CUBIC INTERACTION IN CONFORMAL THEORY OF INTEGER HIGHER-SPIN FIELDS IN FOUR DIMENSIONAL SPACE-TIME

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Based on the infinite-dimensional algebra $hsc^{\infty}(4)$ constructed by us, the cubic interaction in the conformal theory of higher-spin fields in four-dimensional space-time is obtained. The theory contains an infinite number of fields of all integer spins $s \geqslant 2$ and extends the conformal gravity.

1. Introduction

In this paper we continue the investigation of conformally invariant theories of massless higher spin fields, which was initiated in refs. [1–5]. In refs. [1,2] we have constructed N-extended conformally invariant higher spin models in dimension $D=2+1$. These theories are described by the Chern–Simons action associated to the superalgebra $shsc(N|13) \simeq shs^{E}(N|4)$ (shsc stands for super higher spin conformal). In refs. [3–5] a series of new infinite-dimensional Lie superalgebras $shsc^{\infty}(4|N)$, $shsc^{(n)}(4|N)$ are constructed. These are an extension of the superconformal algebra in a four-dimensional space, $SU(2,2|N)$. In the above papers explicit expressions were obtained for the curvatures of all higher spins, which generalize the expressions known for the usual conformal supergravity (see ref. [6]).

The problem we are to solve next lies in constructing an action and constraints for the conformal theory of higher spins in $D=3+1$.

There are several reasons which make the construction of such a theory desirable. The theory of massless higher spin fields could play an essential role in revealing the underlying gauge symmetry of strings at very high energies.

It is known [7–10], however, that no gauge-invariant interaction of massless higher spin fields with Einstein gravity exists in $D=4$.

It was shown in refs. [11–20] that the difficulties encountered in refs. [7–10] could be avoided in the models with a non-zero cosmological constant. The interaction turns out to be non-analytic in Λ and forbids the flat limit $\Lambda \to 0$.

Another approach to the problem consists in constructing the interaction of higher spin fields with Weyl gravity. Similarly to what was the case in supergravity, the conformal higher spin theory (CHST) may play an important role in solving fundamental problems of the theory of higher spins in ADS_4.

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In this paper we solve the problem of localization of the infinite-dimensional algebra $hsc^{\infty}(4)$ in the cubic approximation.

The main facts pertaining to this algebra and the corresponding gauge fields and curvatures are presented in section 2 (for the details, see refs. [3,4]).

In section 3 the linearized $hsc^{\infty}(4)$ curvatures are used to construct a linearized CHST in terms of two-component multispinors.

In section 4 we construct the action of the CHST and prove its gauge-invariance in the cubic order. The action consistently describes the cubic interaction of conformal fields of arbitrary integer spins $s \geqslant 2$.

In the Appendix the free conformal theory of spin s is described in terms of symmetric tensors.

2. Gauge fields and curvatures of the conformal higher spin algebra $hsc^{\infty}(4)$

The algebra $hsc^{\infty}(4)$ gives rise to gauge fields in four-dimensional space–time, devoted as ${}^{(n)}\omega^{(s-1,c)}_{\mu,\alpha(2l),\dot\beta(2j)}$ or simply as ${}^{n}\omega(s,c,l,j)$. In these notations [#1] the subscript s determines a $[d(s)=s^2(s+1)^2(2s+1)/12]$-dimensional irreducible representation (irrep) of so(4, 2); c is the conformal weight of the generator T^c that corresponds to the field ω^c ($[\mathcal{D}, T^c]=cT^c$); (l, j) is the Lorentz signature; $n+1$ labels isomorphic representations of SO(4, 2), and

$$s=2,3,\ldots,\quad n=0,1,\ldots;\quad c=-s+1,-s+2,\ldots,s-1,\quad l,j=\tfrac{1}{2}|c|,\tfrac{1}{2}|c|+1,\ldots,\quad l+j\leqslant s-1\,. \tag{2.1a,b}$$

The SO(4, 2) irrep spaces of dimensions $d(s)$, $s=2, 3, \ldots$, are contained in $hsc^{\infty}(4)$ with infinite multiplicities. The validity of (2.1b) can be easily checked using the decomposition $so(4,2)\to so(3,1)\oplus so(1,1)$.

The fields of the conformal gravity (CG) are, in our notations,

$$(e_{\mu\alpha\dot\beta},\, w_{\mu\alpha(2)},\, \bar w_{\mu\dot\beta(2)},\, b_\mu,\, f_{\mu\alpha\dot\beta}) \sim ({}^{(0)}\omega^{(1,-1)}_{\mu,\alpha,\dot\beta},\, {}^{(0)}\omega^{(1,0)}_{\mu,\alpha(2)},\, {}^{(0)}\omega^{(1,0)}_{\mu,\dot\beta(2)},\, {}^{(0)}\omega^{(1,0)}_{\mu},\, {}^{(0)}\omega^{(1,1)}_{\mu,\alpha,\dot\beta})\,. \tag{2.2}$$

The corresponding generators furnish the $d(2)=15$-dimensional adjoint representation of SO(4, 2). The $hsc^{\infty}(4)$ curvatures have the form (details of the derivation can be found in ref. [3]):

$$\begin{aligned}{}^{(n)}R^{(s-1,c)}_{\mu\nu,\alpha(2l),\dot\beta(2j)} &= \partial_{[\mu}\,{}^{(n)}\omega^{(s-1,c)}_{\nu],\alpha(2l),\dot\beta(2j)} + \sum \delta(m-l'-l''+l)\delta(r-l''+l'-l)\delta(t-l'+l''-l)\delta(p-j'-j''+j)\\ &\times \delta(q+j'-j''-j)\delta(k-j'+j''-j)\delta(|s+s'+s''+n+n'+n''|2)\\ &\times i^{s'+s''-s-2}\begin{bmatrix} n' & s'-1 & c' & l' & j' \\ n'' & s''-1 & c'' & l'' & j'' \\ n & s-1 & c & l & j \end{bmatrix} {}^{(n')}\omega^{(s'-1,c')}_{\mu,\alpha(t)\gamma(m),\dot\beta(k)\dot\delta(p)}\, {}^{(n'')}\omega^{(s''-1,c'')\gamma(m)}_{\nu,\alpha(r)}{}_{\dot\beta(q)}{}^{\dot\delta(p)}\,.\end{aligned} \tag{2.3}$$

The summation in (2.3) goes over all allowed values of the parameters (2.1a), (2.1b) and $\delta(|n|2)=1(0)$ at $n=2k(2k+1)$, $\delta(n)=1(0)$ at $n=0$ ($n\neq 0$).

The structure coefficients take the form

[#1] Notations and conventions. We use the conventions introduced in refs. [12–14]. The Greek indices $\mu, \nu, \rho, \sigma=0, 1, 2, 3$ are contracted by the flat metric $\eta_{\mu\nu}$ (sign $\eta=(+,-,-,-)$). The spinorial Greek indices $\alpha, \beta, \gamma, \ldots, \dot\alpha, \dot\beta, \dot\gamma, \ldots$ take the values 1, 2. The transitions from upper spinorial indices to lower ones and vice versa are carried out with the aid of the symplectic forms $\epsilon_{\alpha\beta}$, $\epsilon_{\dot\alpha\dot\beta}$ ($\epsilon_{12}=\epsilon^{12}=\epsilon_{\dot 1\dot 2}=\epsilon^{\dot 1\dot 2}=1$), $\epsilon_{\alpha\beta}=-\epsilon_{\beta\alpha}$, $\epsilon_{\dot\alpha\dot\beta}=-\epsilon_{\dot\beta\dot\alpha}$, $A^\alpha=\epsilon^{\alpha\beta}A_\beta$, $A_\beta=\epsilon_{\alpha\beta}A^\alpha$. The number of indices is indicated in parentheses (except for the case of a single index). Upper (lower) indices denoted by the same letter are assumed to be fully symmetrized. When this symmetrization is carried out, the maximal possible number of upper and lower indices denoted by the same letter should be contracted.

$$\begin{pmatrix} n_1 & s_1 & c_1 & l_1 & j_1 \\ n_2 & s_2 & c_2 & l_2 & j_2 \\ n_3 & s_3 & c_3 & l_3 & j_3 \end{pmatrix}$$

$$= \sum_{s_i'+s_i''=s_i+n_i} \begin{pmatrix} s_1' & s_2' & s_3' \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & \frac{1}{2}c_3 \\ l_1 & l_2 & l_3 \end{pmatrix} \begin{pmatrix} s_1'' & s_2'' & s_3'' \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & \frac{1}{2}c_3 \\ j_1 & j_2 & j_3 \end{pmatrix} \prod_{i=1}^{3} C^{(s_i'+n_i-l_i+j_i+1)/2,\ (s_i'+n_i+l_i-j_i+1)/2,\ s_i+1}_{(l_i+j_i-s_i'+s_i''+1)/2,\ (l_i+j_i+s_i'-s_i''+1)/2,\ l_i+j_i+1} \tag{2.4}$$

$(s_i', s_i''=0, \frac{1}{2}, 1, ...; s_i' \geqslant l_i; s_i'' \geqslant j_i\ (i=1, 2, 3))$

where the c are Clebsch–Gordan coefficients, and

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_2 & c_3 \\ l_1 & l_2 & l_3 \end{pmatrix} = \delta(c_1+c_2-c_3)\varepsilon(s_1, s_2, s_3)\varepsilon(l_1, l_2, l_3) \frac{\sqrt{(2l_1+1)!(2l_2+1)!(2l_3+1)!}}{(l_1+l_2+l_3+1)!\Delta(l_1, l_2, l_3)}$$

$$\times \sum_{k_1,k_2,k_3} (-1)^{(s_1+s_2-s_3+k_3-k_1+k_2)/2}$$

$$\times \frac{d^{l_1}_{c_1,k_1}(\frac{1}{2}\pi)\, d^{l_2}_{c_2,k_2}(\frac{1}{2}\pi)\, d^{l_3}_{k_3,c_3}(-\frac{1}{2}\pi)}{\Delta(\frac{1}{2}(s_1+k_1), \frac{1}{2}(s_2+k_2), \frac{1}{2}(s_3+k_3))\,\Delta(\frac{1}{2}(s_1-k_1), \frac{1}{2}(s_2-k_2), \frac{1}{2}(s_3-k_3))} \begin{Bmatrix} \frac{1}{2}(s_1+k_1), \frac{1}{2}(s_1-k_1), l_1 \\ \frac{1}{2}(s_2+k_2), \frac{1}{2}(s_2-k_2), l_2 \\ \frac{1}{2}(s_3+k_3), \frac{1}{2}(s_3-k_3), l_3 \end{Bmatrix},$$

$$\Delta(a, b, c) = \sqrt{\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}}.$$

$$\varepsilon(a, b, c) = 1\,(0) \text{ at } c \in \{|a-b|, ..., a+b\}\ (c \notin \{|a-b|, ..., a+b\}), \tag{2.5}$$

$k_i = -l_i, ..., l_i$, and $l_i = |c_i|, |c_i|+1, ...; l_i \leqslant s_i; c_i = -s_i, -s_i+1, ..., s_i$ $(i=1, 2, 3)$ are the coefficients involved in the definition of curvatures of the conformal superalgebra in $D=2+1$, shsc$(1|3)$.

The structure coefficients of the conformal algebra hsc$^\infty(4)$ can be expressed through the quantities known from the theory of angular momentum i.e., the 9j-symbols, Clebsch–Gordan coefficients together with particular values of the Wigner d-functions (our conventions and definitions coincide with those of ref. [21]). The resulting explicit form of the structure coefficients is rather complicated. However, they satisfy certain simple symmetry properties. It is not hard to show that under the interchange of any two rows, the symmetrized table coefficients

$$\text{Sym}\begin{bmatrix} n_1 & s_1 & c_1 & l_1 & j_1 \\ n_2 & s_2 & c_2 & l_2 & j_2 \\ n_3 & s_3 & c_3 & l_3 & j_3 \end{bmatrix} = (-1)^{l_3+n_3+l_3+j_3} \begin{bmatrix} n_1 & s_1 & c_1 & l_1 & j_1 \\ n_2 & s_2 & c_2 & l_2 & j_2 \\ n_3 & s_3 & -c_3 & l_3 & j_3 \end{bmatrix} \tag{2.6}$$

get multiplied with the factor

$$(-1)^{\sum_{i=1}^{3}(n_i+s_i+l_i+j_3)}. \tag{2.7}$$

The algebra hsc$^\infty(4)$ admits the invariant bilinear form

$$(A,B) = \sum (-1)^{n+l+j}\, {}^{(n)}A^{(s,c)}_{\alpha(2l),\dot\beta(2j)}\, {}^{(n)}B^{(s,-c)\alpha(2l)\dot\beta(2j)}. \tag{2.8}$$

The invariance of this bilinear form can be checked using the symmetry properties (2.6), (2.7).

The corresponding MacDowell–Mansouri functions [22], $\int_{M^4} R_A \wedge R_B g^{AB}$, is a topological invariant of the manifold M^4. It would be interesting to find the topological meaning of the invariants of the above type which are associated to higher spin superalgebras and generalize the corresponding supergravity invariants.

Complex conjugation acts on the fields and curvatures according to the formulae

$$\overline{({}^{(n)}\omega^{(s-1,c)}_{\mu,\alpha(2l),\dot\beta(2j)})} = {}^{(n)}\omega^{(s-1,c)}_{\mu,\beta(2j),\dot\alpha(2l)}\,, \quad \overline{({}^{(n)}R^{(s-1,c)}_{\mu\nu,\alpha(2l),\dot\beta(2j)})} = {}^{(n)}R^{(s-1,c)}_{\mu\nu,\beta(2j),\dot\alpha(2l)}\,. \tag{2.9a,b}$$

To study the CHST we are going to use the following expansion procedure. All fields except for the gravitational vierbein are assumed to be equal to zero in the zeroth order, and

$$ {}^{(0)}\dot\omega^{(1,-1)}_{\mu,\alpha,\dot\beta} = \sigma_{\mu\alpha\dot\beta}\,, \tag{2.10}$$

where $\sigma^{\alpha\dot\beta}_{\mu} = (I, \sigma_1, \sigma_2, \sigma_3)^{\alpha\beta}$ (the σ_k are the Pauli matrices) satisfy the following relations:

$$\sigma_{\mu\alpha\dot\beta}\sigma^{\nu\alpha\dot\beta} = 2\delta^{\nu}_{\mu}\,, \quad \sigma_{\mu\alpha\dot\beta}\sigma^{\mu\gamma\dot\delta} = 2\delta^{\gamma}_{\alpha}\delta^{\dot\delta}_{\dot\beta}\,. \tag{2.11}$$

The linearized hsc$^\infty$(4) curvatures are of the form

$$\begin{aligned} {}^{(k)}R^{l(s-1,c)}_{\mu\nu,\alpha(n),\dot\beta(m)} &= \partial_\mu\, {}^{(k)}\omega^{(s-1,c)}_{\nu,\alpha(n),\dot\beta(m)} + a(s,c,n,m)\sigma_{\mu\gamma\dot\beta}\, {}^{(k)}\omega^{(s-1,c+1)\gamma}_{\nu,\alpha(n)}{}_{\dot\beta(m-1)} + a(s,c,m,n)\sigma_{\mu\alpha\dot\delta}\, {}^{(k)}\omega^{(s-1,c+1)}_{\nu,\alpha(n-1),\dot\beta(m)}{}^{\dot\delta} \\ &\quad - b(s,c,n,m)\sigma_{\mu\gamma\dot\delta}\, {}^{(k)}\omega^{(s-1,c+1)\gamma}_{\nu,\alpha(n)}{}_{\dot\beta(m)}{}^{\dot\delta} - b(s,-c-1,n-1,m-1)\sigma_{\mu\alpha\dot\beta}\, {}^{(k)}\omega^{(s-1,c+1)}_{\nu,\alpha(n-1),\dot\beta(m-1)} (\mu\leftrightarrow\nu)\,, \end{aligned} \tag{2.12}$$

$$a(s,c,n,m) = \left(\frac{(n+c+2)(m-c)(2s+m-n)(2s+n-m+2)}{16(n+2)(m+1)}\right)^{1/2}, \tag{2.13}$$

$$b(s,c,n,m) = \left(\frac{(n+c+2)(m+c+2)(2s-n-m-2)(2s+n+m+4)}{16(n+2)(m+2)}\right)^{1/2}. \tag{2.14}$$

It can be shown by a direct calculation that the above curvatures are invariant under the corresponding gauge transformations and satisfy the linearized Bianchi identities. Therefore, the curvatures admit a presentation of the form $R^l = \mathscr{D}^l\omega$ with $\mathscr{D}^l\mathscr{D}^l = 0$.

3. Free conformal theory of higher spin fields

When considering the linearized theory, we shall limit ourselves to the fields ${}^0\omega$, because the other fields ${}^n\omega$ has the same free lagrangian.

The linearized action which describes spin s and extends the action of conformal gravity will be chosen in the form [#2]

$$A^l(s) = \frac{(-1)^s i\beta}{4} \int d^4x\, \varepsilon^{\mu\nu\rho\sigma} (R^{l(s-1,0)}_{\mu\nu,\alpha(2s-2)} R^{l(s-1,0)}_{\rho\sigma}{}^{\alpha(2s-2)} - \text{c.c.}) \tag{3.1}$$

(β is an arbitrary overall normalization). The curvatures $R^l(s,0,s-1,0)$ and $R^l(s,0,0,s-1)$ generalize the linearized curvatures $R^l(2,0,1,0)$ and $R^l(2,0,0,1)$ $(=R^l(\mathrm{M}))$ of CG. They depend on the fields $\omega(s,0,s-1,0)$, $\omega(s,0,0,s-1)$ (which are the analogues of the Lorentz connection) and also on $\omega(s,1,s-\frac{3}{2},\frac{1}{2})$ and $\omega(s,1,\frac{1}{2},s-\frac{3}{2})$ (the analogues of the field $f_{\mu\alpha\dot\beta}$ in CG).

The equations of motion

$$\frac{\delta A^l(s)}{\delta\omega^{(s-1,1)}_{\mu,\alpha(2s-3),\dot\beta}} \sim \epsilon^{\mu\nu\rho\sigma}\sigma_{\nu\alpha}{}^{\dot\beta} R^{l(s-1,0)}_{\rho\sigma}{}^{\alpha(2s-2)} = 0\,, \tag{3.2a}$$

$$\frac{\delta A^l(s)}{\delta\omega^{(s-1,1)}_{\mu,\alpha,\dot\beta(2s-3)}} \sim \epsilon^{\mu\nu\rho\sigma}\sigma_{\nu}{}^{\alpha}{}_{\dot\beta} R^{l(s-1,0)}_{\rho\sigma}{}^{\dot\beta(2s-2)} = 0 \tag{3.2b}$$

are similar to the CG constraint $e^{\mu}_{a} R^{ab}_{\mu\nu}(M) = 0$.

[#2] One could have included into the free action the terms with (see (4.1)) $R(s,c,l,j)$ in which $(s,c,l,j) \neq (s,0,s-1,0)$, $(s,0,0,s-1)$; however, these terms vanish on the conventional constraints.

The solution to the above constraints can be represented in the form

$$R^{l(s-1,0)}_{\mu\nu,\alpha(2s-2)}=\tfrac{1}{4}\sigma_{\mu\nu}{}^{\alpha(2)}C_{\alpha(2s)}\,,\quad R^{l(s-1,0)}_{\mu\nu,\dot\beta(2s-2)}=\tfrac{1}{4}\bar\sigma_{\mu\nu}{}^{\dot\beta(2)}\bar C_{\dot\beta(2s)}\,, \tag{3.3a,b}$$

where C and $\bar C$ are multispinors which represent the spin-s Weyl tensor and have the form

$$C_{\alpha(2s)}=-\partial_\alpha{}^{\dot\beta}\omega^{(s-1,0)}_{\alpha\dot\beta,\alpha(2s-2)}\,,\quad \bar C_{\dot\beta(2s)}=-\partial^\alpha{}_{\dot\beta}\omega^{(s-1,0)}_{\alpha\dot\beta,\dot\beta(2s-2)}\,. \tag{3.4a,b}$$

In the case of spin 2, these coincide with the linearized gravitational Weyl tensor. The notation used in (3.3), (3.4) and further on, is

$$\partial_{\alpha\dot\beta}=\sigma^\mu{}_{\alpha\dot\beta}\partial_\mu\,,\quad \omega_{\alpha\dot\beta,\dots}=\sigma^\mu{}_{\alpha\dot\beta}\omega_{\mu,\dots}\,. \tag{3.5}$$

We have also introduced the quantities

$$\sigma_{\mu\nu}{}^{\alpha(2)}=\sigma_\mu{}^\alpha{}_{\dot\beta}\sigma_\nu{}^{\alpha\dot\beta}\,,\quad \bar\sigma_{\mu\nu}{}^{\dot\beta(2)}=\sigma_{\mu\alpha}{}^{\dot\beta}\sigma_\nu{}^{\alpha\dot\beta}\,, \tag{3.6}$$

which satisfy

$$\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}{}^{\alpha(2)}=-2\mathrm{i}\sigma^{\mu\nu\alpha(2)}\quad \epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma}{}^{\dot\beta(2)}=2\mathrm{i}\bar\sigma^{\mu\nu\dot\beta(2)}\,, \tag{3.7a}$$

$$\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu}{}^{\alpha(2)}\sigma_{\rho\sigma\gamma(2)}=-16\mathrm{i}\delta^\alpha_\gamma\delta^\alpha_\gamma\quad \epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\mu\nu}{}^{\dot\beta(2)}\bar\sigma_{\rho\sigma\dot\delta(2)}=16\mathrm{i}\delta^{\dot\beta}_{\dot\delta}\delta^{\dot\beta}_{\dot\delta}\,. \tag{3.7b}$$

In order to express all fields and curvatures through physical higher spin fields only, one needs to add constraints to the action (3.1). The conventional constraint of conformal gravity (which is the case of $s=2$) is $R(\mathrm{P})=0$. In the general case of $s\geqslant 2$ the constraints can be chosen in the form

$$\epsilon^{\mu\nu\rho\sigma}(\sigma_{\nu\gamma\dot\beta}R^{l,(s-1,s-2),\gamma}_{\rho\sigma,\alpha(s-1),\dot\beta(s-2)}-\sigma_{\nu\alpha\dot\delta}R^{l,(s-1,s-2),\dot\delta}_{\rho\sigma,\alpha(s-2),\dot\beta(s-1)})=0\,,\quad R^{l,(s-1,c)}_{\mu\nu,\alpha(n),\dot\beta(m)}=0 \tag{3.8}$$

for all c, n, m except for $(n,m)=(c,2s-c-2)$ and $(m,n)=(2s-c-2,c)$ and $c=0,1,\dots,s-1$.

The latter curvatures $R^l(s,c,\tfrac{1}{2}c,s-\tfrac{1}{2}c-1)$ and $R^l(s,c,s-\tfrac{1}{2}c-1,\tfrac{1}{2}c)$ can be determined with the help of the linearized Bianchi identities. These, in turn, can be rewritten using the constraints (3.8) as

$$\epsilon^{\mu\nu\rho\sigma}(\partial_\nu R^{l(s-1,c)}_{\rho\sigma,\alpha(2s-c-2)\dot\beta(c)}+\sqrt{(c+1)(s-c-1)}\,\sigma_{\nu\alpha\dot\delta}R^{l(s-1,c+1)}_{\rho\sigma,\alpha(2s-c-3)\dot\beta(c)}{}^{\dot\delta})=0\,, \tag{3.9a}$$

$$\epsilon^{\mu\nu\rho\sigma}(\partial_\nu R^{l(s-1,c)}_{\rho\sigma,\alpha(c)\dot\beta(2s-c-2)}+\sqrt{(c+1)(s-c-1)}\,\sigma_{\nu\gamma\dot\beta}R^{l(s-1,c+1)\gamma}_{\rho\sigma,\alpha(c)}{}_{\dot\beta(2s-c-s)})=0\,,$$

$$c=0,1,\dots,s-1\,. \tag{3.9b}$$

As a result, all curvatures are expressed only through C and $\bar C$ and their derivatives, as

$$R^{l(s-1,c)}_{\mu\nu,\alpha(n),\dot\beta(m)}$$

$$=a(s,c)\theta(c)\Big(\delta(n-2s+c+2)\delta(m-c)\sigma_{\mu\nu}{}^{\alpha(2)}\overbrace{\partial^\alpha{}_{\dot\beta}\dots\partial^\alpha{}_{\dot\beta}}^{c}C_{\alpha(2s)}+\delta(n-c)\delta(m-2s+c+2)\bar\sigma_{\mu\nu}{}^{\dot\beta(2)}\overbrace{\delta_\alpha{}^{\dot\beta}\dots\partial_\alpha{}^{\dot\beta}}^{c}\bar C_{\dot\beta(2s)}\Big)\,,$$

$$a(s,c)=2^{-c-2}\left(\frac{(s-c-1)!}{c!(s-1)!}\right)^{1/2}(-1)^c\,,\quad \theta(c)=1(0)\text{ at }c\geqslant 0\,(c<0)\,. \tag{3.10}$$

The constraints (3.10) generalize the CG linearized relations $R(\mathrm{P})=R(\mathrm{D})=0$, $R(\mathrm{M})=C$ and $R(\mathrm{K})=\mathrm{D}C$ (see ref. [23]).

The relations

$$\omega^{(s-1,-c+1)}_{\alpha\dot\beta,\alpha(2s-c-1),\dot\beta(c-1)}\sim\partial_\alpha{}^{\dot\delta}\omega^{(s-1,-c)}_{\alpha\dot\delta,\alpha(2s-c-2),\dot\beta(c)}\,,\quad \omega^{(s-1,-c+1)}_{\alpha\dot\beta,\alpha(c-1),\dot\beta(2s-c-1)}\sim\partial^\gamma{}_{\dot\beta}\omega^{(s-1,-c)}_{\gamma\dot\beta,\alpha(c),\dot\beta(2s-c-2)}\,, \tag{3.11a,b}$$

$$c=1,2,\dots,s-1\,,$$

which are the consequences of the constraints

$$R^l(s,-c,\tfrac{1}{2}c,s-\tfrac{1}{2}c-1)=R^l(s,-c,s-\tfrac{1}{2}c-1,\tfrac{1}{2}c)=0$$

allow us to express the auxiliary fields $\omega(s,0,s-1,0)$ and $\omega(s,0,0,s-1)$ only through the physical spin-s field $\omega(s,1-s,\frac{1}{2}(s-1),\frac{1}{2}(s-1))$. The Weyl tensor is expressed through the physical fields as

$$C_{\alpha(2s)} \sim \underbrace{\partial_{\alpha}{}^{\dot\beta}\ldots\partial_{\alpha}{}^{\dot\beta}}_{s}\,\omega^{(s-1,1-s)}_{\alpha\beta,\alpha(s-1),\dot\beta(s-1)}, \quad \bar C_{\dot\beta(2s)} \sim \underbrace{\partial^{\alpha}{}_{\dot\beta}\ldots\partial^{\alpha}{}_{\dot\beta}}_{s}\,\omega^{(s-1,1-s)}_{\alpha\beta,\alpha(s-1),\dot\beta(s-1)}. \tag{3.12a,b}$$

The action (3.1) can be written in terms of C and $\bar C$ as

$$A^{l}(s)=\frac{(-1)^{s}}{4}\beta\int d^{4}x\,(C_{\alpha(2s)}C^{\alpha(2s)}+\bar C_{\dot\beta(2s)}\bar C^{\dot\beta(2s)})\,. \tag{3.13}$$

With eq. (3.12) taken into account, this action is equivalent to the known higher-derivative action for conformal higher spin fields which was introduced in ref. [6], see also Appendix A. It differs from eq. (B.4) only due to the multi-spinor notations chosen to write it down.

We have therefore demonstrated that the free CHST can be geometrically formulated in terms of the linearized $hsc^{\infty}(4)$ curvatures, similarly to the free theory of massless higher spin fields in ADS_4 [12,19].

4. The cubic interaction in the conformal theory of higher spin fields

The most general action depending only on curvature, dimensionless and party conserving is given by

$$A=-\frac{i}{4}\sum(-1)^{l+j}\beta(n,s,c,l,j)\int d^{4}x\,\epsilon^{\mu\nu\rho\sigma}\,{}^{(n)}R^{(s-1,c)}_{\mu\nu,\alpha(2l),\dot\beta(2j)}\,{}^{(n)}R^{(s-1,-c)}_{\rho\sigma}{}^{\alpha(2l)}{}^{\dot\beta(2j)}\,,$$

$$n\geq 0,\quad s\geq 2,\quad |c|\leq s-2,\quad l,j=\tfrac{1}{2}|c|,\ldots;\, l+j\leq s-1\,, \tag{4.1}$$

where $\beta(n,s,c,l,j)=-\beta(n,s,c,j,l)$ are arbitrary coefficients. The quadratic part of the action is the sum of the free actions (3.13) (when the constraints (3.10) are satisfied). In the cubic order only terms with $(s,c,l,j)=(s,c,s-\frac{1}{2}|c|-1,\frac{1}{2}|c|)$, $(s,c,\frac{1}{2}|c|,s-\frac{1}{2}|c|-1)$ are sufficient, and all other terms are equal to zero by the constraints (3.10). Therefore, we consider in the cubic approximation only the terms with $\beta(n,s,c,s-\frac{1}{2}|c|-1,\frac{1}{2}|c|)=-\beta(n,s,c,\frac{1}{2}|c|,s-\frac{1}{2}|c|-1)=\beta(n,s,c)$.

The gauge variation of the action [#3] (4.1) is given in the cubic order by the general structure $R^{l}R^{l}\mathscr{E}$, where $\mathscr{E}$ are gauge parameters. We shall analyze it in the 1.5 order formalism, assuming that the linearized curvatures satisfy the constraints (3.10).

The gauge transformations which leave the action (4.1) invariant in the cubic order, are of the general form

$$\delta\omega=\delta_{g}\omega+\Delta\omega\,,\quad \delta_{g}\omega=\mathscr{D}\mathscr{E}\,, \tag{4.2}$$

where $\Delta\omega$ are some deformations of at least second order (the fields ω and gauge parameters $\mathscr{E}$ are of the first order). When checking the cubic order invariance of the action, the only essential deformations are $\Delta\omega(s,2-s,\frac{1}{2}s,\frac{1}{2}s-1)$ and $\Delta\omega(s,2-s,\frac{1}{2}s-1,\frac{1}{2}s)$. As concerns the deformations $\Delta\omega$ of all other fields, they either enter in δA multiplied with the constraints (3.10), or $(\delta A/\delta\omega)\Delta\omega$ are of at least fourth order (for some of the fields, $\delta A/\delta\omega$ is of at least second order). The auxiliary fields ${}^{(n)}\omega(s,2-s,\frac{1}{2}s,\frac{1}{2}s-1)$ and ${}^{(n)}\omega(s,2-s,\frac{1}{2}s-1,\frac{1}{2}s)$ can be expressed at the linearized level through the physical fields with the help of the constraints

$${}^{(n)}R^{l(s-1,1-s)}_{\mu\nu,\alpha(s-1),\dot\beta(s-1)}=0\,, \tag{4.3}$$

which generalize the constraint $R(p)=0\,(s=2)$. Let us assume that these constraints are also valid in second order. Then the corresponding invariance condition with respect to the transformations (4.2) allows one to find the deformations

[#3] Only in the case when $\beta(n,s,c)=\beta(n,s)$ which are independent from C.

$$\Delta^{(n)}\omega^{(s-1,2-s)}_{\alpha\dot\beta,\alpha(s),\dot\beta(s-2)} = -\sum \mathrm{i}^{s_1+s_2-s}\frac{2a(s_1,c)}{\sqrt{s-1}}$$

$$\times\ \delta(|n_1+n_2+n+s_1+s_2+s|2)\,\delta(q-s_1+\tfrac{1}{2}(s+1+c)-l)\,\delta(p-j+\tfrac{1}{2}(s-c-1))$$

$$\times\begin{bmatrix} n_1 & s_1-1 & c & s_1-\tfrac{1}{2}c-1 & \tfrac{1}{2}c \\ n_2 & s_2-1 & 1-s-c & l & j \\ n & s-1 & 1-s & \tfrac{1}{2}(s-1) & \tfrac{1}{2}(s-1)\end{bmatrix} {}^{(n_2)}\mathscr{E}^{(s_2-1,1-c-s)\gamma(q)}_{\alpha(2l-q)}{}_{,\dot\beta(2j-p)}{}^{\dot\delta(p)}$$

$$\times\underbrace{\partial^{\lambda}{}_{\dot\beta}\ldots\partial^{\lambda}{}_{\dot\beta}}_{c-p}\,\underbrace{\partial^{\lambda}{}_{\dot\delta}\ldots\partial^{\lambda}{}_{\dot\delta}}_{p}\,{}^{(n_1)}C_{\alpha(2s_1-c-q)\gamma(q)\lambda(c)}\,,\quad c=0,\ldots,s_1-1\,, \tag{4.4a}$$

$$\Delta^{(n)}\omega^{(s-1,2-s)}_{\alpha\dot\beta,\alpha(s-2),\dot\beta(s)} = \overline{(\Delta^{(n)}\omega_{\beta\dot\alpha,\beta(s),\dot\alpha(s-2)})}\,. \tag{4.4b}$$

These are similar to the deformation $\Delta\omega^{ab}_{\mu}\sim\mathscr{E}^{\nu}R^{ab}_{\mu\nu}(\mathrm{M})$ which is present in the transformation law of a Lorentz connection.

We now proceed directly to the study of the variation $\delta A=\delta_g A+\Delta A$. The variation $\delta_g A$ consists of the terms of the type $\partial^c C_s\partial^{c_1}\bar{C}_{s_1}$ and $\partial^c\bar{C}_s\partial^{c_1}\bar{C}_{s_1}$ where $c=0,\ldots,s-2$, $c_1=0,\ldots,s_1-1$ and C_s and $\bar{C}_s$ denote the spin-s Weyl tensor. (Due to the simple identity, $\mathscr{E}^{\mu\nu\rho\sigma}\sigma_{\mu\nu\alpha(2)}\bar\sigma_{\rho\sigma\dot\beta(2)}\equiv 0$, expressions such as $\partial^c C_s\partial^{c_1}\bar{C}_{s_1}$ do not contribute to $\delta_g A$). The terms with $c=s-1$ are not present in $\delta_g A$ because the terms involving $R(s,s-1,\tfrac{1}{2}(s-1),\tfrac{1}{2}(s-1))$ do not enter in A. Now divide $\delta_g A$ into two parts. The first one, $(\delta_g A)_1$ comprises the terms with $c=0,\ldots,s-2$ and $c_1=0,\ldots,s_1-2$ and it can be brought to the form

$$(\delta_g A)_1=\Bigg\{-4\sum \mathrm{i}^{s_1+s_2+s}\delta(|n_1+n_2+n+s_1+s_2+s|2)\,[\beta(n,s)-(-1)^{s_1+s_2+s+n_2}\beta(n_1,s_1)]$$

$$\times\delta(q-s_1+s-\tfrac{1}{2}(c-c_1)-l)\,\delta(p+\tfrac{1}{2}(c-c_1))\begin{bmatrix} n_1 & s_1-1 & c_1 & s_1-\tfrac{1}{2}c_1-1 & \tfrac{1}{2}c_1 \\ n_2 & s_2-1 & -c-c_1 & l & j \\ n & s-1 & -c & s-\tfrac{1}{2}c-1 & \tfrac{1}{2}c\end{bmatrix} a(s,c)\,a(s_1,c_1)$$

$$\times\int \mathrm{d}^4x\,\underbrace{\partial^{\lambda}{}_{\dot\beta}\ldots\partial^{\lambda}{}_{\dot\beta}}_{c_1-p}\,\underbrace{\partial^{\lambda}{}_{\dot\delta}\ldots\partial^{\lambda}{}_{\dot\delta}}_{p}\,{}^{(n_1)}C_{\alpha(2s_1-c_1-q)\gamma(q)\lambda(c_1)}\,\underbrace{\partial^{\xi\dot\beta}\ldots\partial^{\xi\dot\beta}}_{c}\,{}^{(n)}C^{\alpha(2s-c)}{}_{\xi(c)}\,{}^{(n_2)}\mathscr{E}^{(s_2-1,-c-c_1)\gamma(q)}_{\alpha(2l-q)}{}_{\dot\beta(2j-p)}{}^{\dot\delta(p)}\Bigg\}+\text{c.c.}$$

$$(c=0,\ldots,s-2;\ c_1=0,\ldots,s_1-2)\,, \tag{4.5}$$

which is symmetric in C_s and C_{s_1}. In the process of derivation of (4.5) we have used the symmetry property (3.6), (3.7) of the $\mathrm{hsc}^{\infty}(4)$ structure coefficients. One easily notices that $(\delta_g A)_1=0$ if and only if the equation

$$[\beta(n,s)-(-1)^{s_1+s_2+s+n_2}\beta(n_1,s_1)]\,\delta(|s_1+s_2+s_3+n_1+n_2+n_3|2)=0 \tag{4.6}$$

is satisfied for all $n_1, n_2, n\geqslant 0$ and $s_1, s_2, s\geqslant 2$.

The general solution of (4.6) can be written as

$$\beta(n,s)=(-1)^n\beta\,, \tag{4.7}$$

where β is an arbitrary constant. The whole freedom in the cubic action has thus been reduced to a normalization constant β. The correct normalization of the conformal gravity action follows by setting $\beta=1/2\alpha^2$, where α is the dimensionless coupling constant of conformal gravity.

The second part of the variation $(\delta_g A)_2$ consists of $\partial^c C_s\partial^{s_1-1}C_{s_1}$- and $\partial^c\bar{C}_s\partial^{s_1-1}\bar{C}_{s_1}$-type terms. It is an easy matter to verify that these terms cancel against similar terms in the deformation [#3]

$$\Delta\mathcal{A} = -\frac{1}{\alpha^2}\sum_{n,s}\frac{(-1)^n}{2^s(s-2)!\sqrt{s-1}}\int d^4x\,(\underbrace{\partial^{\alpha}{}_{\dot\beta}\ldots\partial^{\alpha}{}_{\dot\beta}}_{s-1}\,{}^{(n)}C_{\alpha(2s)}\,\Delta\,{}^{(n)}\omega^{(s-1,2-s)\alpha\dot\beta}{}_{\cdot}{}^{\alpha(s)}{}_{\cdot}{}^{\dot\beta(s-2)}+\text{c.c.})\,, \tag{4.8}$$

where $\Delta(\omega)$ are read off from eq. (4.4). The terms of the forms $\partial^{s-1}C_s\partial^{s_1-1}C_{s_1}$ and $\partial^{s-1}\bar{C}_s\partial^{s_1-1}\bar{C}_{s_1}$ are also present in the $\Delta\mathcal{A}$ deformation. These cancel out due to the symmetry property (3.6), (3.7), similarly to the case of $(\delta_g\mathcal{A})_1$.

Therefore, we have proven that the action (4.1), (4.8) is invariant in cubic order under the deformed gauge transformations, $\delta\mathcal{A}=\delta_g\mathcal{A}+\Delta\mathcal{A}=0$. Note that the invariance of the action arises from the symmetry property (3.6), (3.7) of the $hsc^{\infty}(4)$ structure coefficients. Due to this property, it has become possible to solve the equation on the $\beta(n, s)$ coefficients. At the same time this symmetry property is not shared by the structure coefficients of the factor algebras $hsc^{(n)}(4)$ $(n=1, 2, ...)$ of the algebra $hsc^{\infty}(4)$, constructed in refs. [3,4], which contain all spins with multiplicity n. Therefore there exists no invariant action constructed out of the curvatures associated to the above algebras. Similarly, contractions of the superalgebra $shs(N|4)$ (see refs. [15,16] do not generate a consistent interaction of higher spins in ADS_4.

We thus conclude that of the whole series of conformal higher spin superalgebras, the only algebra which gives rise to an invariant interaction is the one containing an infinite number of fields of each spin. We should, however, make the two following remarks.

First, the above conclusion applies only to a given series of algebras and thus has no general meaning. However, the infinite multiplicity of all spins appears natural. In string theory all spins are represented an infinite number of times.

Under spontaneous symmetry breaking all higher spins might become massive, in analogy to string theory.

Note that an infinite number of spin-two fields are present in our theory. However the SO(4, 2) subalgebra is spanned by only those ${}^{n}T(2, c, l, j)$ generators (spin-two gauge fields) of the infinite dimensional algebra which have $n=0$. We identify the corresponding fields with conformal gravity fields. The rest of the spin-two $n>0$ fields should acquire a mass by means of a spontaneous symmetry breaking.

Second, it should be noted that our conclusion concerning the infinite multiplicities of all spins goes beyond the cubic approximation. As discussed in refs. [15,16], those gauge transformations which leave the action invariant in the first non-trivial order in the interaction, do not furnish an algebra. The cubic order invariance of the action does not imply, therefore, that the structure coefficients which enter the definition of the curvature should satisfy the full Jacobi identities.

All that is required of these structure coefficients are the symmetry properties which would allow one to cancel terms in $\delta\mathcal{A}$ of the type $\partial^{n}C\partial^{m}C\mathcal{E}$ and $\partial^{m}\bar{C}\partial^{n}\bar{C}\mathcal{E}$.

In cubic order, it is possible in principle to limit oneself with the fields ${}^{0}\omega$, setting the fields ${}^{n}\omega$ with $n>0$ equal to zero. As discussed in ref. [4], the corresponding generators ${}^{0}T$ do not form a closed algebra (except for those generators associated to spin-two fields which furnish the SO(4, 2) algebra). At the same time, the action (4.1), (4.8) built up from "incomplete curvatures" (which do not satisfy the full Bianchi identities) is cubic order invariant. However, we have been having in mind the construction of a complete interaction (which already in fourth order crucially relies on the full Jacobi identities to be satisfied) and thus have considered only those theories which are built upon closed Lie algebras.

Let us note an essential difference between the proof of the invariance of the cubic interaction in conformal higher spin theory, and the corresponding proof in the ADS_4 higher spin theory [15,16]. When establishing the invariance in the ADS_4 theory, only the linearized constraints were essential. In the conformal theory, on the other hand, the nonlinear constraints ${}^{(n)}R(s, 1-s, \frac{1}{2}(s-1), \frac{1}{2}(s-1))=0$ are essential, along with the linearized constraints.

The extension of the above analysis of the supersymmetric case will be published in a separate paper. Note that since we consider all spins, no restriction on N from the above emerges, contrary to the $N\leqslant 4$ constraint in conformal supergravity. In particular, the $N=5$ model contains the SU(5) grand unification group.

Another important task is to construct higher orders of the interaction. The fundamental problem consists in finding non-linear constraints which generalize the supergravity constraints.

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Appendix

"Pure spin" conformally invariant actions. In the tensor formalism, the spin s is described by a completely symmetric tensor $\phi_{\mu(s)}$. The simplest conformally invariant action for spin s is of the form [6]

$$A(s)=\int \mathrm{d}^4x\, \phi_{\mu(s)}\, \Box^s P^{\mu(s)}_{\nu(s)} \phi^{\nu(s)}\,, \tag{A.1}$$

where the spin projector P is traceless and transverse,

$$\eta_{\mu\mu} P^{\mu(s)}_{\nu(s)}=0\,, \quad \partial_\mu P^{\mu(s)}_{\nu(s)}=0\,. \tag{A.2}$$

The action $A(s)$ possesses the gauge symmetry

$$\delta_g \phi_{\mu(s)}=\partial_\mu \xi_{\mu(s-1)}-\eta_{\mu\mu}\lambda_{\mu(s-2)}\,. \tag{A.3}$$

It can be re-written in a slightly different form as

$$A(s)=(-1)^s\int \mathrm{d}^4x\, C_{\mu(s),\nu(s)}\, C^{\mu(s),\nu(s)}\,, \tag{A.4}$$

where the linearized Weyl tensor associated to spin s is of the form

$$C_{\mu(s),\nu(s)}=\mathscr{P}^{\rho(s),\sigma(s)}_{\mu(s),\nu(s)}\, \underbrace{\partial_\rho \ldots \partial_\rho}_{s}\, \phi_{\sigma(s)}\,. \tag{A.5}$$

In particular, $C_{\mu,\nu}$ coincides with $F_{\mu,\nu}$ for a vector field, while $C_{\mu(2),\nu(2)}$ coincides with the linearized gravitational Weyl tensor.

The Young projector $\mathscr{P}$, is associated with the irreducible traceless tableau

and obeying the (anti)symmetry condition

$$\mathscr{P}^{\rho(s),\sigma(s)}_{\mu(s-1)\nu,\nu(s)}=0\,, \tag{A.6}$$

and traceless,

$$\eta^{\mu\mu}\mathscr{P}^{\rho(s),\sigma(s)}_{\mu(s),\nu(s)}=0\,. \tag{A.7}$$

By virtue of these properties the Weyl tensor $C(s)$ is invariant under the gauge transformations (A.3).

In order to pass from eqs. (A.4) to eq. (A.1) it suffices to integrate by parts and employ the observation that, in view of the properties satisfied by $\mathscr{P}$, the following equality holds:

$$\Box^s P^{\nu(s)}_{\mu(s)}=\mathscr{P}^{\rho(s),\nu(s)}_{\sigma(s),\mu(s)}\, \underbrace{\partial_\rho \ldots \partial_\rho}_{s}\, \underbrace{\partial^\sigma \ldots \partial^\sigma}_{s}\,. \tag{A.8}$$

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