Modern Physics Letters A Vol. 4, No. 8 (1989) 731–744

A SUPERCONFORMAL THEORY OF MASSLESS HIGHER SPIN FIELDS IN $D = 2+1$

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Received 3 January 1989

We construct a superconformal theory of higher spin fields in a space-time of dimension $D = 2+1$. The construction relies on the infinite-dimensional superalgebra shsc $(N/3)$ with the superconformal algebra OSp$(N/4)$ as a maximal finite-dimensional subalgebra. The invariant Chern-Simons action for the higher spin superconformal theory is an extension of the usual conformal supergravity action for particles with maximal spin two.

1. Introduction

In recent works,[1–8] new infinite-dimensional Lie superalgebras have been constructed which extend the usual supergravity superalgebra for the anti-de Sitter space adS_4, and as such, allow one to construct interacting higher spin fields in adS_4.

The most transparent way of constructing these superalgebras is achieved when one uses an operator realization in terms of arbitrary order polynomials in the Heisenberg operators viewed as generating elements.[4] This is an extension of the usual presentation[9] for the finite-dimensional superalgebra in terms of the polynomials of order ≤ 2. The $d = 4$ action can be written down as a generalized MacDowell-Mansouri functional.[10] The interaction of higher spin gauge fields is non-analytical in the cosmological constant which plays the role of an independent dimensionful parameter of the theory, thus allowing the inclusion of new terms, of higher order in fields and their derivatives, into the action and the field transformation laws.

This may lead to an infinite rank theory, in the terminology of Refs. 11 and 12. The proof of the existence of a consistent interaction to all orders appears as a highly non-trivial problem. Consistency of the cubic interactions was shown in Refs. 5 and 6. The essential non-analyticity precludes one from a transition to the flat limit, which rules out the possibility of constructing an analogous theory in a flat background. The situation is quite different, however, within the conformal-invariant approach to the higher spin theory.

In the conformal invariant theory of spin s, the kinetic terms are $\phi\,\Box^s p^s \phi$ (Bose) and $\psi\Box^{s-\frac{1}{2}}\not{p}p^s\psi$ (Fermi),[13] involving higher order derivatives. The dimensions of the fields differ from those in the usual Poincaré-invariant theory. Conformal invariance implies the absence of dimensionful constants, which considerably restricts the possible form of the interaction. Let us note that the existence of higher spin conformal invariant theory is not ruled out by the results of Refs. 14–17. In these works, the incompatibility of the higher spin conformal invariance with the interaction with gravity was shown. The main reasoning invoked the appearance in the gauge variation of the higher spin action of the terms involving proportional Weyl tensors. These terms cannot be cancelled by adding new terms into the metric transformation law. For the conformal theory, the terms involving the Weyl tensors can be eliminated by corresponding alterations of the metric transformation law, thus bypassing the "no-go" statements of the abovementioned works. To construct a complete theory, one has, first of all, to construct a superalgebra which would be a higher spin generalization of the conformal superalgebra SU(2,2/1). Such a superalgebra, shsc(1/4), has been obtained by us, and its form in four space-time dimensions will be published elsewhere. In the present work, we construct the infinite-dimensional superalgebras shsc(N/3) (super higher spin conformal) which extends the superconformal algebra in $D = 2 + 1$. Conformal superalgebras in $D = 2 + 1$, and the adS_4-superalgebras are well known to be isomorphic. Analogously to that, it turns out that shsc(N/3) is isomorphic to shs(N/4), and our generators and those of Ref. 4 differ by a choice of basis in the spaces of irreducible representations of SO(3,2).

Briefly, the program of constructing the infinite-dimensional conformal superalgebras and the gauge theory for higher spins is as follows:

1) A suitable operatorial realization of the finite dimensional conformal superalgebras is to be chosen.

2) An infinite-dimensional associative algebra of all order polynomials of the generating elements chosen in step 1 is to be constructed and the associative multiplication in conformal basis is calculated.

This will be the basis of constructing the conformal infinite-dimensional Lie superalgebra and its localization.

The aim of this article is to realize this program of constructing the global superalgebra shsc(N/3) and its localization to obtain the gauge theory of conformal higher spins in $D = 2 + 1$.

The action invariant under shsc(N/3) can be written down as a Chern-Simons functional. It is noteworthy that the higher spin superconformal theory can only be constructed when all spins are involved, and it is only when spin is not higher than two that the finite-dimensional version occurs (this is the usual conformal supergravity). The three-dimensional conformal theory of higher spins is interesting for the construction of the four-dimensional conformal theory, as well as from the adS_4 higher spin theory point of view, and another possible application of the proposed theory is to the spin membrane model.[18]

2. The $D = 2+1$ Conformal Superalgebra OSp(N/4)

The superalgebra OSp($N/4$) admits a simple operator realization in terms of second order polynomials in the Heisenberg-Clifford algebra generating elements (for the notation, see Appendix A).

$$[a_\alpha, b_\beta] = 2i\varepsilon_{\alpha\beta}, \qquad \{\psi_i, \psi_j\} = 2\delta_{i,j}, \tag{1}$$

$$a_\alpha^\dagger = a_\alpha, \qquad b_\alpha^\dagger = b_\alpha, \qquad \psi_i^\dagger = \psi_i \tag{2}$$

(all other commutators vanish).

The generators of OSp($N/4$) have the form

$$M_{\alpha(2)} = \frac{1}{4i}(a_\alpha b_\alpha + b_\alpha a_\alpha), \tag{3a}$$

$$P_{\alpha(2)} = \frac{1}{4i} a_\alpha a_\alpha, \qquad K_{\alpha(2)} = \frac{1}{4i} b_\alpha b_\alpha, \tag{3b}$$

$$D = \frac{1}{8i}(a_\alpha b^\alpha + b^\alpha a_\alpha), \tag{3c}$$

$$Q_{\alpha i} = \frac{1}{2} a_\alpha \psi_i, \qquad S_{\alpha i} = \frac{1}{2} b_\alpha \psi_i, \qquad T_{i(2)} = \frac{1}{4}\psi_i \psi_i. \tag{3d}$$

The operators (M, P, K, D, Q, S, T) are those of the Lorentz transformations, translations, conformal boosts, dilatations, supersymmetry, special conformal supersymmetry and the internal SO(N) symmetry, respectively. The generators are of the usual statistics.

All the Bose generators are anti-Hermitian, while the Fermi generators are Hermitian. Recall that the OSp($N/4$) generators have definite conformal weights in the adjoint representation

$$\mathrm{ad}D\,(T^c) = [D, T^c] = cT^c. \tag{4}$$

The conformal weights of the generators are, respectively, $(0, -1, 1, 0, -1/2, 1/2, 0)$.

3. The Associative Algebra aq(N/4) of the Weyl Symbols of Operators, and the Representations of SO(3, 2)

In Sec. 2, we have given a realization of OSp($N/4$) as the algebra of quadratic polynomials with the generating elements a, b, ψ.

Our aim now is to extend this to higher order polynomials. We shall work with the Weyl symbols of the operators (see Refs. 20, 21, 4). Now, let a, b, ψ be the symbols of operators[a]

$$[a_\alpha, b_\beta] = [a_\alpha, a_\beta] = [b_\alpha, b_\beta] = \{\psi_i, \psi_j\} = 0. \tag{5}$$

The Weyl symbol of an operator is derived when one substitutes the symbols (5) for the operators (1). The Weyl symbols are multiplied according to

$$(A * B)(z) = \exp\left\langle \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_1} \right\rangle A(z_1)B(z_2)\big|_{z_1 = z_2 = z}, \tag{6a}$$

$$\left\langle \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_1} \right\rangle = i\left(\frac{\partial}{\partial \alpha_{2\alpha}} \frac{\partial}{\partial b_1^\alpha} + \frac{\partial}{\partial b_{2\alpha}} \frac{\partial}{\partial a_1^\alpha} \right) + \frac{\partial_l}{\partial \psi_{2i}} \frac{\partial_l}{\partial_1^i}, \; z = (a_\alpha, b_\alpha, \psi_i). \tag{6b}$$

The polynomials in the variables (5) furnish a representation of SO(3, 2). The representation is given by

$$\overset{*}{T}(A) = [T, A]^* = T * A - A * T, \tag{7}$$

where T stands for the symbols of the SO(3, 2)-operators (3a–c). Irreducible components of this representation are furnished by spaces of definite degree of homogeneity in a_α and b_α.

Let us introduce a conformal basis in these irreducible representation spaces. The Casimir operator $\overset{*}{C}_2 = \overset{*}{M}_{\alpha(2)} \overset{*}{M}^{\alpha(2)}$ of the Lorentz algebra SO(2, 1) and the $\overset{*}{D}$ operator will be diagonal in this basis.

The basis monomials are of the form (throughout this section, we consider a non-extended version with $\psi = 0$, $Q_\alpha = 1/2\, a_\alpha$, $S_\alpha = 1/2\, b_\alpha$)

$$T^{(s,c)}_{\alpha(2l)} = d(s,c,l)\, a_{\alpha(l-c)} b_{\alpha(l+c)} (a^\alpha b_\alpha)^{(s-l)}. \tag{8a}$$

It is convenient to set the overall coefficient equal to

$$d(s,c,l) = \sqrt{\frac{(2l+1)!}{(s-l)!(l+c)!(l-c)!(s+l+1)!}}\,. \tag{8b}$$

(This amounts to normalizing the basis with respect to the bilinear form $\mathrm{tr}\,(A * B)$, see (27) for the definition of the trace.) The conformal weight (4) of the

[a] From this point on, we work with symbols only and thus do not distinguish our notations for operators and the corresponding symbols. All commutators [,]* are understood in the sense of (7).

generator (8) is easily verified to be equal to c, while the homogeneity degree, to s; the operator (8) is seen to transform according to the representation (l) of SO(2, 1). The operators $\overset{*}{K}_{\alpha(2)}$ and $\overset{*}{P}_{\alpha(2)}$ raise and lower the conformal weight by one, respectively. We denote this basis as SO(3, 2) → SO(2, 1) ⊕ SO(2).

The values of (s, c, l) ran over the following set

$$s = 0, \frac{1}{2}, \ldots, \infty; \qquad c = -s, -s+1, \ldots, s; \qquad l = |c|, |c|+1, \ldots, s. \tag{8c}$$

Note that in Ref. 4 the basis was considered in which the four-dimensional Lorentz algebra Casimir operators were diagonal. The generating elements of Ref. 4 are related to ours by

$$q_\alpha = \frac{1}{\sqrt{2}}(l^{i\frac{\pi}{4}} a_\alpha + l^{-i\frac{\pi}{4}} b_\alpha) \tag{9a}$$

$$r_{\dot\alpha} = \frac{1}{\sqrt{2}}(l^{-i\frac{\pi}{4}} a_\alpha + l^{i\frac{\pi}{4}} b_\alpha), \qquad q^\dagger_{\dot\alpha} = r_{\dot\alpha}, \qquad (r_{\dot\alpha})^\dagger = q_\alpha. \tag{9b}$$

Now we are going to derive the associative product of the symbols (8). Direct application of (6) leads to rather cumbersome expressions. The derivation is simplified when one uses new generators, related to the old ones through the unitary transformation

$$q_\alpha = \frac{1}{\sqrt{2}}(a_\alpha + b_\alpha)$$
$$r_\alpha = \frac{1}{\sqrt{2}}(a_\alpha - b_\alpha). \tag{10}$$

New generations satisfy

$$q^\dagger_\alpha = q_\alpha, \qquad r^\dagger_\alpha = r_\alpha, \qquad [q_\alpha, r_\beta]^* = 0,$$
$$[r_\alpha, r_\beta]^* = -2i\varepsilon_{\alpha\beta}, \qquad [q_\alpha, q_\beta]^* = 2i\varepsilon_{\alpha\beta}. \tag{11}$$

The transformation (10) on generating elements gives rise to a unitary transformation of the representation generators

$$T^{(s,k)}_{\alpha(2l)}(q,r) = \sum_{c=-l}^{l} d^l_{k,c}\left(-\frac{\pi}{2}\right) T^{(s,c)}_{\alpha(2l)}(b,a), \tag{12}$$

where the matrix elements of the transformation are given by the Wigner function[22] with $\beta = -\pi/2$. As a consequence of (11), the *-multiplication (6) with respect to q and r are calculated independently in the new basis.

To obtain a final form for the associative product in the initial basis (8), one has to: (i) perform the transformation (10), (ii) notice that the new basis elements are of the form

$$T^{(s,k)}_{\alpha(2l)}(q,r) = \frac{1}{\sqrt{(s+k)!(s-k)!}} C_{\alpha(2l)}{}^{\beta(s+k),\gamma(s-k)} q_{\beta(s+k)} r_{\gamma(s-k)} \tag{13}$$

(see Appendix B for the spinorial $C^{::}_{:}$ Clebsch-Gordan coefficients), and calculate the product following (6) with the use of the expression (B.8) for the spinorial Clebsch-Gordan coefficients; (iii) perform the inverse transformation to the basis one has started with, the inverse transformation matrix elements being given by $d^l_{c,k}(\pi/2)$. This, finally, gives

$$(T^{(s,c)}_{\alpha(2l)} * T^{(s',c')}_{\beta(2l')}) = \sum_{s'',c'',l'',u,V,t} i^{s+s'-s''} \begin{pmatrix} s & s' & s'' \\ c & c' & c'' \\ l & l' & l'' \end{pmatrix} \delta(c+c'-c'')$$

$$\times\, \delta(2u-l-l'+l'')\delta(2V-l+l'-l'')\delta(2t-l'+l-l'')\varepsilon_{\alpha(2u),\beta(2u)} T^{(s'',c'')}_{\alpha(2v),\beta(2t)}\,, \tag{14}$$

with the number coefficient

$$\begin{pmatrix} s & s' & s'' \\ c & c' & c'' \\ l & l' & l'' \end{pmatrix} = \sqrt{(2l+1)!(2l'+1)!}\, C(l,l',l'') \sum_{k,k',k''} (1)^{\frac{1}{2}(s+s'-s''-k-k'+k'')}$$

$$\times \frac{d^l_{c,k}\left(\frac{\pi}{2}\right) d^{l'}_{c',k'}\left(\frac{\pi}{2}\right) d^{l''}_{k'',c''}\left(-\frac{\pi}{2}\right)}{\sqrt{\Delta\left(\frac{s+k}{2},\frac{s-k}{2},l\right)\Delta\left(\frac{s'+k'}{2},\frac{s'-k'}{2},l'\right)}} \begin{Bmatrix} \frac{s+k}{2}, \frac{s-k}{2}, l \\ \frac{s'+k'}{2}, \frac{s'-k'}{2}, l' \\ \frac{s''+k''}{2}, \frac{s''-k''}{2}, l'' \end{Bmatrix}, \tag{15}$$

($C(l,l',l'')$ and $\Delta(l,l',l'')$ are given by the formulae (B.4), (B.9)).

We have expressed the structure constants of the associative algebra aq(0/4) through the 9j-coefficients and some particular values of the Wigner d-functions.

The summation in (14) goes as, formally,

$$u, v, t, s'', l'' = 0, \frac{1}{2}, 1, \ldots, \infty; \qquad c = -\infty, \ldots, -\frac{1}{2}, 0, \frac{1}{2}, \ldots, \infty.$$

However, due to the extension of the definition of the coefficients (15) and δ-functions, the region of the summation gets restricted non-trivially.

We extend the definition of the coefficients (15) by putting them to zero unless at least one of the following conditions is satisfied:

$$s'' \in \{|s-s'|, \ldots, s+s'\}; \qquad l'' \in \{|l-l'|, \ldots, l+l'\}; \qquad c+c'+c''=0;$$
$$l \in \{|c|, \ldots, s\}; \qquad l' \in \{|c'|, \ldots, s'\}; \qquad l'' \in \{|c''|, \ldots, s''\}; \tag{16}$$
$$|c| \leq s; \qquad |c'| \leq s'; \qquad |c''| \leq s''.$$

(In (15), the summation index $k = -l, -l+1, \ldots, l$, and similarly for k' and k''). Note that due to the symmetry of the $9j$-coefficients, the coefficients (15) are symmetric under the interchange of two first columns (the coefficient $(-1)^{s+s'+s''+l+l'+l''}$ arose).

4. The Algebra shsc(N/3).

Let us now build the Lie superalgebra shsc($N/3$), starting with the associative algebra of the previous section.[b] To do that, fix the Grassmann parity of the generations by

$$P(T^{(s,c)}_{i(k),\alpha(2l)}) = 1(0), \qquad 2s\text{—odd (even)}. \tag{17}$$

The shsc($N/3$) gauge fields are

$$\omega_\mu = \frac{1}{2i} \sum_{s,c,l,k} i^{-2s}\, \omega_{\mu,(s,c)}^{i(k),\alpha(2l)}\, T^{(s,c)}_{i(k),\alpha(2l)}. \tag{18}$$

See (8c) for the range of the summation parameters.

We choose the Grassmann shell of the second class[19] (the i^{-2s} factor is introduced for convenience, as in Ref. 4, and consider only T with even-$2s+k$ (aq^{E} ($N/4$) in Ref. 4)).

[b] We will not describe in detail the features specific to the extended version $N > 1$, as the sector of the ψ_i variables coincides with that of Ref. 4. The extended generators are $T_{i(k),\ldots} = (1/k!)\, \psi_{i(k)\ldots}$.

The Hermitian conjugation is defined as

$$\omega_\mu^\dagger = -\omega_\mu, \qquad (\omega_{\mu,(s,c)}^{i(k),\alpha(2l)})^\dagger = (-1)^{\frac{k(k-1)}{2}}\, \omega_{\mu(s,c)}^{i(k),\alpha(2l)}, \tag{19}$$

where $(-1)^{k(k-1)/2}$ arises due to the anti-commutativity of the generating elements ψ_i.

The super-commutator in shsc($N/3$) is defined by the general rule,

$$\{A, B\}^* = A * B - (-1)^{p_A p_B} B * A, \tag{20}$$

where $*$ is the associative product (14).

The shsc($N/3$) curvatures

$$R_{\mu\nu}^A = \partial_\mu \omega_\nu^A - \partial_\gamma \omega_\nu^A + f_{BC}^A \omega_\mu^B \omega_\nu^C \tag{21}$$

are given by

$$\begin{aligned} R_{\mu\nu\, i(k),\alpha(2l)}^{(s,c)} = {} & \partial_\mu \omega_{\nu\, i(k),\alpha(2l)}^{(s,c)} - (\mu \leftrightarrow \nu) \\ & + \sum_{\substack{s',s'',c',c'',l',l'' \\ \mu,v,t,r,p,q}} i^{s'+s''-s+r-|r|_2-1} \frac{k!}{\mu!v!r!} \begin{pmatrix} s' & s'' & s \\ c' & c'' & c \\ l' & l'' & l \end{pmatrix} \delta(c'+c''-c) \\ & \times \delta(k-u-v)\,\delta(2p-l'-l''+l)\delta(2q-l'+l''-l)\,\delta(2t-l''+l'-l) \\ & \times \delta(|4s's''+s'+s''-s+uv+r(u+v)+1|_2) \\ & \times \omega_{\mu\, i(u)\, j(r),\alpha(2q)\gamma(2p)}^{(s',c')}\, \omega_{\gamma\quad\ i(v),\alpha(2t)}^{(s'',c'')j(r)\qquad\gamma(2p)}. \end{aligned} \tag{22a}$$

In this formula, the summation extends formally to the following values of the parameters

$$\begin{gathered} \{s',s'',l',l'',p,q,t\} = 0, \frac{1}{2}, 1, \ldots, \infty; \qquad \{u,v,r\} = 0,1,2,\ldots,\infty; \\ \{c',c''\} = -\infty, \ldots, -\frac{1}{2}, 0, \frac{1}{2}, \ldots, \infty. \end{gathered} \tag{22b}$$

However, the region of the summation is non-trivially restricted due to the above extension of the definition of the coefficients (16), the presence of the δ-function and the factorials in the denominator, and the anti-symmetry with respect to internal indices.

The $\delta(|\ldots|_2)$ factor has emerged when taking into account $1-(-1)^n = 2\delta(|n+1|_2)$, due to the Grassmann parity of the fields $(\varphi s' s'')$, the anti-symmetry with respect to the internal indices $(uv+(u+v)r)$, the property $\chi_\alpha \varphi^\alpha = -\chi^\alpha \varphi_\alpha$ and the symmetry of the coefficients (15) $(s'+s''-s)$.

The curvatures (22) transform homogeneously under the gauge transformations

$$\delta\omega^A_\mu = \partial_\mu \varepsilon^A + f^A_{BC}\omega^B_\mu \varepsilon^c, \tag{23}$$

$$\delta R^A_{\mu\nu} = f^A_{BC} R^B_{\mu\nu}\varepsilon^c, \tag{24}$$

and satisfy the Bianchi identities

$$\varepsilon^{\nu\rho\sigma}(\partial_\nu R^A_{\rho\sigma} + f^A_{BC}\omega^B_\nu R^c_{\rho\sigma}) \equiv 0 \tag{25}$$

with the structure constants f^A_{BC} given by (22).

5. The Chern-Simons Action and the Equations of Motion for the $D = 2+1$ Higher Spin Superconformal Field Theory

The action of the shsc$(N/3)$ invariant theory in three dimensions can be written down in the form of a Chern-Simons functional[c]

$$S = \int \mathrm{tr}\left(\omega \wedge d\omega + \frac{2}{3}\,\omega\wedge\omega\wedge\omega\right), \tag{26}$$

where $\omega = \omega_\mu\, dx^\mu$ and d is the exterior differential.

The trace is defined by

$$\mathrm{tr}\,(\omega(a,b,\psi)) = \omega(0). \tag{27}$$

The equations of motion read

$$R^{(s,c)}_{\mu\nu\, i(k),\alpha(2l)} = 0. \tag{28}$$

These equations are invariant under the gauge transformations (23) and (24).

[c] $A \wedge B = (A_\mu * B_\nu - A_\nu * B_\mu)\, dx_\mu\, dx_\nu$

Some of the fields (the 'auxiliary' fields) can be excluded with the help of (25), by expressing them through 'physical' fields which satisfy the equations of motion with higher derivatives. These equations, however, do not admit non-trivial (i.e., other than the pure gauge) solutions and do not describe physical degrees of freedom (spin $s \geq 1$ massless fields are trivial in three dimensions on the mass-shell). The massive version of the theory becomes non-trivial on the mass shell. It would be interesting to construct a model with the higher spin field mass acquired via a spontaneous symmetry breaking (through the interaction with the singleton?).

Let us show now that, within the sector of fields with spin $s = 1, 3/2, 2$, the above equations generate those of the usual OSp($N/4$) invariant conformal supergravity.

The terms in the decomposition of the gauge field ω_μ that correspond to the maximal finite-dimensional subalgebra OSp($N/4$) of shsc($N/3$), are of the form

$$\omega_\mu = \frac{1}{2i}\Big(\omega^{\alpha(2)}_{\mu(1,-1)} T^{(1,-1)}_{\alpha(2)} + \omega^{\alpha(2)}_{\mu(1,0)} T^{(1,0)}_{\alpha(2)} + \omega^{\alpha(2)}_{\mu(1,1)} T^{(1,1)}_{\alpha(2)}$$

$$+ \omega_{\mu(1,0)} T^{(1,0)} + \omega^{i(2)}_{\mu(0,0)} T_{i(2)} - i\omega^{\alpha i}_{\mu(\frac{1}{2},-\frac{1}{2})} T^{(\frac{1}{2},-\frac{1}{2})}_{\alpha i} - i\omega^{\alpha i}_{\mu(\frac{1}{2},\frac{1}{2})} T^{(\frac{1}{2},\frac{1}{2})}_{\alpha i}\Big). \qquad (29)$$

It is not difficult to notice that the T generators are related to the generators (3) by the transformation (see (8))

$$\left(\frac{1}{2\sqrt{2}i} T^{(1,-1)}_{\alpha(2)}, \frac{1}{2\sqrt{2}i} T^{(1,1)}_{\alpha(2)}, \frac{1}{2i} T^{(1,0)}_{\alpha(2)}, \frac{1}{2\sqrt{2}i} T^{(1,0)}, \frac{1}{2} T^{(0,0)}_{i(2)}, \frac{1}{2} T^{(\frac{1}{2},-\frac{1}{2})}_{i\alpha}, \frac{1}{2} T^{(\frac{1}{2},\frac{1}{2})}_{i\alpha}\right)$$

$$= (P_{\alpha(2)}, K_{\alpha(2)}, M_{\alpha(2)}, D, T_{i(2)}, Q_{i\alpha}, S_{i\alpha}). \qquad (30)$$

The conformal supergravity curvatures

$$R_{\mu\nu\alpha(2)}(P) = D_{[\mu} l_{\gamma]\alpha(2)} - 2i\, \psi_{\mu\alpha i} \psi_{\gamma\alpha}{}^{i}, \qquad (31a)$$

$$R_{\mu\nu\alpha(2)}(K) = D_{[\mu} f_{\gamma]\alpha(2)} - 2i\, \phi_{\mu\alpha i} \phi_{\gamma\alpha}{}^{i}, \qquad (31b)$$

$$R_{\mu\nu\alpha(2)}(M) = \partial_{[\mu} \omega_{\gamma]\alpha(2)} + 2\omega_{\mu\alpha\gamma} \omega_{\gamma\alpha}{}^{\gamma} + l_{[\mu\alpha\gamma} f_{\nu]\alpha}{}^{\gamma} - i\, \psi_{[\mu\alpha i} \phi_{\nu]\alpha}{}^{i}, \qquad (31c)$$

$$R_{\mu\nu}(D) = \partial_{[\mu} b_{\nu]} + l_{[\mu\alpha(2)} f_{\nu]}{}^{\alpha(2)} - i\, \psi_{[\mu\alpha i} \phi_{\nu]}{}^{\alpha i}, \qquad (31d)$$

$$R_{\mu\nu i(2)}(T) = \partial_{[\mu} A_{\nu]i(2)} + 2A_{\mu ik} A_{\nu}{}^{k}{}_{i} - 2i\, \psi_{[\mu\alpha i} \phi_{\nu]}{}^{\alpha}{}_{i}, \qquad (31e)$$

$$R_{\mu\nu\alpha i}(Q) = \partial_{[\mu}\psi_{\nu]\alpha i} + l_{[\mu\alpha\gamma}\phi_{\nu]}{}^{\gamma}{}_{i} + A_{[\mu ik}\psi_{\nu]\alpha}{}^{k}, \tag{31f}$$

$$R_{\mu\nu\alpha i}(S) = \partial_{[\mu}\phi_{\nu]\alpha i} + f_{[\mu\alpha\gamma}\psi_{\nu]}{}^{\gamma}{}_{i} + A_{[\mu ik}\phi_{\nu]\alpha}{}^{k}, \tag{31g}$$

$$D_\mu\omega^c_{\alpha(n)} = \partial_\mu\omega^c_{\alpha(n)} + n\omega_{\mu\alpha\gamma}\omega^c_{\alpha(n-1)}{}^{\gamma} + cb_\mu\omega^c_{\alpha(n)} \tag{31h}$$

follow from the formula (22) by the identification

$$(l^{\alpha(2)}_\mu, f^{\alpha(2)}_\mu, \omega^{\alpha(2)}_\mu, b_\mu, A_{\mu i(2)}, \phi_{\mu\alpha i}, \psi_{\mu\alpha i})$$

$$= (\sqrt{2}\,\omega^{\alpha(2)}_{\mu(1,-1)}, \sqrt{2}\,\omega^{\alpha(2)}_{\mu(1,1)}, \omega^{\alpha(2)}_{\mu(1,0)}\sqrt{2}\,\omega_{\mu(1,0)}, i\omega^{(0,0)}_{\mu i(2)}, \omega^{(\frac{1}{2},\frac{1}{2})}_{\mu i\alpha}, \omega^{(\frac{1}{2},-\frac{1}{2})}_{\mu i\alpha}), \tag{32}$$

and similarly for the curvatures. The fields $(e, f, \omega, b, A, \phi, \psi)$ are, respectively, the drei-bein, the connection for the conformal boosts, the Lorentz connection, the dilatation connection, the SO(N) Yang-Mills field, the connection for the conformal supersymmetries, and the gravitino. These furnish the adjoint representation of OSp(N/4). The set of fields $\omega^{(s,c)}_{\mu\ldots}$ with fixed degree of homogeneity with respect to the spinorial generating elements a_α, b_α, describes spin $S+1$ (additional 1 is due to a vector index).

In conclusion, let us note that it is possible to carry out, following Ref. 11, a generalized canonical quantization and find the S matrix for our superconformal theory of arbitrary spins with first and second class Fermi-Bose constraints. The structure of the S-matrix corresponding to the action (26) is the standard one for rank one theories[11] (ghost sector of the Lagrangian is analogous to that of the Yang-Mills theory with our choice of indices of the fields and structure coefficients taken into account).

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Appendix

A. Notations and Conventions

We follow the conventions of Refs. 1–8. The two-component spinorial indices are raised and lowered by means of $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, $\varepsilon^{\alpha\beta}$, $\varepsilon_{12} = \varepsilon^{12} = 1$, as $A^\alpha = \varepsilon^{\alpha\beta} A_\beta$, $A_\beta = \varepsilon_{\alpha\beta} A^\alpha$. The internal SO($N$) indices $(i, j, k, \ldots)$ are raised and lowered by $\delta^{i,j}$, $\delta_{i,j}$.

A symmetrization (anti-symmetrization) is implied for any set of upper or lower spinorial (internal) indices denoted by like letters. The usual summation convention is understood for each pair of a lower and an upper index denoted by the same letter. We use notations such as $A_{[\mu\nu]} = A_{\mu\nu} - A_{\nu\mu}$,

$$\begin{gathered} A_{\underbrace{\alpha\ldots\alpha}_{n}} = A_{\alpha(n)},\ A_{\underbrace{i\ldots i}_{n}} = A_{i(n)},\ \underbrace{\varepsilon_{\alpha\beta}\ldots\varepsilon_{\alpha\beta}}_{n} = \varepsilon_{\alpha(n),\beta(n)}, \\ \underbrace{q_\alpha\ldots q_\alpha}_{n} = q_{\alpha(n)},\ \underbrace{r_\alpha\ldots r_\alpha}_{n} = r_{\alpha(n)},\ \underbrace{\delta^\gamma_\alpha\ldots\delta^\gamma_\alpha}_{n} = \delta^{\gamma(n)}_{\alpha(n)}, \qquad \text{etc.} \end{gathered} \tag{A.1}$$

The three-dimensional world indices $\mu, \nu, \ldots = 0, 1, 2$. The metric has the signature $(+, -, -)$.

For a change of notation from the Lorentz indices $(a, b, \ldots)$ to the spinorial ones the matrices $\sigma_a^{\alpha(2)} = (I, \sigma_1, \sigma_3)$ are to be employed. (σ_1 and σ_3 are the Pauli matrices.) We often use the notation $\delta(n) = 1(0)$, $n = 0(n \neq 0)$;

$$|n|_2 = \mathrm{mod}_2(n) = n - \left[\frac{n}{2}\right]2. \tag{A.2}$$

B. The Spinorial Clebsch-Gordan (C.-G.) Coefficients

A spinor $T_{\alpha(2l),\beta(2l')}$ can be decomposed into irreducible symmetric multispinors according to[23]

$$T_{\alpha(2l),\beta(2l')} = \sum_{\mu,s,t,l''} \delta(2u-l-l'+l'')\delta(2s-l+l'-l'')$$

$$\times\, \delta(2t+l-l'-l'')\, C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} T^{(l,l')}_{\gamma(2l'')}, \tag{B.1}$$

$$T^{(l,l')}_{\gamma(2l'')} = C_{\gamma(2l''),}{}^{\alpha(2l),\beta(2l')} T_{\alpha(2l),\beta(2l')}, \tag{B.2}$$

where the spinorial C.-G. coefficients are

$$C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} = C(l,l',l'')\, \varepsilon_{\alpha(2u),\beta(2u)}\, \delta^{\gamma(2l'')}_{\alpha(2s),\beta(2t)}, \tag{B.3}$$

$$C(l,l',l'') = \sqrt{\frac{(2l)!(2l')!(2l''+1)!}{(l+l'-l'')!(l+l''-l')!(l'+l''-l)!(l+l'+l''+1)!}}, \tag{B.4}$$

u, s and t are defined through the δ-functions.

The symmetry properties are expressed as

$$C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} = (-1)^{l+l'-l''} C_{\beta(2l'),\alpha(2l),}{}^{\gamma(2l'')}, \tag{B.5}$$

and the orthogonality properties as

$$\sum_{l''} C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')}\, C^{\delta(2l),\rho(2l'),}{}_{\gamma(2l'')} = \delta^{\delta(2l)}_{\alpha(2l)}\delta^{\rho(2l')}_{\beta(2l')}, \tag{B.6}$$

$$C_{\gamma(2l''),}{}^{\alpha(2l),\beta(2l')} C_{\alpha(2l),\beta(2l'),}{}^{\rho(2l'')} = \delta^{\rho(2l'')}_{\gamma(2l'')}. \tag{B.7}$$

These formulae are analogous to the corresponding usual relations for $C^{ll'l''}_{mm'm''}$.[22] The spinorial representation analog of the intertwining formula for five $C^{ll'l''}_{mm'm''}$ coefficients[22] reads

$$C_{\alpha(2j),}{}^{\lambda(2j_1),\rho(2j_2)} C_{\beta(2k),}{}^{\delta(2k_1),\varepsilon(2k_2)} C_{\lambda(2j_1),\delta(2k_1),}{}^{\xi(2j'_1)} C_{\rho(2j_2),\varepsilon(2k_2),}{}^{\zeta(2j'_2)}$$

$$\times\, C_{\xi(2j'_1),\zeta(2j'_2),}{}^{\gamma(2j')} = \sqrt{(2j'_1+1)(2j'_2+1)(2j+1)(2k+1)}$$

$$\times \begin{Bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \\ j'_1 & j'_2 & j' \end{Bmatrix} C_{\alpha(2j),\beta(2k),}{}^{\gamma(2j')} \tag{B.8}$$

where { } are the 9*j*-coefficients.[22] The triangle coefficients $\Delta(l, l', l'')$ is

$$\Delta(l, l', l'') = \frac{(l + l' - l'')!(l - l' + l'')!(l' + l'' - l)!}{(l + l' + l'' + 1)!} . \tag{B.9}$$

Note that the shs (1|2) commutation relations[4,3] can be rewritten as

$$[T_{\alpha(2l)}, T_{\beta(2l')}\} = \sum_{l''} \frac{i^{l + l' - l'' - 1}}{\sqrt{(2l'' + 1)\Delta(l, l', l'')}} \delta(|4ll' + l + l' - l'' + 1|2) \times C_{\alpha(2l), \beta(2l'),}{}^{\gamma(2l'')} T_{\gamma(2l'')} \tag{B.10}$$

or, in the weight basis, as

$$[T^{l}_{m}, T^{l'}_{m'}\} = \sum_{l'', m''} \frac{i^{l + l' - l'' - 1} C^{ll'l''}_{mm'm''}}{\sqrt{(2l'' + 1)\Delta(l, l', l'')}} \delta(|4ll' + l' + l - l'' + 1|2)\, T^{l''}_{m''} \tag{B.11}$$