A} \hat{U}^{AA_1 \ldots A_n} (-1)^{\mathscr{E}_0^n} \,, \tag{2.36}$$

where

$$\hat{X}^{A_1} \equiv 0 \,, \quad \hat{X}^{A_1 \ldots A_n} \equiv \sum_{m=1}^{n-1} (-1)^{\mathscr{E}_m^n} \Bigg(\frac{1}{2} [\hat{U}^{A_1 \ldots A_m}, \hat{U}^{A_{m+1} \ldots A_n}] + \sum_{k=1}^{\infty} \frac{(n-m+k)!}{k!(n-m)!} (i\hbar)^k \frac{\partial_r^k \hat{U}^{A_1 \ldots A_m}}{\partial \hat{\mathcal{C}}^{B_1} \ldots \partial \hat{\mathcal{C}}^{B_k}} \hat{U}^{B_1 \ldots B_k A_{m+1} \ldots A_n} \Bigg) , \tag{2.37}$$

$$\mathscr{E}_m^n \equiv \sum_{j=m+1}^{n} \mathscr{E}(\hat{\mathcal{C}}^{A_j}) \,. \tag{2.38}$$

Analogously, by substituting expansions (2.23), (2.24) into eq. (2.18) and reducing its l.h.s. to the $\hat{\hat{\mathscr{P}}}\hat{\mathcal{C}}$-normal form we obtain the following relations for the operator-valued coefficients (2.26):

$$[\hat{V}_0, \hat{U}_0] = i\hbar \frac{\partial_r \hat{U}_0}{\partial \hat{\mathcal{C}}^A} \hat{V}^A \,, \tag{2.39}$$

$$[\hat{V}^{A_1 \ldots A_n}, \hat{U}_0] - [\hat{U}^{A_1 \ldots A_n}, \hat{V}_0] + \sum_{k=1}^{\infty} (i\hbar)^k \left(\frac{\partial_r^k \hat{V}^{A_1 \ldots A_n}}{\partial \hat{\mathcal{C}}^{B_1} \ldots \partial \hat{\mathcal{C}}^{B_k}} \hat{U}^{B_1 \ldots B_k} - \frac{\partial_r^k \hat{U}^{A_1 \ldots A_n}}{\partial \hat{\mathcal{C}}^{B_1} \ldots \partial \hat{\mathcal{C}}^{B_k}} \hat{V}^{B_1 \ldots B_k} \right) + (\hat{Y}_{\mathrm{sym}})^{A_1 \ldots A_n} = i\hbar(n+1) \frac{\partial_r \hat{U}_0}{\partial \hat{\mathcal{C}}^A} \hat{V}^{AA_1 \ldots A_n} (-1)^{\mathscr{E}_0^n} \,, \tag{2.40}$$

where

$$\hat{Y}^{A_1} \equiv 0 \,, \quad \hat{Y}^{A_1 \ldots A_n} \equiv \sum_{m=1}^{n-1} \Bigg([\hat{V}^{A_1 \ldots A_m}, \hat{U}^{A_{m+1} \ldots A_n}] + \sum_{k=1}^{\infty} \frac{(n-m+k)!}{k!(n-m)!} (i\hbar)^k \bigg(\frac{\partial_r^k \hat{V}^{A_1 \ldots A_m}}{\partial \hat{\mathcal{C}}^{B_1} \ldots \partial \hat{\mathcal{C}}^{B_k}} \hat{U}^{B_1 \ldots B_k A_{m+1} \ldots A_n} - (-1)^{\mathscr{E}_m^n} \frac{\partial_r^k \hat{U}^{A_1 \ldots A_m}}{\partial \hat{\mathcal{C}}^{B_1} \ldots \partial \hat{\mathcal{C}}^{B_k}} \hat{V}^{B_1 \ldots B_k A_{m+1} \ldots A_n} \bigg) \Bigg) . \tag{2.41}$$

Conditions (2.19), (2.20) of formal Hermiticity of operators (2.23), (2.24) imply the following properties of the operator-valued coefficients (2.25), (2.26) under the Hermitian conjugation:

$$(2.42)\qquad \hat{U}_0 = (\hat{U}_0)^\dagger + \sum_{k=1}^{\infty} (i\hbar)^k \frac{\partial_1^k (\hat{U}^{B_1 \dots B_k})^\dagger}{\partial \hat{C}^{B_1} \dots \partial \hat{C}^{B_k}},$$

$$(2.43)\qquad (-1)^{\mathscr{E}_0^n} \hat{U}^{A_1 \dots A_n} = (\hat{U}^{A_n \dots A_1})^\dagger + \sum_{k=1}^{\infty} \frac{(n+k)!}{n!\,k!} (i\hbar)^k \frac{\partial_1^k (\hat{U}^{A_n \dots A_1 B_1 \dots B_k})^\dagger}{\partial \hat{C}^{B_1} \dots \partial \hat{C}^{B_k}},$$

$$(2.44)\qquad \hat{V}_0 = (\hat{V}_0)^\dagger + \sum_{k=1}^{\infty} (-1)^{\mathscr{E}_0^k} (i\hbar)^k \frac{\partial_1^k (\hat{V}^{B_1 \dots B_k})^\dagger}{\partial \hat{C}^{B_1} \dots \partial \hat{C}^{B_k}},$$

$$(2.45)\qquad \hat{V}^{A_1 \dots A_n} = (\hat{V}^{A_n \dots A_1})^\dagger + \sum_{k=1}^{\infty} \frac{(n+k)!}{n!\,k!} (-1)^{\mathscr{E}_0^k} (i\hbar)^k \frac{\partial_1^k (\hat{V}^{A_n \dots A_1 B_1 \dots B_k})^\dagger}{\partial \hat{C}^{B_1} \dots \partial \hat{C}^{B_k}}.$$

The operatorial coefficients (2.25), (2.26) are polynomials in powers of the operators $\hat{C}^A$:

$$(2.46)\qquad \hat{U}_0 = \hat{T}_{0\alpha_0}(\hat{p}, \hat{q}) \hat{C}_0^{\alpha_0},$$

$$(2.47)\qquad \hat{U}^{A_1 \dots A_n} = \sum_{m=1}^{\infty} \sum_{\{B\} = \mu_m^n} \frac{(-1)^{E_{B_m \dots B_1}^{A_1 \dots A_n}}}{n!\,m!} \hat{U}_{B_m \dots B_1}^{A_1 \dots A_n}(\hat{p}, \hat{q}) \hat{C}^{B_1} \dots \hat{C}^{B_m},$$

$$(2.48)\qquad \hat{V}_0 = \hat{H}_0(\hat{p}, \hat{q}),$$

$$(2.49)\qquad \hat{V}^{A_1 \dots A_n} = \sum_{m=1}^{\infty} \sum_{\{B\} = \nu_m^n} \frac{(-1)^{E_{B_m \dots B_1}^{A_1 \dots A_n}}}{n!\,m!} \hat{V}_{B_m \dots B_1}^{A_1 \dots A_n}(\hat{p}, \hat{q}) \hat{C}^{B_1} \dots \hat{C}^{B_m},$$

where μ_m^n, ν_m^n are the sets of values for the indices $B_1, \dots, B_m$ determined by the conditions

$$(2.50)\qquad \mu_m^n: \sum_{j=1}^{m} \mathrm{gh}(\hat{C}^{B_j}) = \sum_{i=1}^{n} \mathrm{gh}(\hat{C}^{A_i}) + 1,$$

$$(2.51)\qquad \nu_m^n: \sum_{j=1}^{m} \mathrm{gh}(\hat{C}^{B_j}) = \sum_{i=1}^{n} \mathrm{gh}(\hat{C}^{A_i}).$$

The signature factor in (2.47), (2.49) is defined as

$$(2.52)\qquad E_{B_m \dots B_1}^{A_1 \dots A_n} = E^{A_1 \dots A_n} + E_{B_m \dots B_1}.$$

For $n = 1$ or $m = 1$ one has

$$(2.53)\qquad E^{A_1} \equiv 0, \quad E_{B_1} \equiv 0,$$

whereas, for $n > 2$, $m > 2$,

$$(2.54)\qquad E^{A_1 \dots A_n} \equiv \sum_{k=2}^{n} \sum_{j=k}^{n} (\mathscr{E}(\hat{C}^{A_j}) + 1),$$

$$(2.55)\qquad E_{B_m \dots B_1} \equiv \sum_{k=2}^{m} \sum_{j=k}^{m} (\mathscr{E}(\hat{C}^{B_j}) + 1).$$

The operators

$$\hat{T}_{0\alpha_0}, \quad \hat{U}^{A_1\ldots A_n}_{B_m\ldots B_1} \tag{2.56}$$

from (2.46), (2.47), as well as

$$\hat{H}_0, \quad \hat{V}^{A_1\ldots A_n}_{B_m\ldots B_1} \tag{2.57}$$

from (2.48), (2.49) are functions only of the original operators (2.1) and form a structural set (basis) of operators of the gauge algebra. Due to (2.4) every operator (2.56), (2.57) has its ghost number zero, while its statistics follows from (2.27), (2.28) to be

$$\mathscr{E}(\hat{T}_{0\alpha_0}) = \mathscr{E}_{0\alpha_0}, \quad \mathscr{E}(\hat{U}^{A_1\ldots A_n}_{B_m\ldots B_1}) = \sum_{j=1}^{n} \mathscr{E}(\hat{C}^{A_j}) + \sum_{j=1}^{m} \mathscr{E}(\hat{C}^{B_j}) + 1, \tag{2.58}$$

$$\mathscr{E}(\hat{H}_0) = 0, \quad \mathscr{E}(\hat{V}^{A_1\ldots A_n}_{B_m\ldots B_1}) = \sum_{j=1}^{n} \mathscr{E}(\hat{C}^{A_j}) + \sum_{j=1}^{m} \mathscr{E}(\hat{C}^{B_j}). \tag{2.59}$$

If any two neighbouring upper A_{i-1}, A_i or lower B_i, B_{i-1} indices are interchanged, the operators (2.56), (2.57) acquire the signature factors

$$-(-1)^{(\mathscr{E}(\hat{C}^{A_{i-1}})+1)(\mathscr{E}(\hat{C}^{A_i})+1)}, \tag{2.60}$$

or

$$-(-1)^{(\mathscr{E}(\hat{C}^{B_{i-1}})+1)(\mathscr{E}(\hat{C}^{B_i})+1)}, \tag{2.61}$$

respectively. Let us use the following detailed indexing of any quantity

$$\hat{K}^{A_1\ldots A_n}_{B_m\ldots B_1} \equiv \hat{K}^{s_1\ldots s_n|\alpha^1_{s_1}\ldots\alpha^n_{s_n}}_{r_m\ldots r_1|\beta^m_{r_m}\ldots\beta^1_{r_1}}, \tag{2.62}$$

which decodes the condensed, in the sense of (2.15), upper

$$A_i = (s_i, \alpha^i_{s_i}), \qquad i = 1, \ldots, n, \tag{2.63}$$

and lower indices

$$B_j = (r_j, \beta^j_{r_j}), \qquad j = 1, \ldots, m, \tag{2.64}$$

where

$$\begin{cases} 0 \leqslant s_i \leqslant L, & 0 \leqslant r_j \leqslant L, \\ 1 \leqslant \alpha^i_{s_i} \leqslant m_{s_i}, & 1 \leqslant \beta^j_{r_j} \leqslant m_{r_j}. \end{cases} \tag{2.65}$$

With designations (2.62) operators (2.15) may be written as

$$\hat{\bar{\mathscr{P}}}_A = \hat{\bar{\mathscr{P}}}_{s|\alpha_s} \equiv \hat{\bar{\mathscr{P}}}_{s\alpha_s}, \quad \hat{C}^A = \hat{C}^{s|\alpha_s} \equiv \hat{C}^{\alpha_s}_s. \tag{2.66}$$

The sets of values of μ^n_m, ν^n_m described in (2.50), (2.51) are now presented as

$$\mu^n_m : \sum_{j=1}^{m} (r_j + 1) = \sum_{i=1}^{n} (s_i + 1) + 1\,, \tag{2.67}$$

$$\nu^n_m : \sum_{j=1}^{m} (r_j + 1) = \sum_{i=1}^{n} (s_i + 1)\,. \tag{2.68}$$

If operators (2.56), (2.57) with the lower indices belonging to the sets μ^n_m, ν^n_m, respectively, are taken for the quantities (2.62), their Grassmann parity owing to (2.58), (2.59) is

$$\sum_{i=1}^{n} \mathscr{E}_{s_i\alpha^i_{s_i}} + \sum_{j=1}^{m} \mathscr{E}_{r_j\beta^j_{r_j}}\,, \tag{2.69}$$

both for (2.56) and (2.57), $\mathscr{E}_{s\alpha_s}$ being the Grassmann parities from distribution (2.6). Powers (2.54), (2.55) of the signature factors are written as

$$E^{s_1\ldots s_n|\alpha^1_{s_1}\ldots\alpha^n_{s_n}} = \sum_{k=2}^{n} \sum_{j=k}^{n} (\mathscr{E}_{s_j\alpha^j_{s_j}} + s_j)\,, \tag{2.70}$$

$$E_{r_m\ldots r_1|\beta^m_{r_m}\ldots\beta^1_{r_1}\ldots} = \sum_{k=2}^{m} \sum_{j=k}^{m} (\mathscr{E}_{r_j\beta^j_{r_j}} + r_j)\,. \tag{2.71}$$

Let us use the special notation

$$\hat{Z}^{\alpha_{s-1}}_{s\alpha_s} \equiv \hat{U}^{s-1}{}_{s}{}^{|\alpha_{s-1}}_{|\alpha_s}(\hat{p}, \hat{q}) \tag{2.72}$$

for the operators (2.56) when $n = m = 1$ in the sectors

$$A_1 = (s-1, \alpha_{s-1})\,, \qquad B_1 = (s, \alpha_s)\,.$$

For $n = 1$, expansions (2.48), (2.49) take in the zeroth sector $A_1 = (0, \alpha_0)$ the explicit form

$$\hat{U}^{0|\alpha_0} = \tfrac{1}{2}\, \hat{U}^{0}_{00}{}^{|\alpha_0}_{|\beta_0\gamma_0}(\hat{p}, \hat{q})\, \hat{C}^{\gamma_0}_0 \hat{C}^{\beta_0}_0 (-1)^{\mathscr{E}_{0\beta_0}} + \hat{Z}^{\alpha_0}_{1\alpha_1}(\hat{p}, \hat{q})\, \hat{C}^{\alpha_1}_1\,, \tag{2.73}$$

$$\hat{V}^{0|\alpha_0} = \hat{V}^{0}_{0}{}^{|\alpha_0}_{|\beta_0}(\hat{p}, \hat{q})\, \hat{C}^{\beta_0}_0\,. \tag{2.74}$$

By substituting (2.46), (2.73) into the lowest-order equation (2.35) we obtain

$$[\hat{T}_{0\alpha_0}, \hat{T}_{0\beta_0}] = i\hbar \hat{T}_{0\gamma_0} \hat{U}^{\gamma_0}_{\alpha_0\beta_0}\,, \tag{2.75}$$

$$\hat{T}_{0\alpha_0} \hat{Z}^{\alpha_0}_{1\alpha_1} = 0\,, \tag{2.76}$$

where the concise notation

$$\hat{U}^{\gamma_0}_{\alpha_0\beta_0} \equiv \hat{U}^{0}_{00}{}^{|\gamma_0}_{|\alpha_0\beta_0}(\hat{p}, \hat{q}) \tag{2.77}$$

is used, the operators (2.77) being antisymmetric with respect to the lower indices

$$\hat{U}^{\gamma_0}_{\alpha_0\beta_0} = -\hat{U}^{\gamma_0}_{\beta_0\alpha_0}(-1)^{\varepsilon_{0\alpha_0}\varepsilon_{0\beta_0}}. \tag{2.78}$$

The substitution of (2.46), (2.48), (2.74) into the lowest-order equation (2.39) results in

$$[\hat{H}_0, \hat{T}_{0\beta_0}] = i\hbar \hat{T}_{0\gamma_0} \hat{V}^{\gamma_0}_{\beta_0}, \tag{2.79}$$

where the notation

$$\hat{V}^{\gamma_0}_{\beta_0} \equiv \hat{V}^{0}_{0}{}^{|\gamma_0}_{|\beta_0}(\hat{p}, \hat{q}) \tag{2.80}$$

has been used. Equations (2.75), (2.79) are nothing but involution equations of the operatorial first-class constraints $\hat{T}_{0\alpha_0}(\hat{p}, \hat{q})$ among themselves and with the original Hamiltonian $\hat{H}_0(\hat{p}, \hat{q})$. Equation (2.76) tells that $\hat{Z}^{\alpha_0}_{1\alpha_1}(\hat{p}, \hat{q})$ are right-handed operator-valued null vectors of the operator-valued first-class constraints. Now consider eq. (2.36) at $n = 1$ in the sector $A_1 = (0, \alpha_0)$:

$$[\hat{U}^{0|\alpha_0}, \hat{U}_0] + i\hbar \frac{\partial_r \hat{U}^{0|\alpha_0}}{\partial \hat{C}^{\beta_0}_0} \hat{U}^{0|\beta_0} + i\hbar \frac{\partial_r \hat{U}^{0|\alpha_0}}{\partial \hat{C}^{\beta_1}_1} \hat{U}^{1|\beta_1} + \\ + (i\hbar)^2 \frac{\partial^2_r \hat{U}^{0|\alpha_0}}{\partial \hat{C}^{\beta_0}_0 \partial \hat{C}^{\gamma_0}_0} \hat{U}^{00|\beta_0\gamma_0} = 2i\hbar \hat{T}_{0\beta_0} \hat{U}^{00|\beta_0\alpha_0}(-1)^{\varepsilon_{0\alpha_0}}, \tag{2.81}$$

where

$$\hat{U}^{1|\alpha_1} = \tfrac{1}{6} \hat{U}^{1}_{000}{}^{|\alpha_1}_{|\alpha_0\beta_0\gamma_0}(\hat{p}, \hat{q}) \hat{C}^{\gamma_0}_0 \hat{C}^{\beta_0}_0 \hat{C}^{\alpha_0}_0 (-1)^{\varepsilon_{0\beta_0}} + \\ + \tfrac{1}{2} \big(\hat{U}^{1}_{01}{}^{|\alpha_1}_{|\beta_0\beta_1}(\hat{p}, \hat{q}) \hat{C}^{\beta_1}_1 \hat{C}^{\beta_0}_0 (-1)^{\varepsilon_{0\beta_0}} - \\ - \hat{U}^{1}_{10}{}^{|\alpha_1}_{|\beta_1\beta_0}(\hat{p}, \hat{q}) \hat{C}^{\beta_0}_0 \hat{C}^{\beta_1}_1 (-1)^{\varepsilon_{1\beta_1}} \big) + \hat{Z}^{\alpha_1}_{2\alpha_2}(\hat{p}, \hat{q}) \hat{C}^{\alpha_2}_2, \tag{2.82}$$

$$(-1)^{\varepsilon_{0\nu_0}} \hat{U}^{00|\mu_0\nu_0} = \tfrac{1}{12} \hat{U}^{00}_{000}{}^{|\mu_0\nu_0}_{|\alpha_0\beta_0\gamma_0}(\hat{p}, \hat{q}) \hat{C}^{\gamma_0}_0 \hat{C}^{\beta_0}_0 \hat{C}^{\alpha_0}_0 (-1)^{\varepsilon_{0\beta_0}} + \\ + \tfrac{1}{4} \big(\hat{U}^{00}_{01}{}^{|\mu_0\nu_0}_{|\beta_0\beta_1}(\hat{p}, \hat{q}) \hat{C}^{\beta_1}_1 \hat{C}^{\beta_0}_0 (-1)^{\varepsilon_{0\beta_0}} - \hat{U}^{00}_{10}{}^{|\mu_0\nu_0}_{|\beta_1\beta_0}(\hat{p}, \hat{q}) \hat{C}^{\beta_0}_0 \hat{C}^{\beta_1}_1 (-1)^{\varepsilon_{1\beta_1}} \big) + \\ + \tfrac{1}{2} \hat{U}^{00}_{2}{}^{|\mu_0\nu_0}_{|\alpha_2}(\hat{p}, \hat{q}) \hat{C}^{\alpha_2}_2. \tag{2.83}$$

From (2.81) the following relations result for the coefficients in the ghost-power operatorial expansions (2.73), (2.82), (2.83): the $\hat{C}_0^3$ terms produce

$$(2.84)\quad ([\hat{U}^{\alpha_0}_{\beta_0\gamma_0}, \hat{T}_{0\delta_0}] - i\hbar \hat{U}^{\alpha_0}_{\beta_0\mu_0}\hat{U}^{\mu_0}_{\gamma_0\delta_0})(-1)^{\mathcal{E}_{0\beta_0}\mathcal{E}_{0\delta_0}} + \text{cycl. perm. } (\beta_0, \gamma_0, \delta_0) = = \tfrac{1}{2} i\hbar \hat{\Pi}^{\alpha_0}_{\mu_0\nu_0}\hat{U}^{\nu_0\mu_0}_{\beta_0\gamma_0\delta_0} - i\hbar \hat{Z}^{\alpha_0}_{1\alpha_1}\hat{U}^{\alpha_1}_{\beta_0\gamma_0\delta_0},$$

where the concise notations have been used

$$(2.85)\quad \hat{\Pi}^{\alpha_0}_{\mu_0\nu_0} \equiv \delta^{\alpha_0}_{\mu_0}\hat{T}_{0\nu_0} - \delta^{\alpha_0}_{\nu_0}\hat{T}_{0\mu_0}(-1)^{\mathcal{E}_{0\mu_0}\mathcal{E}_{0\nu_0}} - i\hbar \hat{U}^{\alpha_0}_{\mu_0\nu_0}, \qquad \hat{T}_{0\alpha_0}\hat{\Pi}^{\alpha_0}_{\mu_0\nu_0} \equiv 0,$$

$$(2.86)\quad \hat{U}^{\mu_0\nu_0}_{\beta_0\gamma_0\delta_0} \equiv \hat{U}^{00|\mu_0\nu_0}_{000|\beta_0\gamma_0\delta_0}(\hat{p}, \hat{q})(-1)^{\mathcal{E}_{0\beta_0}\mathcal{E}_{0\delta_0}},$$

$$(2.87)\quad \hat{U}^{\alpha_1}_{\beta_0\gamma_0\delta_0} \equiv \hat{U}^{1|\alpha_1}_{000|\beta_0\gamma_0\delta_0}(\hat{p}, \hat{q})(-1)^{\mathcal{E}_{0\beta_0}\mathcal{E}_{0\delta_0}},$$

whereas the $\hat{C}_0\hat{C}_1$ terms produce

$$(2.88)\quad [\hat{Z}^{\alpha_0}_{1\alpha_1}, \hat{T}_{0\beta_0}] - i\hbar \hat{U}^{\alpha_0}_{\beta_0\gamma_0}\hat{Z}^{\gamma_0}_{1\alpha_1}(-1)^{\mathcal{E}_{1\alpha_1}\mathcal{E}_{0\beta_0}} = -\tfrac{1}{2} i\hbar \hat{\Pi}^{\alpha_0}_{\mu_0\nu_0}\hat{U}^{\nu_0\mu_0}_{\alpha_1\beta_0} + i\hbar \hat{Z}^{\alpha_0}_{1\beta_1}\hat{U}^{\beta_1}_{\alpha_1\beta_0},$$

where

$$(2.89)\quad \hat{U}^{\mu_0\nu_0}_{\alpha_1\beta_0} \equiv \hat{U}^{00|\mu_0\nu_0}_{10|\alpha_1\beta_0}(\hat{p}, \hat{q}),$$

$$(2.90)\quad \hat{U}^{\beta_1}_{\alpha_1\beta_0} \equiv \hat{U}^{1|\beta_1}_{10|\alpha_1\beta_0}(\hat{p}, \hat{q}).$$

The $\hat{C}_2$ terms give

$$(2.91)\quad \hat{Z}^{\alpha_0}_{1\alpha_1}\hat{Z}^{\alpha_1}_{2\alpha_2} = \tfrac{1}{2}\hat{\Pi}^{\alpha_0}_{\mu_0\nu_0}\hat{U}^{\nu_0\mu_0}_{\alpha_2},$$

where

$$(2.92)\quad \hat{U}^{\mu_0\nu_0}_{\alpha_2} \equiv \hat{U}^{00|\mu_0\nu_0}_{2|\alpha_2}(\hat{p}, \hat{q}).$$

Under the multiplication of eq. (2.84) by $\hat{T}_{0\alpha_0}$ on the left-hand side, its right-hand side disappears owing to (2.76), (2.85), and one has

$$(2.93)\quad \hat{T}_{0\alpha_0}\{([\hat{U}^{\alpha_0}_{\beta_0\gamma_0}, \hat{T}_{0\delta_0}] - i\hbar \hat{U}^{\alpha_0}_{\beta_0\mu_0}\hat{U}^{\mu_0}_{\gamma_0\delta_0})(-1)^{\mathcal{E}_{0\beta_0}\mathcal{E}_{0\delta_0}} + + \text{cycl. perm. } (\beta_0, \gamma_0, \delta_0)\} = 0.$$

Equation (2.93) is nothing but a necessary condition for the compatibility of involution (2.75). It is a consequence of the cyclic Jacobi identity (1.4) for the operator of constraints

$$(2.94)\quad [[\hat{T}_{0\beta_0}, \hat{T}_{0\gamma_0}], \hat{T}_{0\delta_0}](-1)^{\mathcal{E}_{0\beta_0}\mathcal{E}_{0\delta_0}} + \text{cycl. perm. } (\beta_0, \gamma_0, \delta_0) \equiv 0.$$

Thus eq. (2.84), to be referred to as the lowest Jacobi relation, provides the formal fulfilment of the necessary compatibility conditions for the involution relations (2.75) for the constraints.

Now multiply (2.88) by $\hat{T}_{0\alpha_0}$ on the l.h.s. Again in virtue of (2.76), (2.85), the right-hand side of (2.88) does not contribute and we get

$$\hat{T}_{0\alpha_0}[\hat{Z}^{\alpha_0}_{1\alpha_1}, \hat{T}_{0\beta_0}] - i\hbar\hat{T}_{0\alpha_0}\hat{U}^{\alpha_0}_{\beta_0\gamma_0}\hat{Z}^{\gamma_0}_{1\alpha_1}(-1)^{\varepsilon_{1\alpha_1}\varepsilon_{0\beta_0}} = 0 . \tag{2.95}$$

This relation is necessary for involution (2.75) and eq. (2.76) to be compatible. Indeed we obtain from (2.76) that

$$[\hat{T}_{0\alpha_0}\hat{Z}^{\alpha_0}_{1\alpha_1}, \hat{T}_{0\beta_0}] = 0 , \tag{2.96}$$

whence (2.95) results, in virtue of (2.75). Therefore, eq. (2.88) guarantees the formal fulfilment of the necessary compatibility conditions of involution (2.75) and eq. (2.76).

At last, eq. (2.91) shows that $\hat{Z}^{\alpha_1}_{2\alpha_2}$ are «weak» right-handed operator-valued null vectors of the operators $\hat{Z}^{\alpha_0}_{1\alpha_1}$. We bear in mind that in the classical limit (2.91) leads to

$$Z^{\alpha_0}_{1\alpha_1}Z^{\alpha_1}_{2\alpha_2}|_{T_0=0} = 0 . \tag{2.97}$$

The conditions for compatibility of eqs. (2.91), necessary in view of (2.76), are formally fulfilled owing to (2.85).

Equations (2.84), (2.88), (2.91) saturate the contents of (2.81). Now turn to eq. (2.40), at $n = 1$ in the sector $A_1 = (0, \alpha_0)$:

$$\begin{aligned}[][\hat{V}^{0|\alpha_0}, \hat{U}_0] - [\hat{U}^{0|\alpha_0}, \hat{H}_0] + i\hbar\frac{\partial_r\hat{V}^{0|\alpha_0}}{\partial\hat{C}^{\beta_1}_0}\hat{U}^{0|\beta_0} - i\hbar\frac{\partial_r\hat{U}^{0|\alpha_0}}{\partial\hat{C}^{\beta_0}_0}\hat{V}^{0|\beta_0} - \\ - i\hbar\frac{\partial_r\hat{U}^{0|\alpha_0}}{\partial\hat{C}^{\beta_1}_1}\hat{V}^{1|\beta_1} - (i\hbar)^2\frac{\partial^2_r\hat{U}^{0|\alpha_0}}{\partial\hat{C}^{\beta_0}_0\partial\hat{C}^{\gamma_0}_0}\hat{V}^{00|\beta_0\gamma_0} = -2i\hbar\hat{T}_{0\beta_0}\hat{V}^{00|\beta_0\alpha_0}(-1)^{\varepsilon_{0\alpha_0}},\end{aligned} \tag{2.98}$$

where

$$\hat{V}^{1|\alpha_1} = \tfrac{1}{2}\hat{V}^{1|\alpha_1}_{00|\beta_0\gamma_0}(\hat{p}, \hat{q})\hat{C}^{\gamma_0}_0\hat{C}^{\beta_0}_0(-1)^{\varepsilon_{0\beta_0}} + \hat{V}^{1|\alpha_1}_{1|\beta_1}(\hat{p}, \hat{q})\hat{C}^{\beta_1}_1 , \tag{2.99}$$

$$(-1)^{\varepsilon_{0\nu_0}}\hat{V}^{00|\mu_0\nu_0} = \tfrac{1}{4}\hat{V}^{00|\mu_0\nu_0}_{00|\alpha_0\beta_0}(\hat{p}, \hat{q})\hat{C}^{\beta_0}_0\hat{C}^{\alpha_0}_0(-1)^{\varepsilon_{0\alpha_0}} + \tfrac{1}{2}\hat{V}^{00|\mu_0\nu_0}_{1|\alpha_1}(\hat{p}, \hat{q})\hat{C}^{\alpha_1}_1 ; \tag{2.100}$$

the $\hat{C}^2_0$ terms in (2.98) give

$$\begin{aligned}[][\hat{V}^{\alpha_0}_{\beta_0}, \hat{T}_{0\gamma_0}] - [\hat{V}^{\alpha_0}_{\gamma_0}, \hat{T}_{0\beta_0}](-1)^{\varepsilon_{0\beta_0}\varepsilon_{0\gamma_0}} + [\hat{U}^{\alpha_0}_{\beta_0\gamma_0}, \hat{H}_0] - i\hbar\hat{V}^{\alpha_0}_{\mu_0}\hat{U}^{\mu_0}_{\beta_0\gamma_0} + \\ + i\hbar\hat{U}^{\alpha_0}_{\beta_0\mu_0}\hat{V}^{\mu_0}_{\gamma_0} - i\hbar\hat{U}^{\alpha_0}_{\gamma_0\mu_0}\hat{V}^{\mu_0}_{\beta_0}(-1)^{\varepsilon_{0\beta_0}\varepsilon_{0\gamma_0}} = \tfrac{1}{2}i\hbar\hat{\Pi}^{\alpha_0}_{\mu_0\nu_0}\hat{V}^{\nu_0\mu_0}_{\beta_0\gamma_0} - i\hbar\hat{Z}^{\alpha_0}_{1\alpha_1}\hat{V}^{\alpha_1}_{\beta_0\gamma_0} ,\end{aligned} \tag{2.101}$$

where

$$\hat{V}^{\mu_0\nu_0}_{\alpha_0\beta_0} \equiv \hat{V}^{00|\mu_0\nu_0}_{00|\alpha_0\beta_0}(\hat{p}, \hat{q}) , \tag{2.102}$$

$$\hat{V}^{\alpha_1}_{\alpha_0\beta_0} \equiv \hat{V}^{1|\alpha_1}_{00|\alpha_0\beta_0}(\hat{p}, \hat{q}) . \tag{2.103}$$

The $\hat{C}_1$ terms in (2.98) give

$$[\hat{Z}_{1\alpha_1}^{\alpha_0}, \hat{H}_0] - i\hbar\hat{V}_{\beta_0}^{\alpha_0}\hat{Z}_{1\alpha_1}^{\beta_0} = \tfrac{1}{2}i\hbar\hat{\Pi}_{\mu_0\nu_0}^{\alpha_0}\hat{V}_{\alpha_1}^{\nu_0\mu_0} - i\hbar\hat{Z}_{1\beta_1}^{\alpha_0}\hat{V}_{\alpha_1}^{\beta_1}, \tag{2.104}$$

where

$$\hat{V}_{\alpha_1}^{\mu_0\nu_0} \equiv \hat{V}_{1|\alpha_1}^{00|\mu_0\nu_0}(\hat{p}, \hat{q}), \tag{2.105}$$

$$\hat{V}_{\alpha_1}^{\beta_1} \equiv \hat{V}_{1|\alpha_1}^{1|\beta_1}(\hat{p}, \hat{q}). \tag{2.106}$$

By multiplying (2.101), (2.104) on the left by $\hat{T}_{0\alpha_0}$ we obtain relations that guarantee formally fulfilling of the conditions necessary for (2.79) to be compatible with (2.75) and (2.76). For (2.79) and (2.75) the indicated compatibility conditions are

$$\begin{aligned}[][[\hat{T}_{0\beta_0}, \hat{T}_{0\gamma_0}], \hat{H}_0] &\equiv -[[\hat{H}_0, \hat{T}_{0\beta_0}], \hat{T}_{0\gamma_0}] + [[\hat{H}_0, \hat{T}_{0\gamma_0}], \hat{T}_{0\beta_0}](-1)^{\mathscr{E}_{0\beta_0}\mathscr{E}_{0\gamma_0}} = \\ &= i\hbar[\hat{T}_{0\alpha_0}\hat{U}_{\beta_0\gamma_0}^{\alpha_0}, \hat{H}_0] \equiv i\hbar\hat{T}_{0\alpha_0}[\hat{U}_{\beta_0\gamma_0}^{\alpha_0}, \hat{H}_0] - i\hbar[\hat{H}_0, \hat{T}_{0\alpha_0}]\hat{U}_{\beta_0\gamma_0}^{\alpha_0}.\end{aligned} \tag{2.107}$$

The conditions for (2.79) to be compatible with (2.76) have the form

$$[\hat{T}_{0\alpha_0}\hat{Z}_{1\alpha_1}^{\alpha_0}, \hat{H}_0] \equiv \hat{T}_{0\alpha_0}[\hat{Z}_{1\alpha_1}^{\alpha_0}, \hat{H}_0] - [\hat{H}_0, \hat{T}_{0\alpha_0}]\hat{Z}_{1\alpha_1}^{\alpha_0} = 0. \tag{2.108}$$

Equations (2.101), (2.104) saturate the contents of (2.98).

Thus we have completely clarified the meaning of the lowest equations out of (2.36), (2.40), which correspond to $n = 1$ in the sector $A_1 = (0, \alpha_0)$. These equations bear information of two kinds. First, they contain the lowest Jacobi relations (2.84), (2.101) which provide the formal compatibility in the involution (2.75) among the constraints and in the involution (2.79) between the constraints and Hamiltonian, as well as the lowest Jacobi relations (2.88) and (2.104) that provide the formal compatibility of eq. (2.76) for the null vectors (2.72) at $s = 1$ with involutions (2.75), (2.73). Second, they contain eq. (2.91) for the subsequent weak, in sense of (2.97), null vectors (2.72) at $s = 2$.

The information contained in the higher-order equations (2.36), (2.40) is of the same two kinds, namely, they include the necessary compatibility conditions for the previous equations, *i.e.* the higher Jacobi identities, and equations for the subsequent weak null vectors (2.72) at $s = 3, \ldots, L$. For $n = 1$ all the equations for weak null vectors are contained in (2.36), whereas for higher sectors $A_1 = (s, \alpha_s)$, $s = 1, \ldots, L-2$, these equations are

$$\hat{Z}_{s-1\alpha_{s-1}}^{\alpha_{s-2}}\hat{Z}_{s\alpha_s}^{\alpha_{s-1}} = \hat{T}_{0\alpha_0}\hat{U}_{\alpha_s}^{\alpha_0\alpha_{s-2}} + O(\hbar), \tag{2.109}$$

where $s = 3, \ldots, L$,

$$\mathscr{E}(\hat{Z}_{s\alpha_s}^{\alpha_{s-1}}) = \mathscr{E}_{s-1\alpha_{s-1}} + \mathscr{E}_{s\alpha_s}. \tag{2.110}$$

In the classical limit the right-hand sides of (2.109) vanish on the hypersurface of constraints $T_0 = 0$.

Thus eqs. (2.36), (2.40) form, generally, an infinite set of structural relations of the operator-valued gauge algebra created by the first-class constraints $\hat{T}_{0\alpha_0}$ and the Hamiltonian $\hat{H}_0$. The set of structural operators (2.56), (2.57) forms a basis of the operatorial gauge algebra which corresponds to the $\hat{\mathscr{P}}\hat{\mathscr{C}}$-normal form chosen for the ghosts.

In the general case the gauge algebra generated by the first-class constraints is open and reducible. The opennes (or unclosedness) of the algebra manifests itself in the appearance of nonzero terms that contain operators (2.86), (2.102), in the r.h.s. of the lowest Jacobi relations (2.84), (2.101). The reducibility of the gauge algebra is in the presence of nontrivial (see below) operator-valued null vectors (2.72).

It is said that a gauge algebra is of L-th stage of reducibility if, in the classical limit (1.14), one has the following relations:

(2.111) $$T_{0\alpha_0} Z^{\alpha_0}_{1\alpha_1} = 0\,,$$

(2.112) $$Z^{\alpha_{s-2}}_{s-1\alpha_{s-1}} Z^{\alpha_{s-1}}_{s\alpha_s}\big|_{T_0=0} = 0\,, \qquad s = 2, \ldots, L,$$

(2.113) $$\mathscr{E}(Z^{\alpha_{s-1}}_{s\alpha_s}) = \mathscr{E}_{s-1\alpha_{s-1}} + \mathscr{E}_{s\alpha_s}\,,$$

(2.114) $$\operatorname{rank}_\pm \left\| \frac{\partial_1 T_{0\alpha_0}}{\partial\varphi^a} \right\| \Bigg|_{T_0=0} = \gamma_0(L)_\pm\,, \qquad \varphi^a \equiv (p_i, q^i)\,,$$

(2.115) $$\operatorname{rank}_\pm \left\| Z^{\alpha_{s-1}}_{s\alpha_s} \right\| \big|_{T_0=0} = \gamma_s(L)_\pm\,, \qquad s = 1, \ldots, L,$$

where $\gamma_s(L)_\pm$, $s = 0, \ldots, L$, is defined as (2.6)-(2.8).

Equations (2.111), (2.112) are just the classical counterpart of (2.76), (2.91), (2.109). Therefore, the essence of the above definition of the L-th stage of reducibility is in fixing the values for the ranks (2.114), (2.115) on the hypersurface of constraints. The rank (2.114) fixes the genuine number

(2.116) $$n^*_\pm(L) = n_\pm - \gamma_0(L)_\pm$$

of the physical degrees of freedom in the theory. The ranks (2.114), (2.115) indicate also that the quantities

(2.117) $$\frac{\partial_1 T_{0\alpha_0}}{\partial\varphi^a}\,, \qquad Z^{\alpha_0}_{1\alpha_1}, \ldots, Z^{\alpha_{L-1}}_{L\alpha_L}$$

form a perfect sequence on the hypersurface of constraints.

Consider now the first iteration steps for finding the structural operators of the gauge algebra within the framework of the lowest-order equations (2.35), (2.39), (2.81), (2.98). Let the operator-valued original Hamiltonian and constraints $\hat{T}_{0\alpha_0}$ be primordial quantities. Assume that these operators do generate a gauge algebra. Then the structural equation are iterated as follows. Involution (2.75) determines operators (2.77), while eq. (2.76) determines operators (2.72) at $s = 1$. Next, operators (2.89), (2.90) are determined by eq. (2.88), while operators (2.92) by eq. (2.91). Similarly, the involution (2.79) determines operators (2.80). Operators (2.102), (2.103) are then to be found from eq. (2.101) and operators (2.105), (2.106) from (2.104). This process goes further on within the higher-order equations (2.36), (2.40). Certainly, the structural operators (2.56), (2.57) are not determined by the respective equations (2.36), (2.40) in an unique way. The natural arbitrariness here is due to the possibility of the canonical transformations

$$\hat{\Omega}_{\min} \to \hat{\mathfrak{A}}^{-1}\hat{\Omega}_{\min}\hat{\mathfrak{A}}\,, \qquad \hat{H}_{\min} \to \hat{\mathfrak{A}}^{-1}\hat{H}_{\min}\hat{\mathfrak{A}}\,, \tag{2.118}$$

where $\hat{\mathfrak{A}}$ is an unitary operator

$$\hat{\mathfrak{A}}^{-1} = \hat{\mathfrak{A}}^{\dagger}\,, \tag{2.119}$$

depending on (2.16).

Besides, there is an arbitrariness inside, the definition of the operator $\hat{H}_{\min}$ owing just to eq. (2.17):

$$\hat{H}_{\min} \to \hat{H}_{\min} + (i\hbar)^{-1}[\hat{\Lambda}, \hat{\Omega}_{\min}]\,, \tag{2.120}$$

where the operator $\hat{\Lambda}$ depends on (2.16) and possesses the properties

$$\mathscr{E}(\hat{\Lambda}) = 1\,, \qquad \mathrm{gh}(\hat{\Lambda}) = -1\,, \qquad \hat{\Lambda}^{\dagger} = -\hat{\Lambda}\,. \tag{2.121}$$

3. – Unitarizing Hamiltonian.

Now consider the new operators

$$\text{i)}\ (\hat{\pi}_{s\alpha_s}, \hat{\lambda}_s^{\alpha_s})\,, \qquad \text{ii)}\ (\hat{\bar{C}}_{s\alpha_s}, \hat{\bar{\mathscr{P}}}_s^{\alpha_s})\,, \qquad \alpha_s = 1, \dots, m_s;\ s = 0, \dots, L; \tag{3.1}$$

and also

$$\text{i)}\ (\hat{\pi}^{s'}_{s\alpha_s}, \hat{\lambda}_s^{s'\alpha_s})\,, \qquad \text{ii)}\ (\hat{\bar{C}}^{s'}_{s\alpha_s}, \hat{\bar{\mathscr{P}}}_s^{s'\alpha_s})\,, \qquad \alpha_s = 1, \dots, m_s;\ s' = 1, \dots, L;\ s = s', \dots, L. \tag{3.2}$$

Operators i) and ii) from (3.1) are Lagrange multipliers and ghosts, respectively, of an auxiliary sector. Operators (3.2) form the extra-ghost sector.

The statistics of operators (3.1), (3.2) is as follows:

$$\mathscr{E}(\hat{\pi}_{s\alpha_s}) = \mathscr{E}(\hat{\lambda}_s^{\alpha_s}) = \mathscr{E}_{s\alpha_s} + s\,, \tag{3.3}$$

$$\mathscr{E}(\hat{\bar{C}}_{s\alpha_s}) = \mathscr{E}(\hat{\mathscr{P}}_s^{\alpha_s}) = \mathscr{E}_{s\alpha_s} + s + 1\,, \tag{3.4}$$

$$\mathscr{E}(\hat{\pi}_{s\alpha_s}^{s'}) = \mathscr{E}(\hat{\lambda}_s^{s'\alpha_s}) = \mathscr{E}_{s\alpha_s} + s - s'\,, \tag{3.5}$$

$$\mathscr{E}(\hat{\bar{C}}_{s\alpha_s}^{s'}) = \mathscr{E}(\hat{\mathscr{P}}_s^{s'\alpha_s}) = \mathscr{E}_{s\alpha_s} + s - s' + 1\,. \tag{3.6}$$

Their ghost numbers are defined as

$$\operatorname{gh}(\hat{\pi}_{s\alpha_s}) = -\operatorname{gh}(\hat{\lambda}_s^{\alpha_s}) = -s\,, \tag{3.7}$$

$$\operatorname{gh}(\hat{\bar{C}}_{s\alpha_s}) = -\operatorname{gh}(\hat{\mathscr{P}}_s^{\alpha_s}) = -(s+1)\,, \tag{3.8}$$

$$\operatorname{gh}(\hat{\pi}_{s\alpha_s}^{s'}) = -\operatorname{gh}(\hat{\lambda}_s^{s'\alpha_s}) = -(s-s')\,, \tag{3.9}$$

$$\operatorname{gh}(\hat{\bar{C}}_{s\alpha_s}^{s'}) = -\operatorname{gh}(\hat{\mathscr{P}}_s^{s'\alpha_s}) = -(s-s'+1)\,. \tag{3.10}$$

Operators of each pair i) and ii) in (3.1), (3.2) are subjected to the canonical equal-time commutation relations, so that the only nonvanishing commutators among them are

$$[\hat{\lambda}_s^{\alpha_s}, \hat{\pi}_{r\beta_r}] = i\hbar\delta_{sr}\delta_{\beta_r}^{\alpha_s}\hat{1}\,, \tag{3.11}$$

$$[\hat{\mathscr{P}}_s^{\alpha_s}, \hat{\bar{C}}_{r\beta_r}] = i\hbar\delta_{sr}\delta_{\beta_r}^{\alpha_s}\hat{1}\,, \tag{3.12}$$

$$[\hat{\lambda}_s^{s'\alpha_s}, \hat{\pi}_{r\beta_r}^{r'}] = i\hbar\delta_{sr}\delta^{s'r'}\delta_{\beta_r}^{\alpha_s}\hat{1}\,, \tag{3.13}$$

$$[\hat{\mathscr{P}}_s^{s'\alpha_s}, \hat{\bar{C}}_{r\beta_r}^{r'}] = i\hbar\delta_{sr}\delta^{s'r'}\delta_{\beta_r}^{\alpha_s}\hat{1}\,. \tag{3.14}$$

At last, operators (3.1), (3.2) transform under the Hermite conjugation in the following way, compatible with (3.11)-(3.14):

$$(\hat{\pi}_{s\alpha_s})^\dagger = \pi_{s\alpha_s}\,, \qquad (\hat{\lambda}_s^{\alpha_s})^\dagger = \hat{\lambda}_s^{\alpha_s}(-1)^{\mathscr{E}(\hat{\lambda}_s^{\alpha_s})}\,, \tag{3.15}$$

$$(\hat{\bar{C}}_{s\alpha_s})^\dagger = \hat{\bar{C}}_{s\alpha_s}(-1)^{\mathscr{E}(\hat{\bar{C}}_{s\alpha_s})}\,, \qquad (\hat{\mathscr{P}}_s^{\alpha_s})^\dagger = \hat{\mathscr{P}}_s^{\alpha_s}\,, \tag{3.16}$$

$$(\hat{\pi}_{s\alpha_s}^{s'})^\dagger = \hat{\pi}_{s\alpha_s}^{s'}\,, \qquad (\hat{\lambda}_s^{s'\alpha_s})^\dagger = \hat{\lambda}_s^{s'\alpha_s}(-1)^{\mathscr{E}(\hat{\lambda}_s^{s'\alpha_s})}\,, \tag{3.17}$$

$$(\hat{\bar{C}}_{s\alpha_s}^{s'})^\dagger = \hat{\bar{C}}_{s\alpha_s}^{s'}(-1)^{\mathscr{E}(\hat{\bar{C}}_{s\alpha_s}^{s'})}\,, \qquad (\hat{\mathscr{P}}_s^{s'\alpha_s})^\dagger = \hat{\mathscr{P}}_s^{s'\alpha_s}\,. \tag{3.18}$$

Together with (2.16), operators (3.1), (3.2) form a complete set $\hat{\Gamma}$ of operators of the extended (relativistic) phase space

$$\hat{\Gamma} \equiv (\hat{P}_M, \hat{Q}^M)\,, \tag{3.19}$$

where

(3.20) $$\hat{P}_M \equiv (\hat{p}_i, \hat{\bar{\mathscr{P}}}_{s\alpha_s}, \hat{\pi}_{s\alpha_s}, \hat{\bar{C}}_{s\alpha_s}, \hat{\pi}^{s'}_{s\alpha_s}, \hat{\bar{C}}^{s'}_{s\alpha_s}),$$

(3.21) $$\hat{Q}^M \equiv (\hat{q}_i, \hat{C}^{\alpha_s}_s, \hat{\lambda}^{\alpha_s}_s, \hat{\mathscr{P}}^{\alpha_s}_s, \hat{\lambda}^{s'\alpha_s}_s, \hat{\mathscr{P}}^{s'\alpha_s}_s).$$

Operators (3.20), (3.21) evidently satisfy the equal-time commutation relations (1.10).

In the preceding section we have studied in every detail the construction of the fermionic, $\hat{\Omega}_{\min}$, and bosonic, $\hat{H}_{\min}$, generating operators of the gauge algebra. Now these operators will, in their turn, serve as a basis for constructing an operatorial version of the unitarizing Hamiltonian. As a preliminary step, define an extension of the operator $\hat{\Omega}_{\min}$ involved in (2.17), (2.19), (2.23):

(3.22) $$\hat{\Omega} = \hat{\Omega}_{\min} + \sum_{s=0}^{L} \hat{\pi}_{s\alpha_s} \hat{\mathscr{P}}^{\alpha_s}_s + \sum_{s'=1}^{L} \sum_{s=s'}^{L} \hat{\pi}^{s'}_{s\alpha_s} \hat{\mathscr{P}}^{s'\alpha_s}_s .$$

For (3.22) we have

(3.23) $$[\hat{\Omega}, \hat{\Omega}] = 0, \quad \mathscr{E}(\hat{\Omega}) = 1, \quad \mathrm{gh}(\hat{\Omega}) = 1,$$

(3.24) $$\hat{\Omega}^\dagger = \hat{\Omega}.$$

Define $\hat{\Psi}$, a gauge fermion operator (*), as possessing the properties

(3.25) $$\mathscr{E}(\hat{\Psi}) = 1, \quad \mathrm{gh}(\hat{\Psi}) = -1, \quad \hat{\Psi}^\dagger = -\hat{\Psi},$$

and depending, like (3.22), on operators from the complete set (3.19).

The unitarizing Hamiltonian is determined by the following basic equation:

(3.26) $$\hat{H}_{\hat{\Psi}} = \hat{H}_{\min} + (i\hbar)^{-1}[\hat{\Psi}, \hat{\Omega}],$$

where $\hat{\Omega}$ is defined by (3.22) and $\hat{H}_{\min}$ is the generating operator involved in (2.18), (2.20), (2.24).

It follows from definition (3.26) that

(3.27) $$[\hat{H}_{\hat{\Psi}}, \hat{\Omega}] = 0, \quad \mathscr{E}(\hat{H}_{\hat{\Psi}}) = 0, \quad \mathrm{gh}(\hat{H}_{\hat{\Psi}}) = 0,$$

(3.28) $$(\hat{H}_{\hat{\Psi}})^\dagger = \hat{H}_{\hat{\Psi}}.$$

The unitarizing Hamiltonian (3.26) is the basis for the operatorial description of dynamical systems subject to first-class constraints.

(*) The admissibility conditions for the classical limit of the symbol of the gauge fermion operator are given in appendix.

The destination of the gauge fermion operator $\hat{\Psi}$ is to generate a set of admissible gauge conditions that remove the degeneracy from the dynamics. A «minimal» version of the gauge fermion that produces linear gauges is

$$\hat{\Psi}_1 = \sum_{s=0}^{L} (\hat{\bar{\mathcal{P}}}_{s\alpha_s} \hat{\lambda}_s^{\alpha_s} + \hat{\bar{C}}_{s\alpha_s} \hat{\chi}_s^{\alpha_s}) + \sum_{s'=1}^{L} \sum_{s=s'}^{L} (\hat{\bar{C}}^{s'}_{s\alpha_s} \hat{\Phi}_s^{s'\alpha_s} + \hat{\bar{\chi}}^{s'}_{s\alpha_s} \hat{\lambda}_s^{s'\alpha_s}), \tag{3.29}$$

where

$$\hat{\chi}_0^{\alpha_0} = \omega_{0i}^{\alpha_0} \hat{q}^i + \hat{p}_i \omega_0^{i\alpha_0}, \tag{3.30}$$

$$\hat{\chi}_s^{\alpha_s} = \omega_{s\alpha_{s-1}}^{\alpha_s} \hat{C}_{s-1}^{\alpha_{s-1}}, \qquad s = 1, \ldots, L, \tag{3.31}$$

$$\hat{\Phi}_s^{s'\alpha_s} = \sigma_{s\alpha_{s-1}}^{s'\alpha_s} \hat{\lambda}_{s-1}^{s'-1\alpha_{s-1}}, \qquad \hat{\lambda}_s^{0\alpha_s} \equiv \hat{\lambda}_s^{\alpha_s}, \tag{3.32}$$

$$\hat{\bar{\chi}}^{s'}_{s\alpha_s} = \hat{\bar{C}}^{s'-1}_{s-1\alpha_{s-1}} \bar{\omega}_{s\alpha_s}^{s'\alpha_{s-1}}, \qquad \hat{\bar{C}}^0_{s\alpha_s} \equiv \hat{\bar{C}}_{s\alpha_s} \tag{3.33}$$

are linear gauge operators.

The minimal gauge fermion (3.29) produces the so-called singular gauge conditions which correspond to δ-functionals in the path integrands. To cover nonsingular Gaussian gauges, one should add the following terms to (3.29):

$$\hat{\Psi}_2 = \tfrac{1}{2} \sum_{s=0}^{L} \hat{\bar{C}}^s_{s\alpha_s} \varkappa_s^{\alpha_s\beta_s} \hat{\pi}^s_{s\beta_s} + \tfrac{1}{2} \sum_{s'=1}^{L} \sum_{s=s'}^{L} (\hat{\bar{C}}^{s'-1}_{s\alpha_s} \tau_{s\beta_s}^{s'\alpha_s} \hat{\mathcal{P}}_s^{s'\beta_s} + \hat{\pi}^{s'-1}_{s\alpha_s} \varrho_{s\beta_s}^{s'\alpha_s} \hat{\lambda}_s^{s'\beta_s}). \tag{3.34}$$

Using (3.22), (3.29), (3.34), we obtain the following expression for the unitarizing Hamiltonian (3.26) corresponding to linear nonsingular gauges:

$$\begin{aligned} \hat{H}_{\hat{\Psi}_1+\hat{\Psi}_2} = \hat{H}_{\min} + \sum_{s=0}^{L} \big((i\hbar)^{-1} [\hat{\Omega}_{\min}, \hat{\bar{\mathcal{P}}}_{s\alpha_s} \hat{\lambda}_s^{\alpha_s}] + \hat{\bar{\mathcal{P}}}_{s\alpha_s} \hat{\mathcal{P}}_s^{\alpha_s} + {}\\ + \hat{\bar{C}}_{s\alpha_s} (i\hbar)^{-1} [\hat{\chi}_s^{\alpha_s}, \hat{\Omega}_{\min}] + \hat{\pi}_{s\alpha_s} \hat{\chi}_s^{\alpha_s} + \tfrac{1}{2} \hat{\pi}^s_{s\alpha_s} \varkappa_s^{\alpha_s\beta_s} \hat{\pi}^s_{s\beta_s} \big) + {}\\ + \sum_{s'=1}^{L} \sum_{s=s'}^{L} \big(\hat{\pi}^{s'}_{s\alpha_s} \hat{\Phi}_s^{s'\alpha_s} + \hat{\bar{\chi}}^{s'}_{s\alpha_s} \hat{\mathcal{P}}_s^{s'\alpha_s} + \hat{\bar{C}}^{s'}_{s\alpha_s} (i\hbar)^{-1} [\hat{\Phi}_s^{s'\alpha_s}, \hat{\pi}^{s'-1}_{s-1\alpha_{s-1}}] \hat{\mathcal{P}}_{s-1}^{s'-1\alpha_{s-1}} + {}\\ + \hat{\pi}^{s'-1}_{s-1\alpha_{s-1}} (i\hbar)^{-1} [\hat{\mathcal{P}}_{s-1}^{s'-1\alpha_{s-1}}, \hat{\bar{\chi}}^{s'}_{s\alpha_s}] \hat{\lambda}_s^{s'\alpha_s} + \tfrac{1}{2} \hat{\pi}^{s'-1}_{s\alpha_s} (\tau_s^{s'} + \varrho_s^{s'})^{\alpha_s}_{\beta_s} \hat{\mathcal{P}}_s^{s'\beta_s} \big). \end{aligned} \tag{3.35}$$

4. – Operatorial dynamics.

Now assume that we have at hand a unitarizing Hamiltonian (3.26) with an admissible gauge fermion. The time evolution of operators (3.20), (3.21) is governed by the Heisenberg equations of motion induced by the Hamiltonian (3.26):

$$i\hbar \partial_t \hat{P}_M = [\hat{P}_M, \hat{H}_{\hat{\Psi}}], \qquad i\hbar \partial_t \hat{Q}^M = [\hat{Q}^M, \hat{H}_{\hat{\Psi}}]. \tag{4.1}$$

Hence by virtue of (3.27) we have

$$i\hbar\partial_t\hat{\Omega} = [\hat{\Omega}, \hat{H}_{\hat{\Psi}}] = 0\,. \tag{4.2}$$

In other words, operator (3.2) is an integral of motion.

Let $(\hat{P}_M, \hat{Q}^M)_{\hat{\Psi}}$ denote a solution of the operatorial equations of motion (4.1) with the gauge fermion $\hat{\Psi}$. Let $\Delta\hat{\Psi}$ be an admissible (finite) form variation of the gauge fermion. The varied solution of (4.1) determined by the same initial data, but a different Hamiltonian $\hat{H}_{\hat{\Psi}+\Delta\hat{\Psi}}$ corresponding to the varied gauge, is written as

$$(\hat{P}_M, \hat{Q}^M)_{\hat{\Psi}+\Delta\hat{\Psi}} = \hat{G}^{-1}_{\Delta\hat{\Psi}}(\hat{P}_M, \hat{Q}^M)_{\hat{\Psi}}\hat{G}_{\Delta\hat{\Psi}}\,, \tag{4.3}$$

where $\hat{G}_{\Delta\hat{\Psi}}(t)$ satisfies the equation

$$i\hbar\partial_t\hat{G}_{\Delta\hat{\Psi}} = (i\hbar)^{-1}[(\hat{\Omega})_{\hat{\Psi}}, (\Delta\hat{\Psi})_{\hat{\Psi}}]\hat{G}_{\Delta\hat{\Psi}}\,, \tag{4.4}$$

and coincides with the operatorial unity at the initial time moment. The designation $(\hat{F})_{\hat{\Psi}}$ here indicates that the operator $\hat{F}$ is considered as a function of the operators $(\hat{P}_M, \hat{Q}^M)_{\hat{\Psi}}$.

Equation (4.3) states that the gauge variation $\hat{\Psi} \to \hat{\Psi} + \Delta\hat{\Psi}$ is induced by a canonical transformation whose generating operator is $\hat{G}_{\Delta\hat{\Psi}}$. Since at the initial moment $\hat{G}_{\Delta\hat{\Psi}} = \hat{1}$, it follows from (3.27), (4.4) that

$$[(\hat{\Omega})_{\hat{\Psi}}, \hat{G}_{\Delta\hat{\Psi}}] = 0 \tag{4.5}$$

and hence, owing to (4.3), also that

$$(\hat{\Omega})_{\hat{\Psi}+\Delta\hat{\Psi}} = \hat{G}^{-1}_{\Delta\hat{\Psi}}(\hat{\Omega})_{\hat{\Psi}}\hat{G}_{\Delta\hat{\Psi}} = (\hat{\Omega})_{\hat{\Psi}}\,. \tag{4.6}$$

Define a class of operators $E_{\hat{\Psi}}$ which commute with $\hat{\Omega}$,

$$[\hat{E}_{\hat{\Psi}}, \hat{\Omega}] = 0\,, \tag{4.7}$$

whose dependence on the gauge fermion $\hat{\Psi}$ is entirely governed by a law, similar to (4.3):

$$\hat{E}_{\hat{\Psi}+\Delta\hat{\Psi}} = \hat{G}^{-1}_{\Delta\hat{\Psi}}\hat{E}_{\hat{\Psi}}\hat{G}_{\Delta\hat{\Psi}}\,. \tag{4.8}$$

Due to (4.5), it follows from (4.8) that

$$[\hat{E}_{\hat{\Psi}+\Delta\hat{\Psi}}, \hat{\Omega}] = 0\,. \tag{4.9}$$

(We have taken (4.7) into account.) The infinitesimal form of law (4.8) is

$$\hat{E}_{\hat{\Psi}+\delta\hat{\Psi}} - \hat{E}_{\hat{\Psi}} = -(i\hbar)^{-1}\left[\hat{\Omega},(i\hbar)^{-1}\left[\int_{t_0}^{t}(\delta\hat{\Psi}(t'))_{\hat{\Psi}}\,dt', \hat{E}_{\hat{\Psi}}\right]\right]. \tag{4.10}$$

The physical states of the theory

$$|\text{Phys}\rangle\,, \qquad \langle\text{Phys}| = (|\text{Phys}\rangle)^{\dagger} \tag{4.11}$$

are subjected to the conditions

$$\hat{\Omega}|\text{Phys}\rangle = \mathbf{0} = \langle\text{Phys}|\hat{\Omega}\,. \tag{4.12}$$

Now it follows from (4.10), (4.12) that

$$\delta_{\hat{\Psi}}\langle\alpha, \text{Phys}|\hat{E}_{\hat{\Psi}}|\text{Phys}, \beta\rangle = \mathbf{0}\,, \tag{4.13}$$

i.e. the physical matrix elements of the operators involved in (4.7), (4.8) do not depend on the gauge.

We proceed in an usual way. For each operator $\hat{F}$ in the initial representation, we put into correspondence an operator $\hat{F}'$ in the representation depending on external sources $J_M(t)$, $K^M(t)$:

$$\hat{F}'(t) = \hat{\mathscr{Z}}^{-1}(t, -\infty)\hat{F}(t)\hat{\mathscr{Z}}(t, -\infty) \tag{4.14}$$

with the generating functional $\hat{\mathscr{Z}}(t, -\infty)$ obeying the equation

$$i\hbar\partial_t\hat{\mathscr{Z}} = -(J_M\hat{Q}^M + \hat{P}_M K^M)\hat{\mathscr{Z}}, \hat{\mathscr{Z}}|_{t=-\infty} = \hat{1}. \tag{4.15}$$

Operators $\hat{P}'_M$, $\hat{Q}'^M$ obey, in the new representation, the equations to be obtained from (4.1) by the replacement

$$\hat{H}_{\hat{\Psi}} \to \hat{H}'_{\hat{\Psi}'} - J_M\hat{Q}'^M - \hat{P}'_M K^M\,. \tag{4.16}$$

For the operator $\hat{\Omega}'$ we have in the new representation, in place of (4.2),

$$i\hbar\partial_t\hat{\Omega}' = [(J_M\hat{Q}'^M + \hat{P}'_M K^M), \hat{\Omega}']\,. \tag{4.17}$$

Equation (4.17) is nothing but the operatorial Ward relation.

Operators $\hat{P}'_M$, $\hat{Q}'^M$ depend upon the gauge $\hat{\Psi}'$ according to the law

$$(\hat{P}'_M, \hat{Q}'^M)_{\hat{\Psi}'+\Delta\hat{\Psi}'} = \hat{G}'^{-1}_{\Delta\hat{\Psi}'}(\hat{P}'_M, \hat{Q}'^M)_{\hat{\Psi}'}\hat{G}'_{\Delta\hat{\Psi}'}\,, \tag{4.18}$$

where $\hat{G}'_{\Delta\hat{\Psi}'}(t)$ is given by the equation

$$i\hbar\partial_t \hat{G}'_{\Delta\Psi'} = (i\hbar)^{-1}[(\hat{\Omega}')_{\hat{\Psi}'}, (\Delta\hat{\Psi})_{\hat{\Psi}'}]\hat{G}'_{\Delta\hat{\Psi}'} . \tag{4.19}$$

In place of (4.5), we have in the new representation

$$(i\hbar\partial_t \hat{1} - (i\hbar)^{-1}[(\hat{\Omega}')_{\hat{\Psi}'}, (\Delta\hat{\Psi}')_{\hat{\Psi}'}])[(\hat{\Omega}')_{\hat{\Psi}'}, \hat{G}'_{\Delta\hat{\Psi}'}] = \\ = [[(J_M\hat{Q}'^M + \hat{P}'_M K^M)_{\hat{\Psi}'}, (\hat{\Omega}')_{\hat{\Psi}'}], \hat{G}'_{\Delta\hat{\Psi}'}] . \tag{4.20}$$

5. – Generating functional. Path integral.

The generating functional is defined in the usual way as

$$\mathscr{Z}(J, K) = \langle 0, \text{Phys}|\hat{\mathscr{Z}}(+\infty, -\infty)|\text{Phys}, 0\rangle , \tag{5.1}$$

where $|\text{Phys}, 0\rangle$ is the ground state of the unitarizing Hamiltonian (3.26), which simultaneously satisfies eq. (4.12).

Starting from the equations of motion for the operators in representation (4.14), which depends on the external sources, and exploiting (1.21)-(1.23), we obtain via the standard methods [17-20] the following variation derivative equations for the generating functional (5.1):

$$\left\{\partial_t Q^M - \lim_{\varepsilon\to+0} \frac{\partial_l \tilde{H}_{\tilde{\Psi}}(P(t+\varepsilon), Q(t))}{\partial P_M} + K^M(t)\right\}\Bigg|_{P=(\hbar/i)(\delta_r/\delta K), Q=(\hbar/i)(\delta_l/\delta J)} \cdot \mathscr{Z}(J, K) = 0 , \tag{5.2}$$

$$\left\{\partial_t P_M + \lim_{\varepsilon\to+0} \frac{\partial_r \tilde{H}_{\tilde{\Psi}}(P(t+\varepsilon), Q(t))}{\partial Q^M} - J_M(t)\right\}\Bigg|_{P=(\hbar/i)(\delta_r/\delta K), Q=(\hbar/i)(\delta_l/\delta J)} \cdot \mathscr{Z}(J, K) = 0 , \tag{5.3}$$

where $\tilde{H}_{\tilde{\Psi}}(P, Q)$ is the $\hat{P}\hat{Q}$-symbol (in the sense of (1.11) of the unitarizing Hamiltonian (3.26) considered as a function of operators (3.20), (3.21)). The solution to eqs. (5.2), (5.3) obtained by the functional Fourier transformation has the following path integral form:

$$\mathscr{Z} = \lim_{\varepsilon\to+0} \int \exp\left[\frac{i}{\hbar} S^{(\varepsilon)}\right] \mathrm{d}\Gamma , \tag{5.4}$$

where $S^{(\varepsilon)}$ is the «action»

$$S^{(\varepsilon)} = \int \left(P_M \dot{Q}^M - \tilde{H}_{\tilde{\Psi}}(P(t+\varepsilon), Q(t)) + J_M Q^M + P_M K^M\right) \mathrm{d}t , \tag{5.5}$$

and $d\Gamma$ is the phase volume element

$$(5.6) \qquad d\Gamma = (\text{const}) \prod_t \prod_M \frac{dP_M(t)\, dQ^M(t)}{2\pi\hbar} .$$

The normalizing factor (const) is fixed by the condition

$$(5.7) \qquad \mathscr{Z}(J = 0, K = 0) = 1 .$$

The inversion of the operator in the quadratic part of action (5.5) is defined as the causal propagator.

Since the phase variables P and Q enter into $\hat{H}_{\tilde{\Psi}}$ in (5.5) at nonequal time moments, $t + \varepsilon$ and t, the functional integral (5.4) contains the full information about the operator ordering in Hamiltonian (3.26). The ε-regularization in (5.2), (5.3), (5.5) conforms to the type of the symbol. It is this conformity that provides the correctness of the result.

We have thus seen that eqs. (5.2), (5.3) for the generating functional and the effective «action» in the path integrand contain the ε-regularized $\hat{P}\hat{Q}$-symbol of the unitarizing Hamiltonian (3.26). If an explicit expression for operator (3.26) is, in fact, at our disposal, its symbol may be, certainly, immediately found. Bearing in mind, however, the symbol-to-operator correspondence (1.6)-(1.8), it is evident that, in order to create a gauge algebra and to build the unitarizing Hamiltonian, one may handle directly the symbols. The counterpart of eq. (3.26) may be written for the corresponding symbols in the form

$$(5.8) \qquad \tilde{H}_{\tilde{\Psi}} = \tilde{H}_{\min} + (i\hbar)^{-1}[\tilde{\Psi}, \tilde{\Omega}]_* ,$$

where

$$(5.9) \qquad \tilde{\Omega} = \tilde{\Omega}_{\min} + \sum_{s=0}^{L} \pi_{s\alpha_s} \mathscr{P}_s^{\alpha_s} + \sum_{s'=1}^{L} \sum_{s=s'}^{L} \pi_{s\alpha_s}^{s'} \mathscr{P}_s^{s'\alpha_s} .$$

The symbols $\tilde{\Omega}_{\min}$, $\tilde{H}_{\min}$ of the operators $\hat{\Omega}_{\min}$, $\hat{H}_{\min}$ involved into (2.17), (2.18) satisfy the equations

$$(5.10) \qquad [\tilde{\Omega}_{\min}, \tilde{\Omega}_{\min}]_* = 0 , \quad \text{gh}\,(\tilde{\Omega}_{\min}) = 1 ,$$

$$(5.11) \qquad [\tilde{H}_{\min}, \tilde{\Omega}_{\min}]_* = 0 , \quad \text{gh}\,(\tilde{H}_{\min}) = 0 ,$$

in the minimal sector $\Gamma_{\min}$.

Equations (5.10), (5.11) for the fermion $\tilde{\Omega}_{\min}$ and the boson $\tilde{H}_{\min}$ may be solved directly using the power series, obtained from (2.23), (2.24) by replacing every operator by its symbol in them. This results in equations for the symbols of the structural operators that are obtained from (2.35)-(2.41) by replacing every operator product (and commutator) by the corresponding $*$-product (and $*$-commutators) (1.12), (1.8), with the order of factors unchanged. For instance, there are the following involution relations for the symbols of the constraint operators and the original Hamiltonian:

$$[\tilde{T}_{0\alpha_0}, \tilde{T}_{0\beta_0}]_* = i\hbar \tilde{T}_{0\gamma_0} * \tilde{U}^{\gamma_0}_{\alpha_0\beta_0}, \tag{5.12}$$

$$[\tilde{H}_0, \tilde{T}_{0\beta_0}]_* = i\hbar \tilde{T}_{0\gamma_0} * \tilde{V}^{\gamma_0}_{\beta_0}, \tag{5.13}$$

and the following equations for symbols of strong operator-valued null vectors:

$$\tilde{T}_{0\alpha_0} * \tilde{Z}^{\alpha_0}_{1\alpha_1} = 0 . \tag{5.14}$$

Following the correspondence principle, we restrict the class of solutions of the operatorial equations (2.17), (2.18) by imposing the requirement that the symbols of the solutions should admit the series expansion in powers of $\hbar$

$$\tilde{\Omega}_{\min} = \Omega_{\min} + \sum_{n=1}^{\infty} \hbar^n \tilde{\Omega}^{(n)}_{\min}, \tag{5.15}$$

$$\tilde{H}_{\min} = H_{\min} + \sum_{n=1}^{\infty} \hbar^n \tilde{H}^{(n)}_{\min} . \tag{5.16}$$

Substituting these expansions into (5.10), (5.11) and using (1.17), we obtain the following relations for the coefficients:

$$\{\Omega_{\min}, \Omega_{\min}\} = 0 , \tag{5.17}$$

$$\frac{i^2}{2!}\{\Omega_{\min}, \Omega_{\min}\}_2 + 2i\{\Omega_{\min}, \tilde{\Omega}^{(1)}_{\min}\} = 0 , \tag{5.18}$$

$$\frac{i^n}{n!}\{\Omega_{\min}, \Omega_{\min}\}_n + \sum_{m=1}^{n-1} \frac{i^m}{m!} 2\{\Omega_{\min}, \tilde{\Omega}^{(m)}_{\min}\}_m + \sum_{m=2}^{n-1} \sum_{l=1}^{m-1} \frac{i^l}{l!}\{\tilde{\Omega}^{(m-l)}_{\min}, \tilde{\Omega}^{(n-m)}_{\min}\}_l = 0, \qquad n \geqslant 3 , \tag{5.19}$$

as well as

$$\{H_{\min}, \Omega_{\min}\} = 0 , \tag{5.20}$$

$$\frac{i^2}{2!}\{H_{\min}, \Omega_{\min}\}_2 + i(\{H_{\min}, \tilde{\Omega}^{(1)}_{\min}\} + \{\tilde{H}^{(1)}_{\min}, \Omega_{\min}\}) = 0 , \tag{5.21}$$

$$\frac{i^n}{n!}\{H_{\min}, \Omega_{\min}\}_n + \sum_{m=1}^{n-1} \frac{i^m}{m!} (\{H_{\min}, \tilde{\Omega}^{(n-m)}_{\min}\}_m + \{\tilde{H}^{(n-m)}_{\min}, \Omega_{\min}\}_m) + \sum_{m=2}^{n-1} \sum_{l=1}^{m-1} \frac{i^l}{l!} \frac{1}{2} (\{\tilde{H}^{(m-l)}_{\min}, \tilde{\Omega}^{(n-m)}_{\min}\}_l + \{\tilde{H}^{(n-m)}_{\min}, \tilde{\Omega}^{(m-l)}_{\min}\}_l) = 0, \qquad n \geqslant 3 . \tag{5.22}$$

Here $\{A, B\} \equiv \{A, B\}_1$ is the Poisson superbracket (1.20), and $\{A, B\}_n$ is the binary operation defined as (1.18), (1.16).

Consider in more detail the lowest-order equations (5.17), (5.20), which generate the classical gauge algebra. Substituting the expansion

(5.23) $$\Omega_{\min} = U_0 + \sum_{n=1}^{\infty} \bar{\mathscr{P}}_{A_n}, \ldots, \bar{\mathscr{P}}_{A_1} U^{A_1 \ldots A_n}$$

into (5.17), we obtain the classical analog of (2.35)-(2.37)

(5.24) $$\frac{1}{2}\{U_0, U_0\} = -\frac{\partial_r U_0}{\partial C^A} U^A ,$$

(5.25) $$\{U^{A_1 \ldots A_n}, U_0\} + \frac{\partial_r U^{A_1 \ldots A_n}}{\partial C^A} U^A + + (X_{\text{sym}})^{A_1 \ldots A_n} = -(n+1) \frac{\partial_r U_0}{\partial C^A} U^{AA_1 \ldots A_n}(-1)^{\mathscr{E}_0^n} ,$$

where

(5.26) $$X^{A_1} \equiv 0 , \qquad X^{A_1 \ldots A_n} \equiv \sum_{m=1}^{n-1} (-1)^{\mathscr{E}_m^n} \Big(\frac{1}{2}\{U^{A_1 \ldots A_m}, U^{A_{m+1} \ldots A_n}\} + + (n-m+1) \frac{\partial_r U^{A_1 \ldots A_m}}{\partial C^A} U^{AA_{m+1} \ldots A_n}\Big) .$$

In the same way, the expansion

(5.27) $$H_{\min} = V_0 + \sum_{n=1}^{\infty} \bar{\mathscr{P}}_{A_n}, \ldots, \bar{\mathscr{P}}_{A_1} V^{A_1 \ldots A_n} ,$$

when substituted into (5.20), leads to the classical analog of (2.39)-(2.41)

(5.28) $$\{V_0, U_0\} = \frac{\partial_r U_0}{\partial C^A} V^A ,$$

(5.29) $$\{V^{A_1 \ldots A_n}, U_0\} - \{U^{A_1 \ldots A_n}, V_0\} + \frac{\partial_r V^{A_1 \ldots A_n}}{\partial C^A} U^A - \frac{\partial_r U^{A_1 \ldots A_n}}{\partial C^A} V^A + + (Y_{\text{sym}})^{A_1 \ldots A_n} = (n+1) \frac{\partial_r U_0}{\partial C^A} V^{AA_1 \ldots A_n}(-1)^{\mathscr{E}_0^n} ,$$

where

(5.30) $$Y^{A_1} \equiv 0 , \qquad Y^{A_1 \ldots A_n} \equiv \sum_{m=1}^{n-1} \Big(\{V^{A_1 \ldots A_m}, U^{A_{m+1} \ldots A_n}\} + + (n-m+1)\Big(\frac{\partial_r V^{A_1 \ldots A_m}}{\partial C^A} U^{AA_{m+1} \ldots A_n} - (-1)^{\mathscr{E}_m^n} \frac{\partial_r U^{A_1 \ldots A_m}}{\partial C^A} V^{AA_{m+1} \ldots A_n}\Big)\Big) .$$

Using the classical analogs of (2.46), (2.48), (2.73), (2.74) in the lowest-order equations (5.24), (5.28), one comes to the relations

$$\{T_{0\alpha_0}, T_{0\beta_0}\} = T_{0\gamma_0} U^{\gamma_0}_{\alpha_0\beta_0}, \tag{5.31}$$

$$\{H_0, T_{0\beta_0}\} = T_{0\gamma_0} V^{\gamma_0}_{\beta_0}, \tag{5.32}$$

$$T_{0\alpha_0} Z^{\alpha_0}_{1\alpha_1} = 0, \tag{5.33}$$

which are the classical limit of (5.12)-(5.14).

Consider the classical limit of action (5.5) at $\varepsilon = 0$:

$$S^{(0)}_{\text{classical}} = \int (P_M \dot{Q}^M - H_\Psi + J_M Q^M + P_M K^M)\,\mathrm{d}t, \tag{5.34}$$

where

$$H_\Psi = H_{\min} + \{\Psi, \Omega\} \tag{5.35}$$

is the classical unitarizing Hamiltonian, and

$$\Omega = \Omega_{\min} + \sum_{s=0}^{L} \pi_{s\alpha_s} \mathscr{P}^{\alpha_s}_{s} + \sum_{s'=1}^{L} \sum_{s=s'}^{L} \pi^{s'}_{s\alpha_s} \mathscr{P}^{s'\alpha_s}_{s}, \tag{5.36}$$

so that, in virtue of (5.17), (5.20), we have

$$\{\Omega, \Omega\} = 0, \quad \{H_\Psi, \Omega\} = 0. \tag{5.37}$$

Owing to (5.37), the external-source–independent part of action (5.34) is invariant under the canonical BRS transformations

$$\delta P_M = \{P_M, \Omega\}\mu, \quad \delta Q^M = \{Q^M, \Omega\}\mu, \tag{5.38}$$

where μ is a fermionic parameter.

If one uses action (5.34) to define a formal analog of (5.4)

$$\mathscr{Z}_{\text{formal}} \equiv \int \exp\left[\frac{i}{\hbar} S^{(0)}_{\text{classical}}\right] \mathrm{d}\Gamma, \tag{5.39}$$

one is able to show, by choosing μ in (5.38) as

$$\mu = \frac{i}{\hbar}\int \delta\Psi\,\mathrm{d}t, \tag{5.40}$$

that the formal S-matrix which corresponds to (5.39) does not depend (on the mass shell) on the choice of the gauge fermion Ψ in (5.35).

The formal expression (5.39) for the generating functional (or the S-matrix) accomplishes the concept of naive quantization via the path integral in the phase space. The first step in developing this quantization is to form the classical action (5.34) under the assumption that there is a classical gauge algebra that operates on virtual trajectories in the phase space, and is generated by the classical constraints $T_{0\alpha_0}$ and the Hamiltonian H_0 according to the eqs. (5.17), (5.20) or, equivalently, (5.24)-(5.26), (5.28)-(5.30). Then, using the classical action, the formal expression (5.30) is constructed.

The functional integral (5.39), generally speaking, depends essentially on the way of calculation—the choice of a finite-dimensional approximation of the virtual phase trajectories. The arising ambiguities are due to the loss of information about the ordering of operators in the Hamiltonian (passing to the formal limit $\varepsilon = 0$ directly in the action).

It was shown above that the consistent operatorial quantization leads to an accurate expression (5.4) for the generating functional (S-matrix). The structure of action (5.5) in the modified functional integral (5.4) shows that it is a quantum, not classical, gauge algebra that, as a matter of fact, is operating on the virtual phase trajectories, being generated by symbols of the operator-valued first-class constraints and the original Hamiltonian. When $\varepsilon > 0$ the functional integral in (5.4) does not depend on the way of calculations, it being the conformity of the ε-regularization and the type of the symbols that guarantee that the information about operator ordering inside the Hamiltonian is accurately taken into account (this has been explained above).

Formal expression (5.39) cannot, in fact, be applied to systems with a finite number of degrees of freedom, due to ambiguities which it contains in this case. However, in the relativistic quantum field theory the noncommutativity of the equal-time operators in the Hamiltonian gives rise only to noncovariant power divergences like 3-dimensional δ-function or its derivatives in coinciding points. It is thought that these contributions are inessential as long as a Lorentz-covariant ultraviolet regularization is used. As for the gauge-invariant dimensional regularization, it automatically annihilates any power divergences altogether. It is, therefore, considered that, at least within a certain class of ultraviolet regularizations, the naive quantization through the formal phase space path integral (5.39) is justified as applied to relativistic field theory. One should not, however, forget that the formal expression (5.39) itself has no firm foundation in terms of the standard first principles of quantum mechanics. Before the adequate operatorial formulation presented above was given, this expression might have been taken seriously only as far as one believed that such a formulation was in principle possible. On the other hand, the accurate expression (5.5) is a direct consequence of the standard first principles of quantum mechanics, namely the operator equation of motion induced by a Hermitian unitarizing Hamiltonian, and canonical equal-time commutation relations.

6. – Closing and abelizing the operatorial gauge algebra.

It is known in classical mechanics that one may locally abelize first-class constraints—in other words, make them commute among themselves—if one appropriately rotates the basis, using to this end a reversible matrix, which, generally, depends on the phase variables. In a more general context it is natural to treat the corresponding rotation as an effect of the canonical transformation in the minimal sector which leave invariant the form of eqs. (5.17), (5.20), generating the classical gauge algebra. It is the dependence of the canonical transformation generating function on the ghost variables that is responsible for the efficient rotation of the basis of constraints. One may state, therefore, that the closure and abelization of a gauge algebra induced by classical first-class constraints is accomplished by a ghost-dependent canonical transformation of the corresponding generating equations.

An analogous situation is to be expected in quantum mechanics. Here one has eqs. (2.17), (2.18) generating the operatorial gauge algebra. We shall look for a unitary transformation in the minimal sector (2.16) that would make the fermionic generating operator $\hat{\Omega}_{\min}$, involved in (2.17), linear in the ghost canonical momenta, thus providing the closure or abelization to the commutation relations of the operator-valued constraints. The same transformation—this time, however, accompanied by a form transformation of the gauge fermion—makes also the bosonic generating operator $\hat{H}_{\min}$, involved in (2.18), linear in the ghost momenta. This corresponds to the closure or abelization of the commutation relations for the new operatorial constraints and the original Hamiltonian.

Let the operator $\hat{\mathfrak{A}}$, such that

$$\mathscr{E}(\hat{\mathfrak{A}}) = 0\,, \quad \mathrm{gh}\,(\hat{\mathfrak{A}}) = 0\,, \tag{6.1}$$

be unitary

$$\hat{\mathfrak{A}}^\dagger \hat{\mathfrak{A}} = \hat{\mathfrak{A}}\hat{\mathfrak{A}}^\dagger = \hat{1} \tag{6.2}$$

and obey the equation

$$\hat{\Omega}_{\min}\hat{\mathfrak{A}} = \hat{\mathfrak{A}}\hat{\Omega}^*_{\min}\,, \tag{6.3}$$

where

$$\hat{\Omega}^* = \hat{U}^*_0 + \hat{\mathscr{P}}_A \hat{U}^{*A}; \tag{6.4}$$

within this section the asterisk marks operators in the new representation.

Due to (2.17), it follows from (6.3) that the fermionic operator (6.4) satisfies the equation

$$[\hat{\Omega}^*_{\min}, \hat{\Omega}^*_{\min}] = 0\,, \quad \mathrm{gh}\,(\hat{\Omega}^*_{\min}) = 1\,, \tag{6.5}$$

so that the coefficient operators in (6.4) are subject to the structural relations

$$\frac{1}{2}[\hat{U}_0^*, \hat{U}_0^*] = -i\hbar \frac{\partial_r \hat{U}_0^*}{\partial \hat{C}^A} \hat{U}^{*A}, \tag{6.6}$$

$$[\hat{U}^{*A}, \hat{U}_0^*] = -i\hbar \frac{\partial_r \hat{U}^{*A}}{\partial \hat{C}^B} \hat{U}^{*B}, \tag{6.7}$$

$$[\hat{U}^{*A}, \hat{U}^{*B}] = 0. \tag{6.8}$$

On the other hand, it follows from (6.3), using (2.19), (6.2), that

$$(\hat{\Omega}^*_{\min})^\dagger = \hat{\Omega}^*_{\min}, \tag{6.9}$$

whence

$$\hat{U}_0^* = (\hat{U}_0^*)^\dagger + i\hbar \frac{\partial_l (\hat{U}^{*A})^\dagger}{\partial \hat{C}^A}, \tag{6.10}$$

$$(-1)^{\varepsilon(C^A)} \hat{U}^{*A} = (\hat{U}^{*A})^\dagger. \tag{6.11}$$

There take place the following Taylor expansions in powers of ghosts for the coefficient operators from (6.4):

$$\hat{U}_0^* = \hat{T}^*_{0\alpha_0}(\hat{p}, \hat{q}) \hat{C}_0^{\alpha_0}, \tag{6.12}$$

$$\hat{U}^{*A} = \sum_{m=1}^{\infty} \sum_{\{B\}=\mu_m^1} \frac{(-1)^{E_{B_m \dots B_1}}}{m!} \hat{U}^{*A}_{B_m \dots B_1}(\hat{p}, \hat{q}) \hat{C}^{B_1} \dots \hat{C}^{B_m}, \tag{6.13}$$

where the set of values μ_m^1 is defined as (2.50) at $n = 1$, $A_1 = A$; the signature is given by (2.55), the structure operators acquire the signature factor (2.61) under the permutation of any two neighbouring lower indices B_{i+1}, B_i. We are also using the corresponding designation

$$\hat{Z}^{*\alpha_{s-1}}_{s\alpha_s} \equiv \hat{U}^{*s-1}{}_{s}{}^{\alpha_{s-1}}_{\alpha_s}(\hat{p}, \hat{q}) \tag{6.14}$$

for the operator-valued null vectors in the new representation.

Thus operator (6.4), linear in the ghost momenta, is—by definition—a fermionic generating operator of the closed gauge algebra. We assume that this gauge algebra is of L-th stage of reducibility in the sense of the definition (2.111)-(2.115) formulated in the application to the constraints involved in (6.12) and the null vectors (6.14).

Therefore, eqs. (6.2), (6.3) give the unitary transformation which relates the fermionic generating operators of the open basis of the gauge algebra to those of the closed one.

Let us expand the transformation operator into the $\hat{\bar{\mathscr{P}}}\hat{C}$-normal series in powers of the ghosts (2.15):

$$\hat{\mathfrak{A}} = \hat{F}_0 + \sum_{n=1}^{\infty} \hat{\bar{\mathscr{P}}}_{A_n}, \ldots, \hat{\bar{\mathscr{P}}}_{A_1} \hat{F}^{A_1 \ldots A_n}, \tag{6.15}$$

where the coefficients

$$\hat{F}_0, \qquad \hat{F}^{A_1 \ldots A_n}, \qquad\qquad n = 1, \ldots, \tag{6.16}$$

possess the same statistics, ghost number and generalized symmetry properties with respect to the upper indices as the operators (2.26) for the same n.

Substituting expansion (6.15) into eq. (6.3), we get the following relations for the operators (6.16):

$$\hat{U}_0 \hat{F}_0 + i\hbar \frac{\partial_r \hat{U}_0}{\partial \hat{C}^A} \hat{F}^A = \hat{F}_0 \hat{U}_0^*, \tag{6.17}$$

$$\begin{aligned} \hat{U}^{A_1 \ldots A_n} \hat{F}_0 + \hat{U}_0 \hat{F}^{A_1 \ldots A_n} (-1)^{\varepsilon_0^n} + (n+1) i\hbar \frac{\partial_r \hat{U}_0}{\partial \hat{C}^A} \hat{F}^{A A_1 \ldots A_n} (-1)^{\varepsilon_0^n} + \\ + \sum_{k=1}^{\infty} (i\hbar)^k \frac{\partial_r^k \hat{U}^{A_1 \ldots A_n}}{\partial \hat{C}^{B_1} \ldots \partial \hat{C}^{B_k}} \hat{F}^{B_1 \ldots B_k} + (\hat{M}_{\mathrm{sym}})^{A_1 \ldots A_n} = \hat{F}^{A_1 \ldots A_n} \hat{U}_0^* + \\ + i\hbar \frac{\partial_r \hat{F}^{A_1 \ldots A_n}}{\partial \hat{C}^A} \hat{U}^{*A} + (\hat{M}^*_{\mathrm{sym}})^{A_1 \ldots A_n}, \end{aligned} \tag{6.18}$$

where

$$\hat{M}^{A_1} \equiv 0, \qquad \hat{M}^{*A_1} \equiv \hat{F}_0 \hat{U}^{*A_1}, \tag{6.19}$$

$$\begin{aligned} \hat{M}^{A_1 \ldots A_n} \equiv \sum_{m=1}^{n-1} (-1)^{\varepsilon_m^n} \Big(\hat{U}^{A_1 \ldots A_m} \hat{F}^{A_{m+1} \ldots A_n} + \sum_{k=1}^{\infty} \frac{(n-m+k)!}{k!(n-m)!} \cdot \\ \cdot (i\hbar)^k \frac{\partial_r^k \hat{U}^{A_1 \ldots A_m}}{\partial \hat{C}^{B_1} \ldots \partial \hat{C}^{B_k}} \hat{F}^{B_1 \ldots B_k A_{m+1} \ldots A_n} \Big), \end{aligned} \tag{6.20}$$

$$\hat{M}^{*A_1 \ldots A_n} \equiv \hat{F}^{*A_1 \ldots A_{n-1}} \hat{U}^{*A_n} \tag{6.21}$$

(see also (2.33), (2.34), (2.38)).

In virtue of (6.2), the following conditions must be fulfilled by the operators (6.16):

$$\hat{F}_0 \hat{\bar{F}}_0 = \hat{1}, \tag{6.22}$$

$$\begin{aligned} \hat{F}_0 \hat{\bar{F}}^{A_1 \ldots A_n} + \hat{F}^{A_1 \ldots A_n} \hat{\bar{F}}_0 + \\ + \sum_{k=1}^{\infty} (i\hbar)^k \frac{\partial_r^k \hat{F}^{A_1 \ldots A_n}}{\partial \hat{C}^{B_1} \ldots \partial \hat{C}^{B_k}} \hat{\bar{F}}^{B_1 \ldots B_k} + (\hat{N}_{\mathrm{sym}})^{A_1 \ldots A_n} = 0, \end{aligned} \tag{6.23}$$

where

$$\hat{\hat{F}}_0 \equiv (\hat{F}_0)^\dagger + \sum_{k=1}^{\infty}(-1)^{\varepsilon_0^k}(i\hbar)\frac{\partial_l^k(\hat{F}^{B_1\dots B_k})^\dagger}{\partial\hat{C}^{B_1}\dots\partial\hat{C}^{B_k}}, \tag{6.24}$$

$$\hat{\hat{F}}^{A_1\dots A_n} \equiv (\hat{F}^{A_n\dots A_1})^\dagger + \sum_{k=1}^{\infty}\frac{(n+k)!}{n!k!}(-1)^{\varepsilon_0^k}(i\hbar)^k\frac{\partial_l^k(\hat{F}^{A_n\dots A_1 B_1\dots B_k})^\dagger}{\partial\hat{C}^{B_1}\dots\partial\hat{C}^{B_k}}, \tag{6.25}$$

$$\hat{N}^{A_1} \equiv 0\,, \qquad \hat{N}^{A_1\dots A_n} \equiv \sum_{m=1}^{n-1}\Big(\hat{F}^{A_1\dots A_m}\hat{\hat{F}}^{A_{m+1}\dots A_n} + \sum_{k=1}^{\infty}\frac{(n-m+k)!}{k!(n-m)!}(i\hbar)^k\frac{\partial_r^k\hat{F}^{A_1\dots A_m}}{\partial\hat{C}^{B_1}\dots\partial\hat{C}^{B_k}}\hat{\hat{F}}^{B_1\dots B_k A_{m+1}\dots A_n}\Big) \tag{6.26}$$

(see also (2.33), (2.34), (2.38)).

Besides, also in virtue of (6.2), the relations must be held which are obtained from (6.22), (6.23), (6.26) by the replacements

$$\hat{F}_0 \leftrightarrow \hat{\hat{F}}_0\,, \qquad \hat{F}^{A_1\dots A_n} \leftrightarrow \hat{\hat{F}}^{A_1\dots A_n}. \tag{6.27}$$

The following expansions in powers of ghosts take place for the operators (6.16):

$$\hat{F}_0 = \hat{F}_0(\hat{p}, \hat{q})\,, \tag{6.28}$$

$$\hat{F}^{A_1\dots A_n} = \sum_{m=1}^{\infty}\sum_{\{B\}=\nu_m^n}\frac{(-1)^{E^{A_1\dots A_n}_{B_m\dots B_1}}}{n!m!}\hat{F}^{A_1\dots A_n}_{B_m\dots B_1}(\hat{p}, \hat{q})\,\hat{C}^{B_m}\dots\hat{C}^{B_1} \tag{6.29}$$

(see (2.51)-(2.55)). The structural operators in (6.29) have the same statistics and the same generalized antisymmetry with respect to the upper and lower indices as the operators (2.57) for the same n, m.

The solvability of eqs. (6.17), (6.18) under conditions (6.22), (6.27) is guaranteed by eqs. (2.35), (2.36), (2.42), (2.43), (6.6)-(6.8), (6.10), (6.11).

Consider in more detail the lowest equations out of (6.17), (6.18). Equation (6.17) gives

$$\hat{T}_{0\alpha_0}\hat{F}_0 + i\hbar\hat{T}_{0\beta_0}\hat{F}^{\beta_0}_{\alpha_0} = \hat{F}_0\hat{T}^*_{0\alpha_0}, \tag{6.30}$$

where

$$\hat{F}^{\beta_0}_{\alpha_0} \equiv \hat{F}^{0|\beta_0}_{0|\alpha_0}(\hat{p}, \hat{q})\,. \tag{6.31}$$

Now turn to the lowest equation out of (6.18) at $n = 1$ in the lowest sector $A_1 = (0, \alpha_0)$:

$$\hat{U}^{0|\alpha_0}\hat{F}_0 - \hat{U}_0\hat{F}^{0|\alpha_0}(-1)^{\varepsilon_{0\alpha_0}} - 2i\hbar\hat{T}_{0\beta_0}\hat{F}^{00|\beta_0\alpha_0}\cdot(-1)^{\varepsilon_{0\alpha_0}} + i\hbar\frac{\partial_r\hat{U}^{0|\alpha_0}}{\partial\hat{C}_0^{\beta_0}}\hat{F}^{0|\beta_0} + i\hbar\frac{\partial_r\hat{U}^{0|\alpha_0}}{\partial\hat{C}_1^{\beta_1}}\hat{F}^{1|\beta_1} + (i\hbar)^2\frac{\partial_r^2\hat{U}^{0|\alpha_0}}{\partial\hat{C}_0^{\beta_0}\partial\hat{C}_0^{\gamma_0}}\hat{F}^{00|\beta_0\gamma_0} = i\hbar\frac{\partial_r\hat{F}^{0|\alpha_0}}{\partial\hat{C}_0^{\beta_0}}\hat{U}^{*0|\beta_0} + \hat{F}^{0|\alpha_0}\hat{U}_0^* + \hat{F}_0\hat{U}^{*0|\alpha_0}. \tag{6.32}$$

The $\hat{C}_0^2$-component of eq. (6.32) gives

$$\hat{U}^{\alpha_0}_{\mu_0\nu_0}\hat{F}_0 + i\hbar\hat{U}^{\alpha_0}_{\mu_0\beta_0}\hat{F}^{\beta_0}_{\nu_0} - i\hbar\hat{U}^{\alpha_0}_{\nu_0\beta_0}\hat{F}^{\beta_0}_{\mu_0}(-1)^{\mathscr{E}_{0\mu_0}\mathscr{E}_{0\nu_0}} + \hat{T}_{0\mu_0}\hat{F}^{\alpha_0}_{\nu_0}(-1)^{\mathscr{E}_{0\alpha_0}\mathscr{E}_{0\mu_0}} - \hat{T}_{0\nu_0}\hat{F}^{\alpha_0}_{\mu_0}(-1)^{(\mathscr{E}_{0\alpha_0}+\mathscr{E}_{0\mu_0})\mathscr{E}_{0\nu_0}} - \tfrac{1}{2}i\hbar\hat{\Pi}^{\alpha_0}_{\beta_0\gamma_0}\hat{F}^{\gamma_0\beta_0}_{\mu_0\nu_0} + i\hbar\hat{Z}^{\ \alpha_0}_{1\beta_1}\hat{F}^{\beta_1}_{\mu_0\nu_0} + \hat{F}^{\alpha_0}_{\mu_0}\hat{T}^*_{0\nu_0} - \hat{F}^{\alpha_0}_{\nu_0}\hat{T}^*_{0\mu_0}(-1)^{\mathscr{E}_{0\mu_0}\mathscr{E}_{0\nu_0}} = \hat{F}_0\hat{U}^{*\alpha_0}_{\mu_0\nu_0} + i\hbar\hat{F}^{\alpha_0}_{\beta_0}\hat{U}^{*\beta_0}_{\mu_0\nu_0}, \tag{6.33}$$

where

$$\hat{F}^{\alpha_0\beta_0}_{\mu_0\nu_0} \equiv \hat{F}^{00|\alpha_0\beta_0}_{00|\mu_0\nu_0}(\hat{p}, \hat{q}), \tag{6.34}$$

$$\hat{U}^{*\alpha_0}_{\mu_0\nu_0} \equiv \hat{U}^{*0|\alpha_0}_{00|\mu_0\nu_0}(\hat{p}, \hat{q}) \tag{6.35}$$

(see also (2.72), (2.85)).

The $\hat{C}_1$-component of eq. (6.32) gives

$$\hat{Z}^{\ \alpha_0}_{1\alpha_1}\hat{F}_0 - \tfrac{1}{2}i\hbar\hat{\Pi}^{\alpha_0}_{\beta_0\gamma_0}\hat{F}^{\gamma_0\beta_0}_{\alpha_1} + i\hbar\hat{Z}^{\ \alpha_0}_{1\beta_1}\hat{F}^{\beta_1}_{\alpha_1} = \hat{F}_0\hat{Z}^{*\alpha_0}_{1\alpha_1} + i\hbar\hat{F}^{\alpha_0}_{\beta_0}\hat{Z}^{*\beta_0}_{1\alpha_1}, \tag{6.36}$$

where

$$\hat{F}^{\alpha_0\beta_0}_{\alpha_1} \equiv \hat{F}^{00|\alpha_0\beta_0}_{1|\alpha_1}(\hat{p}, \hat{q}), \tag{6.37}$$

$$\hat{F}^{\beta_1}_{\alpha_1} \equiv \hat{F}^{1|\beta_1}_{1|\alpha_1}(\hat{p}, \hat{q}) \tag{6.38}$$

(see also (2.85)).

Equation (6.30) gives the transformation law for the constraint operators. Equation (6.33) gives the transformation law for the structural operators of the involutions of constraints. Equation (6.36) gives the transformation law for the operatorial null vectors of the constraints.

Now assume that the transformation operator (6.15), subject to (6.2), (6.3), is known, and define the operator $\hat{\mathscr{H}}^*_{\min}$ by the equation

$$\hat{H}_{\min}\hat{\mathfrak{A}} = \hat{\mathfrak{A}}\hat{\mathscr{H}}^*_{\min}. \tag{6.39}$$

Then we have, owing to (2.8),

$$[\hat{\mathscr{H}}^*_{\min}, \hat{\Omega}^*_{\min}] = 0. \tag{6.40}$$

Unfortunately, this does not imply yet that $\hat{\mathscr{H}}^*_{\min}$ is the bosonic operator generating closed gauge algebra since it may, generally, contain arbitrarily high powers of the ghost moments.

Expand the operator $\hat{\mathscr{H}}^*_{\min}$ into the $\hat{\bar{\mathscr{P}}}\hat{C}$-normal series in powers of ghosts

$$\hat{\mathscr{H}}^*_{\min} = \hat{\mathscr{V}}^*_0 + \sum_{n=1}^{\infty}\hat{\bar{\mathscr{P}}}_{A_n}\dots\hat{\bar{\mathscr{P}}}_{A_1}\hat{\mathscr{V}}^{*A_1\dots A_n}, \tag{6.41}$$

where

$$\hat{\mathscr{V}}^{*A_1\ldots A_n} = (\hat{\mathscr{V}}^{*}_{\mathrm{sym}})^{A_1\ldots A_n} \tag{6.42}$$

(see also (2.33), (2.34)).

Substituting (6.41) into (6.40), we have

$$[\hat{\mathscr{V}}^*_0, \hat{U}^*_0] = i\hbar \frac{\partial_r \hat{U}^*_0}{\partial \hat{C}^A} \hat{\mathscr{V}}^{*A}, \tag{6.43}$$

$$[\hat{\mathscr{V}}^{*A_1}, \hat{U}^*_0] - [\hat{U}^{*A_1}, \hat{\mathscr{V}}^*_0] + \\ + i\hbar \frac{\partial_r \hat{\mathscr{V}}^{*A_1}}{\partial \hat{C}^A} \hat{U}^{*A} - \sum_{k=1}^{\infty} (i\hbar)^k \frac{\partial_r^k \hat{U}^{*A_1}}{\partial \hat{C}^{B_1} \ldots \partial \hat{C}^{B_k}} \hat{\mathscr{V}}^{*B_1\ldots B_k} = 2i\hbar \frac{\partial_r \hat{U}^*_0}{\partial \hat{C}^A} \hat{\mathscr{V}}^{*AA_1}(-1)^{\mathscr{E}^0_1}, \tag{6.44}$$

$$[\hat{\mathscr{V}}^{*A_1\ldots A_n}, \hat{U}^*_0] + i\hbar \frac{\partial_r \hat{\mathscr{V}}^{*A_1\ldots A_n}}{\partial \hat{C}^A} \hat{U}^{*A} + (\hat{\mathscr{Y}}^*_{\mathrm{sym}})_{A_1\ldots A_n} = \\ = i\hbar(n+1) \frac{\partial_r \hat{U}^*_0}{\partial \hat{C}^A} \hat{\mathscr{V}}^{*AA_1\ldots A_n}(-1)^{\mathscr{E}^n_0}, \tag{6.45}$$

where $n \geqslant 2$ and

$$\hat{\mathscr{Y}}^{*A_1\ldots A_n} \equiv [\hat{\mathscr{V}}^{*A_1\ldots A_{n-1}}, \hat{U}^{*A_n}] - \\ - \sum_{k=1}^{\infty} \frac{(n-1+k)!}{k!(n-1)!} (-1)^{\mathscr{E}^n_1} (i\hbar)^k \frac{\partial_r^k \hat{U}^{*A_1}}{\partial \hat{C}^{B_1} \ldots \partial \hat{C}^{B_k}} \hat{\mathscr{V}}^{*B_1\ldots B_k A_2 \ldots A_n}. \tag{6.46}$$

Let us next define the operator

$$\hat{H}^*_{\min} \equiv \hat{\mathscr{H}}^*_{\min} + (i\hbar)^{-1}[\hat{\Lambda}, \hat{\Omega}^*_{\min}], \tag{6.47}$$

where

$$\mathscr{E}(\hat{\Lambda}) = 1, \quad \mathrm{gh}(\hat{\Lambda}) = -1, \quad \hat{\Lambda}^\dagger = -\hat{\Lambda}, \tag{6.48}$$

and $\hat{\Lambda}$ is a function of operators of the minimal sector. For (6.47) we have

$$[\hat{H}^*_{\min}, \hat{\Omega}^*_{\min}] = 0, \quad \mathrm{gh}(\hat{H}^*_{\min}) = 0, \tag{6.49}$$

$$(\hat{H}^*_{\min})^\dagger = \hat{H}^*_{\min}. \tag{6.50}$$

Expand the operator $\hat{\Lambda}$ into $\hat{\mathscr{P}}\hat{C}$-normal power series in ghosts

$$\hat{\Lambda} = \sum_{n=1}^{\infty} \hat{\mathscr{P}}_{A_n}, \ldots, \hat{\mathscr{P}}_{A_1} \hat{W}^{A_1\ldots A_n}, \tag{6.51}$$

where

$$(\hat{W}_{\mathrm{sym}})^{A_1\ldots A_n} = \hat{W}^{A_1\ldots A_n}. \tag{6.52}$$

The properties (6.48) impose the following conditions on the coefficients in (6.51):

(6.53) $$\mathscr{E}(\hat{W}^{A_1 \dots A_n}) = \sum_{j=1}^{n} \mathscr{E}(\hat{C}^{A_j}) + 1 ,$$

(6.54) $$\mathrm{gh}\,(\hat{W}^{A_1 \dots A_n}) = \sum_{j=1}^{n} \mathrm{gh}\,(\hat{C}^{A_j}) - 1 ,$$

(6.55) $$-(-1)^{\mathscr{E}_0^n} \hat{W}^{A_1 \dots A_n} = (\hat{W}^{A_n \dots A_1})^\dagger + \sum_{k=1}^{\infty} \frac{(n+k)!}{n!\,k!} (i\hbar)^k \frac{\partial_1^k (\hat{W}^{A_n \dots A_1 B_1 \dots B_k})^\dagger}{\partial \hat{C}^{B_1} \dots \partial \hat{C}^{B_k}}$$

(see also (2.38) at $m = 0$).

Let us subject the operatorial coefficients of expansion (6.51) to the following equations:

(6.56) $$\frac{\partial_r \hat{U}_0^*}{\partial \hat{C}^A} \hat{W}^A + \hat{\mathscr{V}}_0^* = \hat{V}_0^* ,$$

(6.57) $$2 \frac{\partial_r \hat{U}_0^*}{\partial \hat{C}^A} \hat{W}^{AA_1} (-1)^{\mathscr{E}_0^1} + \frac{\partial_r \hat{W}^{A_1}}{\partial \hat{C}^A} \hat{U}^{*A} + (i\hbar)^{-1} [\hat{W}^{A_1}, \hat{U}_0^*] + \hat{\mathscr{V}}^{*A_1} = \hat{V}^{*A_1} ,$$

(6.58) $$(n+1) \frac{\partial_r \hat{U}_0^*}{\partial \hat{C}^A} \hat{W}^{AA_1 \dots A_n} (-1)^{\mathscr{E}_0^n} + \frac{\partial_r \hat{W}^{A_1 \dots A_n}}{\partial \hat{C}^A} \hat{U}^{*A} + (i\hbar)^{-1} [\hat{W}^{A_1 \dots A_n}, \hat{U}_0^*] + (i\hbar)^{-1} \hat{Y}^*_{\mathrm{sym}})^{A_1 \dots A_n} + \hat{\mathscr{V}}^{*A_1 \dots A_n} = 0 , \qquad n \geqslant 2 ,$$

where

(6.59) $$\hat{Y}^{*A_1 \dots A_n} \equiv [\hat{W}^{A_1 \dots A_{n-1}}, \hat{U}^{*A_n}] (-1)^{\mathscr{E}_{n-1}^n} + \sum_{k=1}^{\infty} \frac{(n-1+k)!}{k!(n-1)!} (-1)^{\mathscr{E}_1^n} (i\hbar)^k \frac{\partial_r^k \hat{U}^{*A_1}}{\partial \hat{C}^{B_1} \dots \partial \hat{C}^{B_k}} \hat{W}^{B_1 \dots B_k A_2 \dots A_n} ,$$

and the r.h.s. of (6.56), (6.57) obey the conditions

(6.60) $$\mathscr{E}(\hat{V}_0^*) = 0 , \quad \mathrm{gh}\,(\hat{V}_0^*) = 0 ,$$

(6.61) $$\mathscr{E}(\hat{V}^{*A}) = \mathscr{E}(\hat{C}^A) , \quad \mathrm{gh}\,(\hat{V}^{*A}) = \mathrm{gh}\,(\hat{C}^A) ,$$

(6.62) $$[\hat{V}_0^*, \hat{U}_0^*] = i\hbar \frac{\partial_r \hat{U}_0^*}{\partial \hat{C}^A} \hat{V}^{*A} ,$$

(6.63) $$[\hat{V}^{*A}, \hat{U}_0^*] - [\hat{U}^{*A}, \hat{V}_0^*] + i\hbar \frac{\partial_r \hat{V}^{*A}}{\partial \hat{C}^B} \hat{U}^{*B} - i\hbar \frac{\partial_r \hat{U}^{*A}}{\partial \hat{C}^B} \hat{V}^{*B} = 0 ,$$

(6.64) $$[\hat{V}^{*A}, \hat{U}^{*B}] - [\hat{V}^{*B}, \hat{U}^{*A}] (-1)^{\mathscr{E}(\hat{C}^A)\mathscr{E}(\hat{C}^B)} = 0 ,$$

(6.65) $$\hat{V}_0^* = (\hat{V}_0^*)^\dagger + (-1)^{\mathscr{E}(\hat{C}^A)} i\hbar \frac{\partial_1 (\hat{V}^{*A})^\dagger}{\partial \hat{C}^A} ,$$

(6.66) $$\hat{V}^{*A} = (\hat{V}^{*A})^\dagger .$$

The solvability of eqs. (6.56)-(6.58) under conditions (6.52)-(6.55) is provided by eqs. (6.6)-(6.8), (6.10), (6.11), (6.60)-(6.66). If eqs. (6.56)-(6.58) are satisfied, eq. (6.46) takes the form

$$\hat{H}^*_{\min} = \hat{V}^*_0 + \hat{\bar{\mathscr{P}}}_A \hat{V}^{*A} . \tag{6.67}$$

This is linear in ghost momenta. Thus eq. (6.67) is the bosonic operator generating the closed gauge algebra (see (6.5), (6.49)).

The following expansions in powers of ghosts take place:

$$\hat{V}^*_0 = \hat{H}^*_0(\hat{p}, \hat{q}) , \tag{6.68}$$

$$\hat{V}^{*A} = \sum_{m=1}^{\infty} \sum_{\{B\}=\nu^1_m} \frac{(-1)^{E_{B_m \dots B_1}}}{m!} \hat{V}^{*A}_{B_m \dots B_1}(\hat{p}, \hat{q}) \hat{C}^{B_1} \dots \hat{C}^{B_m} , \tag{6.69}$$

where the set of values of ν^1_m is defined as (2.5) at $n = 1$, $A_1 = A$; the signature is given as (2.55); the structural operators acquire the signature factor (2.61) under the permutation of any two neighbouring lower indices B_{i+1}, B_i. We are also exploiting the concise notation

$$\hat{V}^{*\alpha_0}_{\ \ \mu_0} \equiv \hat{V}^{*0}_{\ 0}\Big|^{\alpha_0}_{\mu_0}(\hat{p}, \hat{q}) . \tag{6.70}$$

This completes the procedure of closing the operatorial gauge algebra. For the new constraint operators, involved in (6.12), for the original Hamiltonian in (6.68) and for the operatorial null vectors (6.14) we have

$$[\hat{T}^*_{0\alpha_0}, \hat{T}^*_{0\beta_0}] = i\hbar \hat{T}^*_{0\gamma_0} \hat{U}^{*\gamma_0}_{\alpha_0\beta_0} , \tag{6.71}$$

$$[\hat{H}^*_0, \hat{T}^*_{0\beta_0}] = i\hbar \hat{T}^*_{0\gamma_0} \hat{V}^{*\gamma_0}_{\ \ \beta_0} , \tag{6.72}$$

$$\hat{T}^*_{0\alpha_0} \hat{Z}^{*\alpha_0}_{1\alpha_1} = 0 , \qquad \hat{Z}^{*\ \ \alpha_{s-2}}_{s-1\alpha_{s-1}} \hat{Z}^{*\alpha_{s-1}}_{s\alpha_s} = 0 , \tag{6.73}$$

where $s = 2, \dots, L$. The coefficients in involutions (6.71), (6.72) are given by (6.35), (6.70), respectively. The classical limits of the symbols of the new operatorial constraints and null vectors are subjected to the conditions

$$\operatorname{rank}_\pm \left\| \frac{\partial_l T^*_{0\alpha_0}}{\partial \varphi^a} \right\| \Bigg|_{T^*=0} = \gamma_0(L)_\pm , \qquad \varphi^a \equiv (p_i, q^i) , \tag{6.74}$$

$$\operatorname{rank}_\pm \|Z^{*\alpha_{s-1}}_{s\alpha_s}\|\big|_{T^*=0} = \gamma_s(L)_\pm , \qquad s = 1, \dots, L, \tag{6.75}$$

which fix the L-th stage of reducibility.

A closed gauge algebra is called Abelian if expansions (6.13) contain only those terms which are linear in ghost operators, while all operators (6.69) disappear. In this simplest case we have

$$(\hat{\Omega}^*_{\min})_{\text{Abelian}} = \sum_{s=0}^{L} \hat{T}^*_{s\alpha_s} \hat{C}^{\alpha_s}_s , \tag{6.76}$$

$$(\hat{H}^*_{\min})_{\text{Abelian}} = \hat{H}^*_0(\hat{p}, \hat{q}) , \tag{6.77}$$

where $\hat{T}^*_{0\alpha_0}$ are the operators involved in (6.12),

$$\hat{T}^*_{s\alpha_s} \equiv \hat{\bar{\mathcal{P}}}_{s-1\alpha_{s-1}} \hat{Z}^{*\alpha_{s-1}}_{s\alpha_s} , \qquad s = 1, \dots, L. \tag{6.78}$$

For every operator $\hat{T}^*_{s\alpha_s}$, $s = 0, \dots, L$, from (6.76) the following relations hold in the Abelian case:

$$[\hat{T}^*_{s\alpha_s}, \hat{T}^*_{r\beta_r}] = 0 , \qquad [\hat{H}^*_0, \hat{T}^*_{r\beta_r}] = 0 , \tag{6.79}$$

$$\hat{T}^*_{s-1\alpha_{s-1}} \hat{Z}^{*\alpha_{s-1}}_{s\alpha_s} = 0 . \tag{6.80}$$

Hence, the r.h.s. of (6.76) may be naturally understood as the sum of contributions coming from the Abelian first-class constraints $\hat{T}^*_{s\alpha_s}$ of the $(L - s)$-th stage of reducibility for each $s = 0, \dots, L$. At $s > 0$, these constraints defined as (6.78) depend on the ghost moments $\bar{\mathcal{P}}_{s-1}$ and are thereby able to produce a gauge variation of the ghost co-ordinates $\hat{C}_{s-1}$. The simplest example of the Abelian constraints has thus made explicit the fact that at every stage of reducibility some of the ghost variables of the preceding stage acquire the gauge arbitrariness and thereby behave as gauge variables.

Our last task is to consider how the dynamics is affected by the closure of the gauge algebra. Define the transformed unitarizing Hamiltonian $\hat{H}^*_{\Psi^*}$ by the equation

$$\hat{H}_{\Psi} \hat{\mathfrak{A}} = \hat{\mathfrak{A}} \hat{H}^*_{\Psi^*} . \tag{6.81}$$

Using (3.22), (3.26), (6.3), (6.39), (6.47), we have

$$\hat{H}^*_{\Psi^*} = \hat{H}^*_{\min} + (i\hbar)^{-1}[\hat{\Psi}^*, \hat{\Omega}^*] , \tag{6.82}$$

where $\hat{H}^*_{\min}$ is given by (6.67),

$$\hat{\Omega}^* = \hat{\Omega}^*_{\min} + \sum_{s=0}^{L} \hat{\pi}_{s\alpha_s} \hat{\bar{\mathcal{P}}}^{\alpha_s}_s + \sum_{s'=1}^{L} \sum_{s=s'}^{L} \hat{\pi}^{s'}_{s\alpha_s} \hat{\bar{\mathcal{P}}}^{s'\alpha_s}_s , \tag{6.83}$$

$\hat{\Omega}^*_{\min}$ is defined as (6.4), $\hat{\Psi}^*$ is given by the equation

$$\hat{\mathfrak{A}}(\hat{\Psi}^* + \hat{\Lambda}) = \hat{\Psi} \hat{\mathfrak{A}} , \tag{6.84}$$

$\hat{\Lambda}$ is defined by (6.51)-(6.68).

Obviously, we have for (6.83), (6.82)

$$[\hat{\Omega}^*, \hat{\Omega}^*] = 0\,, \qquad [\hat{H}^*_{\hat{\Psi}^*}, \hat{\Omega}^*] = 0\,. \tag{6.85}$$

To the operators (3.20), (3.21) we put into correspondence the transformed operators $\hat{P}^*_M$, $\hat{Q}^{*M}$:

$$\hat{P}^*_M \hat{\mathfrak{A}} = \hat{\mathfrak{A}} \hat{P}_M\,, \qquad \hat{Q}^{*M} \hat{\mathfrak{A}} = \hat{\mathfrak{A}} \hat{Q}^M\,. \tag{6.86}$$

Then the transformation law

$$\hat{H}_{\hat{\Psi}}(\hat{P}, \hat{Q}) = \hat{H}^*_{\hat{\Psi}^*}(\hat{P}^*, \hat{Q}^*)\,, \qquad \hat{\Omega}(\hat{P}, \hat{Q}) = \hat{\Omega}^*(\hat{P}^*, \hat{Q}^*) \tag{6.87}$$

follows from (6.3), (6.81), (6.83). Owing to (4.1), (6.87), the time evolution of the transformed operators defined by (6.86) is governed by the equations

$$\begin{cases} i\hbar\partial_t \hat{P}^*_M = [\hat{P}^*_M, \hat{H}^*_{\hat{\Psi}^*}(\hat{P}^*, \hat{Q}^*)]\,, \\ i\hbar\partial_t \hat{Q}^{*M} = [\hat{Q}^{*M}, \hat{H}^*_{\hat{\Psi}^*}(\hat{P}^*, \hat{Q}^*)]\,, \end{cases} \tag{6.88}$$

taken together with the canonical equal-time commutation relations between the transformed co-ordinates and momenta.

Due to (4.12), (6.86), the restriction on physical states may be presented in the form

$$\hat{\Omega}^*(\hat{P}^*, \hat{Q}^*)|\mathrm{Phys}\rangle = 0 = \langle\mathrm{Phys}|\hat{\Omega}^*(\hat{P}^*, \hat{Q}^*)\,. \tag{6.89}$$

Thus the dynamics of the transformed operators $\hat{P}^*_M$, $\hat{Q}^{*M}$, eq. (6.86), is governed by a Hermitian Hamiltonian to be obtained from (6.82) by the formal substitution

$$\hat{P}_M \to \hat{P}^*_M\,, \qquad \hat{Q}^M \to \hat{Q}^{*M}\,. \tag{6.90}$$

The bosonic, eq. (6.67), and fermionic, eq. (6.4), generating operators of the closed gauge algebra enter into the new unitarizing Hamiltonian in the very same way as the generating operators (2.24), (2.23) of the open gauge algebra enter into the unitarizing Hamiltonian (3.26). In place of the gauge fermion $\hat{\Psi}$ involved in eq. (3.26), there is the new gauge fermion $\hat{\Psi}^*$ in eq. (6.82), defined by (6.84). In virtue of (6.85), (6.89), however, the physical dynamics does not depend on the choice of the operator $\hat{\Psi}$.

Bearing in mind the above-formulated way to close and abelize the gauge algebra, one should conclude that the qualities of being non-Abelian and unclosed are not intrinsic for the theory, but have rather originated from an improper choice of the operatorial gauge algebra basis. For systems with a finite number of degrees of freedom the choice of the basis of the gauge algebra is quite arbitrary. By using the canonical transformation (6.3) one may always find a new basis in which the gauge algebra is closed and even Abelian. In the field theory, however, we just deal with an «improper» choice of a, generally, non-Abelian and unclosed basis, since the latter, within the relativistic context, is made imperative by the four-dimensional locality and covariance of the dynamical description. The closedness and Abelianness may be indeed referred to as intrinsic properties of the theory in the conventional sense when they occur in a local covariant basis of the gauge algebra.

From the general point of view, the fact there is a regular procedure for closing and abelizing the algebra is of basic significance and is an important aspect of the operatorial quantization of gauge systems.

7. – Conclusion.

The main result of the present paper is a general method for constructing the unitarizing Hamiltonian operator for dynamical systems with first-class constraints. The unitarizing Hamiltonian, eq. (3.26), is built using (3.22) out of the three building blocks of the theory: fermionic, eq. (2.23), and bosonic, eq. (2.24), generating operators of the gauge algebra, and the gauge fermion operator, eqs. (3.29), (3.34). The gauge algebra is accomplished in the minimal sector (2.16) (see also (2.115)), using eqs. (2.46)-(2.49) within the structural relations (2.35), (2.36), (2.39), (2.40) that follow from the generating equations (2.17) and (2.18).

The time evolution of operators (3.20), (3.21) of the relativistic phase space is given by the Heisenberg equations of motion (4.1), determined by the unitarizing Hamiltonian, with the canonical commutation relations (1.10) for the operators taken in coinciding time moments. The generating functional of quantum Green's functions is defined in the usual way with the help of eqs. (5.1), (4.15). Then the operatorial equations of motion (4.1) give, in a standard way, rise to differential—with respect to external sources-equations (5.2), (5.3) for the generating functional (5.1). Solution (5.4) to these equations is written as a path integral in relativistic phase space. The effective action (5.5) in the path integrand contains the symbol of the unitarizing Hamiltonian operator (3.26), wherein the time arguments are displaced apart. This allows for the order of the operator factors in (3.26). The symbols of the unitarizing Hamiltonian operator (3.26) and the generating operators (2.23), (2.24) of the gauge algebra obey eqs. (5.8)-(5.11) which are the counterparts of (3.26), (3.22), (2.17), (2.18) in the sense of the «symbol$\leftrightarrow$operator» correspondence (1.6)-(1.8). The quasi-classical expansions of eqs. (5.10), (5.11) for the symbols of the gauge algebra generating operators are presented by eqs. (5.15)-(5.22). In the classical limit the generating equations (5.17), (5.20) produce the structural relations (5.24), (5.25), (5.28), (5.29) of the classical gauge algebra.

An important aspect of the solution given to the problem of operatorial quantization of systems subject to first-class constraints is in the formulation of a regular procedure for abelizing and closing the operatorial gauge algebra, valid for the general case of any stage of reducibility. The following points are of basic significance here. Equations (6.2), (6.3) give a unitary operator (6.15) of canonical transformation that reduces the fermionic generating operator (2.23) to the form (6.4), linear in the ghost canonical momenta. Next, eq. (6.39) determines the transformed Hamiltonian (6.41). At last, eqs. (6.56)-(6.58) reduce operator (6.47) to the form (6.67) linear in the ghost momenta. Then eqs. (6.6)-(6.8), (6.12), (6.13), together with (6.62)-(6.69), generate a closed gauge algebra. The transformation of operators (3.20), (3.21) of the relativistic phase space, accompanying the closure of the gauge algebra, is given by (6.86). The time evolution of the new operators defined in (6.86) is described by the Heisenberg equations (6.88) which contain the new unitarizing Hamiltonian (6.82) corresponding to the equivalent dynamical system characterized by the closed gauge algebra.

Appendix

Admissibility conditions for the symbol of the gauge fermion operator.

The admissibility conditions, when imposed on the classical limit of the symbol of the gauge fermion operator, must provide removing the degeneracy of the functional integral (5.4) if calculated using the stationary-phase method (the loop expansion).

We face, therefore, the following conditions:

1) admissibility in the sector of initial variables:

$$\text{(A.1)} \qquad \operatorname{rank}_{\pm}\left\|\{\{\mathscr{P}_0^{\alpha_0}, \Psi\}, \varphi^a\}\right\| = \operatorname{rank}_{\pm}\left\|\{\{\mathscr{P}_0^{\alpha_0}, \Psi\}, T_{0\beta_0}\}\right\| = \gamma_0(L)_{\pm}\,,$$

$$\varphi^a \equiv (p_i, q^i);$$

2) admissibility in the algebraic sector of ghosts:

$$\text{(A.2)} \qquad \operatorname{rank}_{\pm}\left\|\{\{\mathscr{P}_s^{\alpha_s}, \Psi\}, \bar{\mathscr{P}}_{s-1\alpha_{s-1}}\}\right\| = \operatorname{rank}_{\pm}\left\|\{\{\mathscr{P}_s^{\alpha_s}, \Psi\}, T_{s\beta_s}\}\right\| = \gamma_s(L)_{\pm}\,,$$

$$T_{s\alpha_s} \equiv \bar{\mathscr{P}}_{s-1\alpha_{s-1}} Z^{\alpha_{s-1}}_{s\alpha_s}\,,$$

where $s = 1, \ldots, L$;

3) admissibility in the sector of Lagrange multipliers:

$$\text{(A.3)} \qquad \operatorname{rank}_{\pm}\left\|\{\{\mathscr{P}_s^{1\alpha_s}, \Psi\}, \pi_{s-1\alpha_{s-1}}\}\right\| = \operatorname{rank}_{\pm}\left\|\{\{\mathscr{P}_s^{1\alpha_s}, \Psi\}, T^1_{s\beta_s}\}\right\| = \gamma_s(L)_{\pm}\,,$$

$$T^1_{s\alpha_s} \equiv \pi_{s-1\alpha_{s-1}} Z^{\alpha_{s-1}}_{s\alpha_s}\,,$$

where $s = 1, \ldots, L$;

4) admissibility in the auxiliarity ghost sector:

$$\text{(A.4)}\qquad \operatorname{rank}_{\pm}\left\|\{\mathscr{P}^{\alpha_{s-1}}_{s-1}, \{\Psi, \pi^{1}_{s\beta_s}\}\}\right\| = \operatorname{rank}_{\pm}\left\|\{\bar{T}^{1\alpha_s}_{s}, \{\Psi, \pi^{1}_{s\beta_s}\}\}\right\| = \gamma_s(L)_{\pm}\,,$$

$$\bar{T}^{1\alpha_s}_{s} \equiv \bar{Z}^{1\alpha_s}_{s\alpha_{s-1}} \mathscr{P}^{\alpha_{s-1}}_{s-1}\,,$$

where $s = 1, \ldots, L$ and also

$$\text{(A.5)}\qquad \bar{Z}^{1\alpha_1}_{1\alpha_0}\{\{\mathscr{P}^{\alpha_0}_{0}, \Psi\}, \varphi^{a}\} = 0\,, \qquad \varphi^{a} \equiv (p_i, q^i),$$

$$\text{(A.6)}\qquad \bar{Z}^{1\alpha_s}_{s\alpha_{s-1}}\{\{\mathscr{P}^{\alpha_{s-1}}_{s-1}, \Psi\}, \bar{\mathscr{P}}_{s-2\alpha_{s-2}}\} = 0\,, \qquad s = 2, \ldots, L,$$

$$\text{(A.7)}\qquad \mathscr{E}(\bar{Z}^{1\alpha_s}_{s\alpha_{s-1}}) = \mathscr{E}_{s-1\alpha_{s-1}} + \mathscr{E}_{s\alpha_s}\,, \qquad \operatorname{gh}(\bar{Z}^{1\alpha_s}_{s\alpha_{s-1}}) = 0\,,$$

$$\text{(A.8)}\qquad \operatorname{rank}_{\pm}\|\bar{Z}^{1\alpha_s}_{s\alpha_{s-1}}\| = \gamma_s(L)_{\pm}\,;$$

5) admissibility in the extraghost sector:

a)

$$\text{(A.9)}\qquad \operatorname{rank}_{\pm}\left\|\{\{\mathscr{P}^{s'\alpha_s}_{s}, \Psi\}, \pi^{s'-1}_{s-1\alpha_{s-1}}\}\right\| = \operatorname{rank}_{\pm}\left\|\{\{\mathscr{P}^{s'\alpha_s}_{s}, \Psi\}, T^{s'}_{s\beta_s}\}\right\| = \gamma_s(L)_{\pm}\,,$$

$$T^{s'}_{s\alpha_s} \equiv \pi^{s'-1}_{s\alpha_{s-1}} \bar{Z}^{s'\alpha_{s-1}}_{s\alpha_s}\,,$$

where $s' = 2, \ldots, L$, $s = s', \ldots, L$, and also

$$\text{(A.10)}\qquad \{\mathscr{P}^{s'-2\alpha_{s-2}}_{s-2}, \{\Psi, \pi^{s'-1}_{s-1\alpha_{s-1}}\}\} Z^{s'\alpha_{s-1}}_{s\alpha_s} = 0\,,$$

$$\text{(A.11)}\qquad \mathscr{E}(Z^{s'\alpha_{s-1}}_{s\alpha_s}) = \mathscr{E}_{s-1\alpha_{s-1}} + \mathscr{E}_{s\alpha_s}\,, \qquad \operatorname{gh}(Z^{s'\alpha_{s-1}}_{s\alpha_s}) = 0\,,$$

$$\text{(A.12)}\qquad \operatorname{rank}_{\pm}\|Z^{s'\alpha_{s-1}}_{s\alpha_s}\| = \gamma_s(L)_{\pm}\,;$$

b)

$$\text{(A.13)}\qquad \operatorname{rank}_{\pm}\left\|\{\mathscr{P}^{s'-1\alpha_{s-1}}_{s-1}, \{\Psi, \pi^{s'}_{s\alpha_s}\}\}\right\| = \operatorname{rank}_{\pm}\left\|\{\bar{T}^{s'\alpha_s}_{s}, \{\Psi, \pi^{s'}_{s\beta_s}\}\}\right\| = \gamma_s(L)_{\pm}\,,$$

$$\bar{T}^{s'\alpha_s}_{s} \equiv \bar{Z}^{s'\alpha_s}_{s\alpha_{s-1}} \mathscr{P}^{s'-1\alpha_{s-1}}_{s-1}\,,$$

where $s' = 2, \ldots, L$, $s = s', \ldots, L$, and also

$$\text{(A.14)}\qquad \bar{Z}^{s'\alpha_s}_{s\alpha_{s-1}}\{\{\mathscr{P}^{s'-1\alpha_{s-1}}_{s-1}, \Psi\}, \pi^{s'-2}_{s-2\alpha_{s-2}}\} = 0\,,$$

$$\text{(A.15)}\qquad \mathscr{E}(\bar{Z}^{s'\alpha_s}_{s\alpha_{s-1}}) = \mathscr{E}_{s-1\alpha_{s-1}} + \mathscr{E}_{s\alpha_s}\,, \qquad \operatorname{gh}(\bar{Z}^{s'\alpha_s}_{s\alpha_{s-1}}) = 0\,,$$

$$\text{(A.16)}\qquad \operatorname{rank}_{\pm}\|\bar{Z}^{s'\alpha_s}_{s\alpha_{s-1}}\| = \gamma_s(L)_{\pm}\,.$$

Conditions (A.1)-(A.16) must be fulfilled at least together with the equations

$$\text{(A.17)}\qquad \{\lambda^{\alpha_s}_{s}, H_{\Psi}\} = 0\,, \qquad \{H_{\Psi}, \pi_{s\alpha_s}\} = 0\,,$$

$$\text{(A.18)}\qquad \{\mathscr{P}^{\alpha_s}_{s}, H_{\Psi}\} = 0\,, \qquad \{H_{\Psi}, \bar{C}_{s\alpha_s}\} = 0\,,$$

$$\text{(A.19)}\qquad \{C^{\alpha_s}_{s}, H_{\Psi}\} = 0\,, \qquad \{H_{\Psi}, \bar{\mathscr{P}}_{s\alpha_s}\} = 0\,,$$

where $s = 0, \ldots, L$, and also together with the equations

$$\{\lambda_s^{s'\alpha_s}, H_\Psi\} = 0\,, \qquad \{H_\Psi, \pi_{s\alpha_s}^{s'}\} = 0\,, \tag{A.19}$$

$$\{\mathscr{P}_s^{s'\alpha_s}, H_\Psi\} = 0\,, \qquad \{H_\Psi, \bar{C}_{s\alpha_s}^{s'}\} = 0\,, \tag{A.20}$$

where $s' = 1, \ldots, L$, $s = s', \ldots, L$.

REFERENCES

[1] P. A. M. Dirac: *Can. J. Math.*, **2**, 129 (1950).
[2] P. A. M. Dirac: *Proc. R. Soc. London, Ser. A*, **249**, 326 (1958).
[3] P. A. M. Dirac: *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, N. Y., 1964).
[4] J. Anderson and P. Bergmann: *Phys. Rev.*, **83**, 1018 (1951).
[5] P. Bergmann and I. Goldberg: *Phys. Rev.*, **98**, 531 (1955).
[6] P. Bergmann: *Rev. Mod. Phys.*, **33**, 510 (1961).
[7] A. Hanson, T. Regge and C. Teitelboim: *Constrained Hamiltonian Systems* (Accademia dei Lincei, Roma, 1976).
[8] K. Sundermeyer: *Constrained Dynamics, Lecture Notes in Physics*, Vol. **169** (Springer-Verlag, Berlin, 1982).
[9] M. Henneaux: *Phys. Rep.*, **126**, 1 (1985).
[10] T. Kugo and I. Ojima: *Phys. Lett. B*, **73**, 459 (1978).
[11] T. Kugo and I. Ojima: *Prog. Theor. Phys. Suppl.* (*Japan*), **66**, 1 (1979).
[12] K. Nishijima: *Nucl. Phys. B*, **238**, 601 (1984).
[13] I. A. Batalin and E. S. Fradkin: preprints of Lebedev Physical Institute No. 132 and No. 201 (1986), to appear in *Nucl. Phys. B* and *Phys. Lett. B*, respectively.
[14] F. A. Berezin: *Sov. Theor. Math. Phys.*, **6**, 194 (1971).
[15] L. D. Faddeev: *Teor. Mat. Fiz.*, **1**, 3 (1969).
[16] E. S. Fradkin: *Hamiltonian formalism in covariant gauge and the measure in quantum gravity*, Acta University Wratislav (1973), *Proceedings of the X Winter School of Theoretical Physics in Karpacs*, No. **207**, p. 93.
[17] E. S. Fradkin and G. A. Vilkovisky: *Phys. Lett. B*, **55**, 224 (1975).
[18] I. A. Batalin and G. A. Vilkovisky: *Phys. Lett. B*, **69**, 309 (1977).
[19] E. S. Fradkin and T. E. Fradkina: *Phys. Lett. B*, **72**, 343 (1978).
[20] E. S. Fradkin and G. A. Vilkovisky: *Lett. Nuovo Cimento*, **13**, 187 (1975).
[21] E. S. Fradkin and M. A. Vasiliev: *Phys. Lett. B*, **72**, 70 (1977).
[22] I. A. Batalin and E. S. Fradkin: *Phys. Lett. B*, **122**, 157 (1983).
[23] I. A. Batalin and E. S. Fradkin: *Lett. Nuovo Cimento*, **38**, 393 (1983).
[24] E. S. Fradkin and G. A. Vilkovisky: *Phys. Rev. D*, **8**, 4241 (1973).
[25] I. V. Tyutin: *Gauge invariance in field theory and statistical physics in operatorial formulation*, preprint of Lebedev Physics Institute, No. 39 (1975).
[26] T. Kugo and I. Ojima: *Prog. Theor. Phys. Suppl.* (*Japan*), **66**, 1 (1979).
[27] I. A. Batalin and E. S. Fradkin: *Phys. Lett. B*, **128**, 303 (1983).
[28] I. A. Batalin and E. S. Fradkin: *Yad. Fiz.*, **39**, 231 (1984).
[29] I. A. Batalin and E. S. Fradkin: *J. Math. Phys.* (*N.Y.*), **25**, 2426 (1984).

[30] E. S. FRADKIN: *Green function method in the theory of quantized fields and in quantum statistics*, in *Quantum Field Theory and Hydrodynamics, Proceedings of P. N. Lebedev Physical Institute*, Vol. **29** (1965), p. 7 (translation: Consultants Bureau, New York, N. Y., 1967).
[31] E. S. FRADKIN: *Nucl. Phys.*, **76**, 588 (1966).
[32] E. S. FRADKIN: *On the functional method in quantum statistics and many particle theory*, in *N. N. Bogoliubov's 60th Anniversary Issue* (Nauka, Moscow, 1969), p. 386.
[33] E. S. FRADKIN: *S-matrix for nonlinear Lagrangians of general from*, in *I. E. Tamm Memorial Issue* (Nauka, Moscow, 1972), p. 146.

Direttore responsabile: RENATO ANGELO RICCI

Stampato in Bologna dalla Tipografia Compositori
coi tipi della Tipografia Monograf

Questo fascicolo è stato licenziato dai torchi il 19-XII-1986

Questo periodico
è iscritto
all'Unione Stampa
Periodica Italiana