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HIGHER-SPIN SYMMETRY IN ONE AND TWO DIMENSIONS (II)

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Constructed are conformal higher spin superalgebras in one and two dimensions, which contain the Virasoro algebra as a subalgebra. The general structure of these superalgebras is investigated.

1. We have considered in Ref. 1 higher spin theories in one and two dimensions. The theories are based on the little conformal higher spin superalgebras $\mathrm{shs}(N|2)$ in $D=1$ and $\mathrm{shs}(N|2)_+ \oplus \mathrm{shs}(M|2)_-$ in $D=1+1$, which generalize the little conformal algebras so(2,1) in $D=1$ and $\mathrm{so}(2,1)_+ \oplus \mathrm{so}(2,1)_-$ in $D=1+1$).

However, the full conformal algebras in one and two dimensions are infinite dimensional algebras Vir and $\mathrm{Vir}_+ \oplus \mathrm{Vir}_-$ respectively (Vir — abbreviation for Virasoro). They contain the little conformal algebras as subalgebras. In the present paper, we shall investigate the general structure and construct conformal higher spin superalgebras containing the Virasoro algebra as a subalgebra. For these superalgebras the notation shsc (super higher spin conformal) is used (as for the higher spin superalgebras generalize the conformal algebras $\mathrm{so}(D,2)$ in $D>2$).

In Sec. 2, we consider the general structure of conformal higher spin superalgebras and their central extensions. In Sec. 3, the algebra of the pseudo-differential operators is considered (in terms of symbols). In Sec. 4, N-extended conformal higher spin superalgebras are obtained on the basis of the Poisson brackets. They generalize the SO(N)-superconformal algebras to higher spins. In Sec. 5, an one-parameter family of conformal higher spin algebras hsc_λ (1) is described. It extends the family L_λ (Sl(2)) in the same way as the Virasoro algebra extends sl(2). In Sec. 6, some physical applications of the above algebras are discussed.

2. Consider the general structure of the superalgebra that extends the Virasoro algebra to higher spins. The basis can be chosen in this superalgebra as

$$T^s_{m,\alpha}, \quad \alpha = 1, \ldots, \alpha_S. \tag{2.1}$$

The conformal spin, s, of a generator defines a representation of the Virasoro algebra (without the central extension)

$$[L_n, T^s_{m,\alpha}] = (sn - m)\, T^s_{n+m,\alpha}. \tag{2.2}$$

Our definition of the conformal spin differs by one from the usual definition.[2,3] The corresponding currents have conformal spin $s + 1$. In this case the Virasoro, supercharge and Kac-Moody generators have conformal spin (1, 1/2, 0), respectively (the corresponding currents are of dimensions (2, 3/2, 1), respectively). This definition looks natural from the standpoint of restricting the Virasoro algebra representations to the little conformal algebra so(2,1). The generators T^s_m, $m = -s, \ldots, s$ span a finite-dimensional representation of so(2,1) of spin s (for $s \geq 0$).

The generators of conformal spin $s = -1, -1/2$ correspond to the fields of spin $s+1 = 0, 1/2$. These one-dimensional scalar and spinor fields can be viewed as singletons of the adS_2-algebra so(2,1),[4,5] similarly to the Rac and Di singletons for so(3,2).

The index n pertaining to the spin-s representation space of the Virasoro algebra, runs over $\mathbb{Z}$ for integer s and over $\mathbb{Z} + 1/2$ for half-integer s.

The index $\alpha = 1, \ldots, \alpha_s$ ranges over different representations with the same spin s, α_s being the number of such representations in the algebra.

Superalgebra commutation relations with respect to the basis (2.1) are of the form

$$[T^s_{n,\alpha}, T^{s'}_{n',\alpha'}\} = \sum_{s'', n'', \alpha''} f_{s,n,\alpha;s',n',\alpha';}{}^{s'',n'',\alpha''}\, T^{s''}_{n'',\alpha''} + C_{s,n,\alpha;s',n',\alpha'}, \tag{2.3}$$

where the Grassmann parity of the generators is defined by

$$\Pi\,(T^s) = O(1) \quad \text{for} \quad s\,\text{(half) integer}. \tag{2.4}$$

The dependence of the structure constants f and the "central matrix" C on n, n', and n'' is determined by transformation properties of the generators (2.2). We assume that the generators $T^1_{n,1}$ span a subalgebra isomorphic to the Virasoro algebra (accordingly, $T^1_{n,1}$ is sometimes denoted as L_n).

The dependence of the central matrix in (2.3) on n and n' is easily found with the help of the Jacobi identities with so(2,1)-generators L_0, $L_{\pm 1}$:

$$C_{s,n,\alpha;s',n',\alpha'} = \delta(n+n')\,\delta(s-s')\,C^s_{\alpha,\alpha'} \times \prod_{p=-s}^{s} (n+p),\ C^s_{\alpha,\alpha'} = C^s_{\alpha',\alpha}. \tag{2.5}$$

This expression generalizes the well-known central terms for spins 0, 1/2 and 1:

$$nC^0, \left(n^2-\frac{1}{4}\right)C^{1/2}, n(n^2-1)\,C^1.$$

In terms of the OPE the superalgebra (2.3) reads ($T^s_\alpha(z) = \sum_{n=-\infty}^{\infty} T^{s-1}_{n,\alpha}/z^{n+s}$)

$$T^s_\alpha(z)\,T^{s'}_{\alpha'}(w) \sim \frac{C^{s,s'}_{\alpha,\alpha'}}{(z-w)^{s+s'}} + \sum_{s''}\sum_{\alpha''} F_{s,\alpha;s',\alpha';}{}^{s'',\alpha''}(z,w)\,T^{s''}_{\alpha''}(w) + \dots, \tag{2.6a}$$

where "structure operators" have the form

$$F_{s,\alpha;s',\alpha';}{}^{s'',\alpha''}(z,w) = \sum_{\substack{p+q=s+s'-s''\\ p\geq 1}} f_{s,\alpha;s',\alpha';}{}^{s'',\alpha''}(p)\,(z-w)^{-p}\partial^q,\ \partial = \frac{d}{dw}. \tag{2.6b}$$

The first term in the rhs (2.6a) generates (2.5) for the central term.

Assume now that the superalgebra (2.3) is represented in the Hilbert space of states of a physical theory. Then, according to general rules, one can build[6–8] a Fermi generating gauge operator Ω. The nilpotence condition $\Omega^2 = 0$, which secures gauge invariance, fixes the values of central terms, which are in this case interpreted[6] as quantum deformations in Wick's involution.

In Ref. 6 explicit formulae are given which express the quantum deformations through the structure constants f of the algebra with which one has started. In particular, for the superalgebra (2.3) (assuming that $(T^s_{n,\alpha})^\dagger = T^s_{-n,\alpha}$) it follows that

$$[T^s_{n,\alpha},(T^{s'}_{n',\alpha'})^\dagger\} = \sum_{\substack{s'',\alpha''\\ n''\geq 0}} [\bar{f}_{s,n,\alpha;s',n',\alpha';}{}^{s'',n'',\alpha''}\,T^{s''}_{n'',\alpha''} + \bar{f}_{s',n',\alpha';s,n,\alpha;}{}^{s'',n'',\alpha''}\,(T^{s''}_{n'',\alpha''})^\dagger]$$

$$+\ \bar{C}_{s,n,\alpha;s',n',\alpha'}\,, \tag{2.7}$$

where

$$\bar{f}_{s,n,\alpha;s',n',\alpha';}{}^{s'',n'',\alpha''} = \Theta\,(n-n')\,f_{s,n,\alpha;s',-n',\alpha';}{}^{s'',n'',\alpha''} \tag{2.8}$$

($\Theta\,(n) = 1(0)$ for $n > (<)\,0$ and $1/2$ for $n = 0$) and in accordance with Eq. (6.17) of Ref. 6

$$\bar{C}_{s_1,n_1,\alpha_1;s_2,n_2,\alpha_2} = \sum_{\substack{s_3,\alpha_3 \\ s_4,\alpha_4}} \sum_{\substack{n_3\geq 0 \\ n_4 \geq 0}} (-1)^{4s_3s_4} \bar{f}_{s_1,n_1,\alpha_1;s_3,n_3,\alpha_3;}{}^{s_4,n_4,\alpha_4} \bar{f}_{s_2,n_2,\alpha_2;s_4,n_4,\alpha_4;}{}^{s_3,n_3,\alpha_3}$$

$$= C_{s_1,n_1,\alpha_1;s_2,-n_2,\alpha_2}\,. \tag{2.9}$$

To calculate the complete "central matrix" C one should know all the structure constants f. However in order to calculate the conformal anomaly (central term of the Virasoro algebra $C_{1,n,1;1,n',1}$) it is sufficient to know only the spectrum of states in the algebra (the range of summation over s and α), since in this case only structure coefficients from Eq. (2.2) enter in (2.8)

$$\bar{f}_{1,n,1;s',n',\alpha';}{}^{s'',n'',\alpha''} = (ns'+n')\,\Theta\,(n-n')\,\delta_{\alpha'}^{\alpha''}\,\delta_{s'}^{s''}\,\delta_{n-n'}^{n''}\,. \tag{2.10}$$

The sought anomaly is then

$$\bar{C}_{1,n,1;1,m,1} = \delta(n-m)\sum_s (-1)^{2s}\,\alpha_s\, C(s,n)\,, \tag{2.11}$$

where the spin s contribution is

$$C(s,n) = \sum_{n'} \Theta\,(n')\,\Theta\,(n-n')\,(ns+n')\,(n(s+1)-n') \tag{2.12}$$

(the multiplicity α_s arises when summing $\Sigma_{\alpha=1}^{\alpha_s}\,\delta_\alpha^\alpha$). Doing the sums in (2.12), we arrive at (omitting the term linear in n, since it can be absorbed into a redefinition of L_0)[a]

$$\tilde{C}(s,n) = \frac{n^3}{6}(6s^2+6s+1)\,. \tag{2.13}$$

Thus, the conformal anomaly assumes the form

$$\bar{C}_{1,n,1;1,m,1} = \delta(n-m)\frac{n^3}{6}\sum_s(-1)^{2s}\,\alpha_s\,(6s^2+6s+1)\,. \tag{2.14}$$

In the case of the Virasoro algebra ($\alpha_1 = 1, \alpha_s = 0$ when $s \neq 1$), the Neveu-Schwarz algebra ($\alpha_1 = \alpha_{1/2} = 1, \alpha_s = 0$ when $s \neq 1,\ 1/2$) and $N = 2$ superconformal algebra ($\alpha_1 = \alpha_0 = 1, \alpha_{1/2} = 2$, otherwise $\alpha_s = 0$) we get

$$\bar{C}^{\text{Vir}}_{n,m} = \delta(n-m)\frac{13}{6}n^3\,, \tag{2.15a}$$

[a] This term is $(-n/6)$. After summation over s the contribution of this term in C is equal to zero in the supersymmetric case (2.16), (2.17) (this is a "graduate number of degrees of freedom").

$$\bar{C}_{n,m}^{N-S} = \delta(n-m)\, n^3 \frac{(26-11)}{12} = \delta(n-m)\frac{5}{4} n^3 , \tag{2.15b}$$

$$\bar{C}_{n,m}^{N=2} = \delta(n-m)\, n^3 \frac{(26-22+2)}{12} = \delta(n-m)\frac{n^3}{2} . \tag{2.15c}$$

These values of the central charge (i.e., the coefficient in front of n^3) coincide with the values of conformal anomalies calculated by other means in Refs. 9 and 10 for the bosonic string in $D = 26$ dimensions and $N = 1$ spinning string in $D = 10$, and in Ref. 10 for the $N = 2$ spinning string in $D = 2$. To calculate the conformal anomaly (2.14) one is to do the summation over the contributions of all spins. This requires choosing an appropriate regularization (the ζ-function one, for instance).

In the supersymmetric theory it is natural to sum over the contributions of each (finite-dimensional algebra) supermultiplet as a whole. The conformal anomaly is then

$$\bar{C}_{1,n,1;1,m,1} = \delta(n-m)\frac{n^3}{6}\sum_{s_{\max}} \tilde{C}(s_{\max})\,\alpha^{s_{\max}} , \tag{2.16}$$

$$\tilde{c}(s_{\max}) = \sum_{s=s_{\min}}^{s_{\max}} (-1)^{2s}\,\alpha_s^{s_{\max}}(6s^2 + 6s + 1) , \tag{2.17}$$

where $s_{\max}$ is the maximal spin of the supermultiplet, $\alpha^{s_{\max}}$ are the multiplicities of these supermultiplets, and $\tilde{C}(s_{\max})$ is the contribution of the supermultiplet ($s = s_{\min}, \ldots, s_{\max}$ are the spins which constitute the supermultiplet, and $\alpha_s^{s_{\max}}$ their multiplicities in this supermultiplet).

Using this natural prescription for the calculation, the conformal anomaly can be seen to vanish in some of the extended superalgebras without resorting to a regularization, since the contributions $\tilde{C}(s_{\max})$ of each supermultiplet vanish separately one by one (see further Sec. 4).

3. The most appropriate method for building up superalgebras of the type of (23) is through the operator realizations. Following the general strategy of constructing infinite-dimensional higher spin algebras,[11-14] one is to choose first an appropriate operator realization of the subalgebra which corresponds to lower spins, and then to extend it to the full higher spin algebra containing representations involving higher spins.

There are two ways of building up such operator realizations. The first one is to consider the infinite-dimensional Heisenberg algebra generated by infinite number of oscillators. Generators of the higher spin algebra are realized as infinite formal series, bilinear in creation/annihilation operators. It is in this way that the Virasoro algebra is usually realized in two-dimensional field theory and string theory.

In this case one actually deals with an embedding of the higher spin algebra into the infinite-dimensional algebra $sp(\infty;\mathbb{C})$ (or $osp(N|\infty;\mathbb{C})$). We hope to return to these realizations in the future.

The other way to build up higher spin algebras is to extend the operator symbol construction[18,19] to the case of non-analytical dependence of symbols on their arguments. In that case one deals with an embedding of the higher spin algebras into the algebra of pseudodifferential operators.[15-17]

Consider formal pseudodifferential operators of the form

$$\hat{\mathscr{L}}(x,\partial)=\sum_{n,m\in\mathbb{Z}}\mathscr{L}_{n,m}x^n\partial^m,\quad \partial=\frac{d}{dx},\tag{3.1}$$

where ∂^{-1} is the inverse of the derivation ∂. It is required to satisfy the following rules for commuting through multiplication with a function

$$\partial^{-1}\cdot U(x)=\sum_{n=0}^{\infty}(-1)^n(\partial^n U)\cdot\partial^{-n-1},\tag{3.2a}$$

$$U(x)\cdot\partial^{-1}=\sum_{n=0}^{\infty}\partial^{-n-1}\cdot(\partial^n U).\tag{3.2b}$$

Define $a^\dagger a$-symbols of the pseudodifferential operators (3.1) (with $x\to\hat{a}^\dagger$, $\partial\to\hat{a}$, $[\hat{a},\hat{a}^\dagger]=1$)

$$\mathscr{L}_{a^\dagger a}=\sum_{n,m\in\mathbb{Z}}\mathscr{L}_{n,m}a^{\dagger n}a^m.\tag{3.3}$$

These symbols are formal Laurent series in the variables a and $a^\dagger$.

The transformations

$$\mathscr{L}_W=\exp\left\{-\frac{1}{2}\frac{\partial}{\partial a^\dagger}\frac{\partial}{\partial a}\right\}\mathscr{L}_{a^\dagger a},\tag{3.4a}$$

$$\mathscr{L}_{aa^\dagger}=\exp\left\{-\frac{\partial}{\partial a^\dagger}\frac{\partial}{\partial a}\right\}\mathscr{L}_{a^\dagger a}\tag{3.4b}$$

allow one to go over from the $a^\dagger a$-symbols to the Weyl and $aa^\dagger$-symbols. The commutation relations (3.2a,b) easily follow from the transformation (3.4b). The relation (3.2a) describes a transition from the $aa^\dagger$-ordering to the $a^\dagger a$-ordering. whereas in terms of symbols this transition is given by Eq. (3.4b). The relation

(3.2b) follows in terms of symbols with the help of the transformation inverse to (3.4b).

The multiplication formulae for these symbols are defined as usual,

1) Weyl symbols

$$A * B = A \exp\left\{\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial a}\frac{\overrightarrow{\partial}}{\partial a^\dagger} - \frac{\overleftarrow{\partial}}{\partial a^\dagger}\frac{\overrightarrow{\partial}}{\partial a}\right)\right\} B; \tag{3.5a}$$

2) $aa^\dagger$-symbols

$$A * B = A \exp\left(-\frac{\overleftarrow{\partial}}{\partial a^\dagger}\frac{\overrightarrow{\partial}}{\partial a}\right) B; \tag{3.5b}$$

3) $a^\dagger a$-symbols

$$A * B = A \exp\left(\frac{\overleftarrow{\partial}}{\partial a}\frac{\overrightarrow{\partial}}{\partial a^\dagger}\right) B. \tag{3.5c}$$

Note that the inclusion of symbols which are non-analytical in a and $a^\dagger$, does not spoil the associativity of the algebra of symbols, since the proof of the associativity of the $*$-multiplication does not involve the explicit dependence of the symbols on their arguments.

We will be working below with the Weyl symbols, which corresponds to using Eq. (3.5a) for the $*$-multiplication.

Define the commutator and the super-commutator of two symbols as

$$[A,B]_* = A * B - B * A, \tag{3.6}$$

$$[A,B\}_* = A * B - (-1)^{\pi(A)\pi(B)} B * A \tag{3.7}$$

with the Grassmann parity defined by the relations (for the elements with fixing degree of homogeneity)

$$A(-a, -a^\dagger) = (-1)^{\pi(A)} A(a, a^\dagger). \tag{3.8}$$

The brackets (3.6) and (3.7) endow the associative algebra of symbols, $\tilde{A}_1$, with two distinct algebraic structures, that of a Lie algebra and Lie superalgebra (these will be denoted, respectively, as $[\tilde{A}_1]$ and $[\tilde{A}_1\}$).

The associative algebra $\tilde{A}_1$ contains aq(2, $\mathbb{C}$) as a subalgebra[11] (in the mathematical literature this is called the Weyl algebra A_1[20]); the Lie algebra $[\tilde{A}_1]$ contains the $[A_1]$ algebra, and the superalgebra $[\tilde{A}_1\}$ contains shs$(1|2; \mathbb{C})$.[11]

In $\tilde{A}_1$ we introduce the following basis

$$\Gamma^s_n = (a^\dagger)^{s+n}(a)^{s-n}, \tag{3.9}$$

where $s,\ n \in \mathbb{Z}$ or $s,\ n \in \mathbb{Z} + 1/2$.

The associative $*$-multiplication, the commutator and super-commutator of two basis elements (3.9) are given by,

$$\Gamma^s_n * \Gamma^{s'}_{n'} = \sum_{s'',n''} U^{ss's''}_{nn'n''}\Gamma^{s''}_{n''},\ -\infty < s'' \leq s+s' \text{ and } (s+s'+s'') \in \mathbb{Z}, \tag{3.10}$$

$$[\Gamma^s_n, \Gamma^{s'}_{n'}]_* = \sum_{s'',n''} \delta(|s+s'-s''+1|_2)\, U^{ss's''}_{nn'n''}\Gamma^{s''}_{n''}, \tag{3.11}$$

$$\{\Gamma^s_n, \Gamma^{s'}_{n'}\}_* = \sum_{s'',n''} \delta(|4ss'+s+s'-s''+1|_2)\, U^{ss's''}_{nn'n''}\Gamma^{s''}_{n''}, \tag{3.12}$$

where $\delta(|n|_2) = 0(1)$ for n odd (even), and

$$U^{ss's''}_{nn'n''} = \delta(n+n'-n'') \sum_{m_1+m_2=s+s'-s''} (-1)^{m_2} m_1!\, m_2!$$
$$\times \binom{s+n}{m_2}\binom{s-n}{m_1}\binom{s'+n'}{m_1}\binom{s'-n'}{m_2}, \tag{3.13}$$

$$\binom{p}{n} = \begin{cases} \dfrac{p(p-1)\dots(p-n+1)}{n!} & \text{for } n > 0\,, \\ 1 & \text{for } n = 0\,, \\ 0 & \text{for } n < 0\,. \end{cases}$$

The algebra $\tilde{A}_1$, as a linear space, can be expanded into a direct sum $\tilde{A}_1 = V_{++} \oplus V_{+-} \oplus V_{-+} \oplus V_{--}$, where subspaces V consist of following sets of the symbols:

V_{++} (V_{--}) – symbols (non-)analytic on both a and $a^\dagger$,
V_{+-} – symbols (non-)analytic on $a^\dagger(a)$,
V_{-+} – symbols (non-)analytic on a $(a^\dagger)$.

Subspace V_{++}, V_{--}, $V_{++} \oplus V_{+-}$, $V_{++} \oplus V_{-+}$, $V_{--} \oplus V_{+-}$, $V_{--} \oplus V_{-+}$, V_{+-}, V_{-+} form subalgebras in $\tilde{A}_1$ under $*$-multiplication. Subspaces of polynomials (only a finite set of the coefficients $\mathscr{L}_{n,m}$ in (3.3) is different from zero) form subalgebras only in the algebras[b] V_{++}, $V_{++} \oplus V_{+-}$ and $V_{++} \oplus V_{-+}$.

4. Consider now a Poisson-bracket version $[\tilde{A}_1\}_{\mathrm{PB}}$ of the superalgebra $[\tilde{A}_1\}$. For functions of the variables a and $a^\dagger$ with a definite Grassmann parity, the super-Poisson bracket is defined by

$$[A,B\}_{\mathrm{PB}} = \begin{cases} 2AB, & \pi(A) = \pi(B) = 1 \\ \dfrac{\partial A}{\partial a}\dfrac{\partial B}{\partial a^\dagger} - \dfrac{\partial A}{\partial a^\dagger}\dfrac{\partial B}{\partial a}, & \text{otherwise}. \end{cases} \tag{4.1}$$

Introducing the notations $L_n^s = -1/2\,\Gamma_{-n}^s$, $s, n \in \mathbb{Z}$ and $G_n^s = -1/2\,\Gamma_{-n}^s$, $s, n \in \mathbb{Z} + 1/2$, we write down commutation relations in the superalgebra $[\tilde{A}_1\}_{\mathrm{PB}}$

$$[L_n^s, L_{n'}^{s'}]_{\mathrm{PB}} = (s'n - sn')\, L_{n+n'}^{s+s'-1}\,, \tag{4.2a}$$

$$[L_n^s, G_{n'}^{s'}]_{\mathrm{PB}} = (s'n - sn')\, G_{n+n'}^{s+s'-1}\,, \tag{4.2b}$$

$$\{G_n^s, G_{n'}^{s'}\}_{\mathrm{PB}} = -\,L_{n+n'}^{s+s'}\,. \tag{4.2c}$$

Generators with $s > 0$ span a subalgebra in $[\tilde{A}_1\}_{\mathrm{PB}}$ which we call the conformal higher spin superalgebra in $D = 1$ and denote as shsc* $(1|1)$ (* for PB, as in Ref. 11 for shs*). Its real form is specified by a natural Hermitean conjugation $(\Gamma_n^s)^\dagger = \Gamma_{-n}^s$.

This superalgebra extends the $N = 1$ conformal superalgebra in $D = 1$, i.e., the Neveu-Schwarz superalgebra (its generators being L_n^1 and $G_n^{1/2}$). The generators $L_{\pm 1,0}^1$ and $G_{\pm 1/2}^{1/2}$ span osp $(1|2)$ – the little conformal superalgebra in $D = 1$. The generators L_n^s, $n \in [-s, s]$ and G_n^s, $n \in [-s, s]$ span the superalgebra shs* $(1|2)$.[11,24] The $N = 1$ superconformal algebra in $D = 1 + 1$ consists of two $D = 1$-superalgebras, shsc* $(1|2) \simeq$ shsc* $(1|1) \oplus$ shsc* $(1|1)$. The superalgebra shsc* $(1|1)$ gives us an example of a superalgebra of the type of Eq. (2.3). It contains all spins $s = 1/2, 1, \ldots$ with multiplicity one ($\alpha_s = 1$).

In order to build N-extended superalgebras, introduce Clifford generating elements $\hat{\psi}_i$

$$\{\hat{\psi}_i, \hat{\psi}_j\} = 2\delta_{ij}, \qquad i, j = 1, \ldots, N. \tag{4.3}$$

The basis in the extended superalgebra is chosen in the form (Grassmann variables ψ_i are symbols of the $\hat{\psi}_i$ operators):

[b] An isomorphism $V_{++} \oplus V_{+-} \simeq V_{++} \oplus V_{-+}$ takes place $(\Gamma_n^s \to (-1)^s \Gamma_{-n}^s)$.

$$L^s_{n,i_1\ldots i_k} = \frac{1}{2}(a^\dagger)^{s-n}(a)^{s+n}\psi_{i_1}\ldots\psi_{i_k}. \tag{4.4}$$

We shall be considering a superalgebra generated by generators of even degree of homogeneity, $k + 2s \geq 2$ with $k + 2s$ even. Denote this superalgebra as shsc^{*E} $(N|1)$ (E for even).

The Poisson-bracket is defined in this case by

$$[A,B\}_{PB} = \frac{\partial A}{\partial a}\frac{\partial B}{\partial a^\dagger} - \frac{\partial A}{\partial a^\dagger}\frac{\partial B}{\partial a} + 2\frac{\partial_r A}{\partial \psi_i}\frac{\partial_l B}{\partial \psi^i}. \tag{4.5}$$

For two basis elements (4.5) the Poisson-brackets reads

$$[L^s_{n,i_1\ldots i_k}, L^{s'}_{n',j_1\ldots j_{k'}}\}_{PB} = (s'n - sn')\,L^{s+s'-1}_{n+n',i_1\ldots i_k j_1\ldots j_{k'}}$$

$$+\,kk'\ \mathrm{alt}\,(\delta_{i_k j_1}\,L^{s+s'}_{n+n',i_1\ldots i_{k-1}j_2\ldots j_{k'}}), \tag{4.6}$$

where alt means anti-symmetrization in all i- and j-indices separately (alt $(A_{i_1}B_{i_2})$ $= 1/2\,(A_{i_1}B_{i_2} - A_{i_2}B_{i_1})$).

Note an important special feature of the N-extended superalgebras shsc^{*E} $(N|1)$. When $N > 2$ they contain generators with conformal spins $s \leq 0$. These generators enter supermultiplets with highest spin $s_{max} \leq (N-1)/2$ (the $\{s_{max}\}$-supermultiplets).

The superalgebra spanned in shsc^{*E} $(N|1)$ by generators of degree of homogeneity two in a, $a^\dagger$ and ψ has the basis

$$L^1_n, L^{1/2}_{n,i}, L^0_{n,i_1 i_2}, \ldots, L^{1-N/2}_{n,i_1\ldots i_N}. \tag{4.7}$$

This subalgebra is isomorphic to the so(N)-extended conformal superalgebra (see Refs. 21, 22). The corresponding composition law follows from the general expression (4.6).

In particular, for $N = 2$ we have the usual $N = 2$ conformal superalgebra with the generators $L_n = L^1_n$, $G^i_n = L^{1/2 i}_n$, $T_n = 1/2\,\varepsilon^{ij}L^0_{n,ij}$

$$[L_n, L_m] = (n-m)L_{n+m},$$

$$[L_n, G^i_m] = \left(\frac{n}{2} - m\right)G^i_{n+m},$$

$$[L_n, T_m] = -mT_{n+m}, \tag{4.8}$$

$$\{G^i_n, G^j_m\} = \frac{1}{2}(n-m)\varepsilon^{ij}T_{n+m} + \delta^{ij}L_{n+m},$$

$$[T_n, G^i_m] = \varepsilon_j{}^i G^j_{n+m}\,(\varepsilon_j{}^i = \varepsilon_{jk}\delta^{ki}),$$

$$[T_n, T_m] = 0.$$

With the help of the general formulae (2.16), (2.17), it is possible to calculate the conformal anomalies for the above N-extended superalgebras. The results of the calculation are presented in the table (C^1 is the coefficient at n^3 in (2.16)).

Table.

N	0		1	2	$N>2$
C^1	without $u(1)$-Kac-Moody	with $u(1)$-Kac-Moody	$-\frac{5}{24}$	$-\frac{1}{4}$	0
	$-\frac{1}{6}$	0			

Here for $N = 0$ two algebras are considered; one with generators L^s_n ($s = 1, 2, \ldots$) and second with adding $u(1)$-Kac-Moody generators L^0_n. For calculations with $N = 0, 1, 2$ the ζ-function regularization has been used. For example, for $N = 0$ (with $u(1)$-Kac-Moody, $s = 0$) we have

$$\bar{C}_{1,n;1,m} = \delta(n-m)\frac{n^3}{6}\sum_{s=0}^{\infty}(6s^2+6s+1)$$

$$\underline{\underline{\text{reg}}}\ \delta(n-m)\frac{n^3}{6}(1+6\zeta(-1)+\zeta(0)+6\zeta(-2)) = 0. \tag{4.9}$$

It turns out that all superalgebras with $N > 2$ have zero conformal anomaly. It is interesting that no regularization was required to establish the vanishing of the conformal anomaly. Each of the contributions of supermultiplets with maximal spin $s_{\max}$ (i.e., a set of generators with degree of homogeneity $2s_{\max} = 2s + k$, $s_{\max} = 1, 2, \ldots$) is separately equal to zero. The contribution in (2.16) of each $\{s_{\max}\}$-supermultiplet is given by

$$\tilde{C}(s_{\max}) = \sum_{k=0}^{N}(-1)^k C^k_N\left(6\left(s_{\max}-\frac{k}{2}\right)^2 + 6\left(s_{\max}-\frac{k}{2}\right)+1\right), \tag{4.10}$$

where $C_N^k = N!/k!(N-k)!$ is the number of components of a totally antisymmetric rank-k tensor (with the internal indices $i(k)$, $i = 1, \ldots, N$), and it has been taken into account that $2s_{\max} = 2s + k$.

The expression on the rhs of Eq (4.10) vanishes when $N > 2$ due to elementary relations for the binomial coefficients

$$\sum_{k=0}^{N} (-1)^k C_N^k = \sum_{k=0}^{N} (-1)^k k C_N^k = \sum_{k=0}^{N} (-1)^k k^2 C_N^k = 0 \qquad (4.11)$$

for all $N > 2$.

These results hold true, besides the shsc$^{*E}(N|1)$ superalgebras with $N > 2$, also for all N-extended superalgebras with the same spectrum (as we have already mentioned, only transformation properties (2.2) of the generators (2.1) are used in the calculation of conformal anomaly).

To conclude this section, note that the construction proposed here can be generalized to induced the Laurent polynomials in a set of variables a_p, $a_p^\dagger$, $p = 1, \ldots, M$. This would give new infinite-dimensional algebras which extend shs* $(N|M)$ similarly to the way the Virasoro algebra extends the little conformal algebra so(2,1). We will consider these superalgebras in separate publication.

5. In the present section we shall describe Virasoro-like generalizations of the algebras L_λ (sl(2)) which extend sl(2). Firstly, let us consider operator realizations of these algebras. There exists an embedding of the algebras L_λ (sl(2)) into V_{++}. For extending it to Virasoro-like generalizations of the algebras L_λ (sl(2)) the algebra $V_{++} \oplus V_{-+}$ should be used. The non-analytic dependence of the symbols on one of their arguments make it possible to extend sl(2) and its representations to Vir and its representations.

Let the basis $T_\pm, T_0$ in sl(2) be chosen so that

$$\begin{aligned} [T_0, T_\pm] &= \pm\, T_\pm\,, \\ [T_+, T_-] &= -\,2\,T_0\,. \end{aligned} \qquad (5.1)$$

The universal enveloping algebra U(sl(2))[20] is defined as an associative algebra with generating elements $T_\pm, T_0$ which satisfy the relations (5.1).

The Casimir operator

$$C = -\,T_0^2 + \frac{1}{2}(T_+T_- + T_-T_+) \qquad (5.2)$$

generates the centre of U(sl(2)).

By considering an ideal $\mathscr{J}_\lambda$ generated by the element of the form

$$C_2 - \lambda \mathbb{1} \tag{5.3}$$

with λ being an arbitrary number (the value of C in the same representation), it is possible to construct a family of factor-algebras

$$U_\lambda(\mathrm{sl}(2)) = U(\mathrm{sl}(2))/\mathcal{J}_\lambda . \tag{5.4}$$

Endowing $U_\lambda(\mathrm{sl}(2))$ with a Lie algebra structure, one obtains a one-parameter family of infinite-dimensional Lie algebras which extend sl(2). This construction is described in Ref. 23. For a certain special value of λ this produces, in particular, the hs(2) algebra.[11,24]

The family of Lie algebras, $U_\lambda(\mathrm{sl}(2))$, admits the following operator realization. As is well-known (see, for instance, Ref. 25), sl(2) can be realized by the Heisenberg generators in various ways (i.e., there exist a number of distinct embeddings $\mathrm{sl}(2) \to [A_1]$). The Schwinger realization,

$$\hat{T}_+ = \frac{1}{2}\hat{a}^{\dagger 2},\ \hat{T}_0 = \frac{1}{4}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger),\ \hat{T}_- = \frac{1}{2}\hat{a}^2,\ [\hat{a}, \hat{a}^\dagger] = 1 \tag{5.5}$$

has been used in Ref. 11. In this case the value of the Casimir is $\lambda = 3/16$. This is the value of the Casimir operator in the representation (5.5) of sl(2) in the Fock space with the basis

$$(\hat{a}^\dagger)^n|0\rangle, \quad n = 0, 1, \ldots . \tag{5.6}$$

However, sl(2) has a one-parameter family of representations in the Fock space (5.6). These are the so-called Gelfand-Dyson representations. The operator realization in this case reads

$$\begin{aligned} \hat{T}_0 &= \hat{a}^\dagger \hat{a} + \rho, \quad \hat{T}_- = \hat{a}, \\ \hat{T}_+ &= \hat{a}^{\dagger 2}\hat{a} + 2\rho\hat{a}^\dagger . \end{aligned} \tag{5.7}$$

The value of the Casimir operator is equal to

$$\lambda = -\rho(\rho - 1) . \tag{5.8}$$

Note that ρ and $1 - \rho$ determine the same value of the Casimir operator.

Now, let us go over to the Weyl symbols (see (3.4a))

$$T_0 = a^\dagger a + \rho', \quad T_- = a \tag{5.9}$$

$$T_+ = a^{\dagger 2} a + 2\rho' a^\dagger, \quad \rho' = \rho - \frac{1}{2}.$$

The symbols (5.9) satisfy the relation

$$-T_0 * T_0 + \frac{1}{2}(T_+ * T_- + T_- * T_+) = -\left(\rho'^2 - \frac{1}{4}\right). \tag{5.10}$$

In order to construct the infinite-dimensional algebras $L_\lambda(\mathrm{sl}(2))$ it is necessary now to extend the construction (5.7) to the case of polynomials of higher order in a and $a^\dagger$ (the way to carry out the extension is described below).

Highest vectors of finite-dimensional representations of sl(2) have the form

$$T^s_S = \underbrace{T_+ * \ldots * T_+}_{s}, \quad s = 0, 1, 2, \ldots, \tag{5.11}$$

$$[T_+, T^s_S]_* = 0, \quad [T_0, T^s_S] = s\, T^s_S. \tag{5.12}$$

Acting with the lowering operator T_- one obtains other basis elements in the spin-s representation

$$T^S_m = \frac{1}{(s-m)!}\underbrace{[T_-, \ldots [T_-}_{s-m}, T^S_S]_* \ldots]_*$$

$$= \frac{1}{(s-m)!}\frac{\partial^{s-m}}{\partial a^{\dagger s-m}}(T^S_S), \quad m = -s, \ldots, S. \tag{5.13}$$

The elements T^S_m define a basis in the Lie algebra isomorphic to the $L_\lambda(\mathrm{sl}(2))$. Relations (5.9–13) define an embedding of $U_\lambda(\mathrm{sl}(2))$ into the Weyl algebra A_1. Note an important difference between the realizations (5.5) and (5.7). Only second order polynomials in the generating elements enter the realization (5.5), whereas the polynomials up to the third order are involved in the realization (5.7).

The representation (5.9) of sl(2) can be easily extended to a representation of the Virasoro algebra,

$$T^1_n = (a^\dagger)^{n+1} a + \rho' (n+1)(a^\dagger)^n, \; n \in \mathbb{Z}, \tag{5.14}$$

where negative powers of $a^\dagger$ are also presented.

In terms of differential operators this representation has the form

$$T^1_n = z^{n+1}\frac{d}{dz} + \rho(n+1) z^n. \tag{5.15}$$

Now we construct an extension of $L_\lambda(\mathrm{sl}(2))$ which would contain the Virasoro algebra as a subalgebra. To do that, it is necessary to extend the spaces of the finite-dimensional irreducible representations of sl(2) with the bases $\{T^s_m, m = -s, \ldots, s\}$ to infinite-dimensional representations of the Virasoro algebra with the bases $\{T^s_m, m \in \mathbb{Z}\}$ (see the Appendix). The missing basis elements T^s_m for $m > s$ and $m < -s$ are constructed with the help of the raising and lowering operators $T^1_{\pm 1}$ and $T^1_{\pm 2}$

$$T^s_{s+1} \sim [T^1_2, T^s_{s-1}]_*\,,$$
$$T^s_{-s-1} \sim [T^1_{-2}, T^s_{1-s}]_*\,, \tag{5.16}$$

$$T^s_m \sim [\underbrace{T^1_1, \ldots, [T^1_1}_{m-s-1}, T^s_{s+1}]_* \ldots]_*, \; m > s+1\,,$$
$$T^s_m \sim [\underbrace{T^1_{-1}, \ldots, [T^1_{-1}}_{-s-1-m}, T^s_{-s-1}]_* \ldots]_*, \; m < -s-1\,. \tag{5.17}$$

We thus get a one-parameter family of $\mathrm{hsc}_\lambda(1)$ algebras extending the Virasoro algebra to the case of higher spins. Among these there is an extension of hs(2) (which occurs for $\lambda = 3/16$).

A more detailed description of the above construction (i.e., the explicit form of the basis elements and structure constants) will be given elsewhere.

6. In conclusion let us discuss the possible physical applications of conformal higher spin superalgebras.

In the paper[26] devoted to conformal theories in D-dimensions, considered are composite operators with higher spins which arise in Wilson operator product expansions of the type $j_\mu(x_1)j_\nu(x_2)$ and $T_{\mu\nu}(x_1)j_\rho(x_2)$ and $T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)$.

These composite operators are in fact conserved higher tensor currents; they allow one to obtain[26] a complete set of equations to be satisfied by Green functions in certain models, and use these equations to find a conformally invariant solution of the theory. With this symmetry is associated a higher spin conformal algebras which in two dimensions must be of general structure as given in (2.3). It would be important to know this algebra in order to be able to find a solution for the Green functions from group-theoretical considerations.

Fields with conformal higher spins frequently appear in the current two-dimensional conformal theory.[2,3] The symmetry algebras of such theories contain spin generators of higher conformal spins along with the Virasoro generators. When considering these models, one usually limits oneself to first several values of higher spins (spin 3 for example). The corresponding algebras are referred to as the W-algebras.[2] On the rhs of the commutators in the W-algebras there emerge new terms nonlinear in the original generators. These nonlinear operators can be considered, for instance, as new generators; hence they should be included into the algebra on equal footing with the original generators. This new algebra will involve on the rhs of the commutators the powers of the generators which again can be treated as new generators with higher spins, and so forth (see Ref. 3). Proceeding in this way, one ultimately obtains an algebra containing all higher spins of the type of Eq. (2.3). Such algebras might occur as symmetry algebras of conformal models involving all higher spins.

The models with $N = 3$ and $N = 4$ conformal higher spin superalgebras are particularly singled out because, as we have demonstrated, conformal anomaly cancels for these algebras.

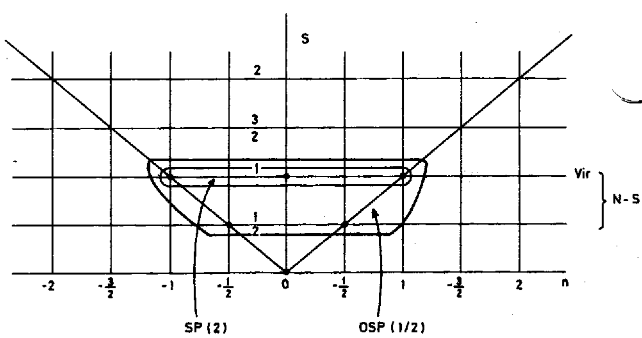
When quantizing the superstring, these arises, as a result of the reducibility of generators, an infinite system of higher spin ghost fields. This might be a consequence of a hidden higher spin symmetry in string theory. In that case the theory should allow a formulation with higher spin fields present from the very beginning which would require the theory to have an infinite-dimensional symmetry with irreducible generators.

Conformal higher spin superalgebras should play in this formulation the role played by the Virasoro algebra in the conventional theories without higher spins.

One may expect that conformal higher spin algebras and their representations play an important role in the second-quantized string theory, i.e., field theory of string.

Appendix

The structure of the $N = 1$ conformal higher spin superalgebras in $D = 1$



S – is the conformal spin of the generators T_n^S ($S = 1/2, 1, \ldots$), n – is the L_0-eigenvalue ($[L_0, T_n^S] = nT_n^S$) ($n \in \mathbb{Z}(\mathbb{Z} + 1/2)$ at $S \in \mathbb{Z}(\mathbb{Z} + 1/2)$), (S, n) – is the generator T_n^S $\{(1, +1), (1, -1), (1, 0), (1/2, 1/2), (1/2, -1/2)\}$ – OSP(1|2) – superalgebra (little conformal superalgebra) $\{(1, n), n \in \mathbb{Z}$ and $(1/2, n), n \in \mathbb{Z} + 1/2\}$ – Neveu-

Schwarz superalgebra $\{(S, n),\ n \in [-S, S],\ S = 1/2, 1 \ldots\}$ – little conformal higher spin superalgebra (it contains only finite-dimensional representations $SO(2,1) \simeq SP(2)$).

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