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QUANTIZATION AND COCYCLES ON THE SUPERTORUS AND LARGE-N LIMITS FOR THE CLASSICAL LIE SUPERALGEBRAS

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The Poisson superbracket Lie superalgebra on the supertorus $T^{2d|N}$ is considered and its quantization is carried out. It is shown that there exists a non-trivial supercentral extension by means of $2d$ arbitrary c-numbers (when N is even), or $2d$ Grassmann numbers (when N is odd). It is shown that the infinite-dimensional superalgebras on the supertorus $T^{2d|N}$ can be considered as certain generalizations and large-M limits of the classical superalgebras $A(M|M)$ and $Q(M)$ (when N is even and odd respectively).

Algebra of symplectic diffeomorphisms on the torus[1] T^2 and its quantum versions,[7] algebras with trigonometric structure constants, as well as their central extensions,[2–4,7] have recently attracted much attention[2–9] in the context of membrane theory,[2–4,8] integrable models,[5] large-N limits,[6–9] etc. The $N=1$ supersymmetric extension of sdiff(T^2) was proposed in Refs. 4 and 8, and its quantum version, $N = 1$ superalgebra with trigonometric structure constants, was obtained in Ref. 7. Different from its bosonic subalgebra sdiff(T^2), however, the superalgebra does not admit any non-trivial C-number central extension,[4] but will nevertheless admit a non-trivial supercentral extension by means of two independent Grassmann numbers, as was emphasized in Ref. 7.

In this note we study superalgebras of orthosymplectic superdiffeomorphisms and construct their quantum versions on the N-extended supertorus $T^{2d|N}$. Their structure constants are easily calculated in terms of the Poisson superbrackets and the Weyl symbols of quantum operators.

For these superalgebras we have found non-trivial supercentral extensions. Curiously enough, it turns out that there exists a non-trivial supercentral extension by means of $2d$ arbitrary c-numbers (Grassmann numbers) when N is even(odd). Corresponding supercocycles are presented in the form of integrals over the supertorus.

Let us consider a supertorus $T^{2d|N}$ with angular coordinates $0 \le \varphi_\alpha < 2\pi$, $\alpha = 1,\dots, 2d$ parametrizing the bosonic part $T^{2d} = S^1 \times \dots \times S^1$ ($2d$ times) and the Grassmannian coordinates ψ_i, $i = 1,\dots, N$ (their Grassmann parities are $P(\varphi_\alpha) = 0$, $P(\psi_i) = 1$). On the supertorus $T^{2d|N}$ one can define an orthosymplectic structure by means of the Poisson superbracket

$$[f,g]_{\mathrm{PB}}=\frac{\partial f}{\partial\varphi_\alpha}\,\omega_{\alpha\beta}\,\frac{\partial g}{\partial\varphi_\beta}+\frac{\partial_{\mathrm{r}} f}{\partial\psi_i}\,\frac{\partial_{\mathrm{l}} g}{\partial\psi^i}\,, \tag{1}$$

where $\omega_{\alpha\beta}$ is a non-degenerate antisymmetric $2d\times 2d$ matrix which can be chosen in the block-diagonal form (however, we did not assume this in our case):

$$(\omega_{\alpha\beta})=\begin{pmatrix} \begin{matrix}0 & 1\\ -1 & 0\end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix}0 & 1\\ -1 & 0\end{matrix}\end{pmatrix}, \tag{2}$$

and the Latin (Grassmann odd) indices $i, j, \ldots$ contracted with the help of δ_{ij}.

In the space of functions on the supertorus, $\mathcal{A}$, one can choose a basis of the form

$$\mathcal{F}_{\mathbf{n},i_1\ldots i_k}=\psi_{i_1}\cdots\psi_{i_k}e^{i\mathbf{n}\cdot\boldsymbol{\varphi}},\quad k=0,1,\ldots,N\,,$$

$$P(\mathcal{F}_{\mathbf{n},i_1\ldots i_k})=\frac{1}{2}(1-(-1)^k)\,,$$

$$\text{where }\mathbf{n}\in\mathbb{Z}\times\ldots\times\mathbb{Z}(2d\text{ times})\text{ and }\mathbf{n}\cdot\boldsymbol{\varphi}=n^\alpha\varphi_\alpha\,. \tag{3}$$

With respect to the Poisson superbrackets (1), $\mathcal{A}$ becomes a Lie superalgebra with the supercommutation relations in the basis (3)

$$\begin{aligned}&[\mathcal{F}_{\mathbf{n},i_1\ldots i_k},\mathcal{F}_{\mathbf{m},j_1\ldots j_l}\}_{\mathrm{PB}}\\ &=\mathbf{m}\times\mathbf{n}\;\mathcal{F}_{\mathbf{n}+\mathbf{m},i_1\ldots i_k j_1\ldots j_l}\\ &\quad+kl\;\mathrm{Alt}(\delta_{i_k j_1}\mathcal{F}_{\mathbf{n}+\mathbf{m},i_1\ldots i_{k-1}j_2\ldots j_l})\,,\end{aligned} \tag{4}$$

where Alt means complete antisymmetrization with respect to all the indices $i_1\ldots i_k$ and $j_1\ldots j_l$ separately

$$(\text{e.g., Alt }(\delta_{i_2 j_1}\mathcal{F}_{\mathbf{n}+\mathbf{m},i_1 j_2})=\frac{1}{4}(\delta_{i_2 j_1}\mathcal{F}_{\mathbf{n}+\mathbf{m},i_1 j_2}-\delta_{i_1 j_1}\mathcal{F}_{\mathbf{n}+\mathbf{m},i_2 j_2}+\delta_{i_1 j_2}\mathcal{F}_{\mathbf{n}+\mathbf{m},i_2 j_1}-\delta_{i_2 j_2}\mathcal{F}_{\mathbf{n}+\mathbf{m},i_1 j_1}))\,,$$

and

$$\mathbf{m}\times\mathbf{n}=m^\alpha\omega_{\alpha\beta}n^\beta\,. \tag{5}$$

The algebra (4) is isomorphic to the algebra of all orthosymplectic diffeomorphisms on $T^{2d|N}$ generated by the vector superfields

$$\begin{aligned}F_{\mathbf{n},i_1\ldots i_k}&=[\mathcal{F}_{\mathbf{n},i_1\ldots i_k},\cdot\}_{\mathrm{PB}}\\ &=i\psi_{i_1}\cdots\psi_{i_k}e^{i\mathbf{n}\cdot\boldsymbol{\varphi}}n^\alpha\omega_{\alpha\beta}\frac{\partial}{\partial\varphi_\beta}\\ &\quad+ke^{i\mathbf{n}\cdot\boldsymbol{\varphi}}\,\mathrm{Alt}\left(\psi_{i_1}\cdots\psi_{i_{k-1}}\frac{\partial_{\mathrm{l}}}{\partial\psi^{i_k}}\right).\end{aligned} \tag{6}$$

The Poisson superbracket algebra admits non-trivial central extensions, namely the following takes place.

The Poisson bracket superalgebra on $T^{2d|N}$ has a non-trivial supercentral extension given by

$$[f,g\}_{\text{CPB}} = [f,g\}_{\text{PB}} + \theta_\alpha C^\alpha(f,g), \tag{7}$$

with non-trivial basic cocycles

$$C^\alpha(f,g) = \int d\Phi\, f \frac{\partial}{\partial\varphi_\alpha} g = \int d\Phi\, f[\varphi_\beta, g]_{\text{PB}}\, \omega^{\alpha\beta}\,, \; (\omega_{\alpha\beta}\omega^{\beta\gamma} = \delta_\alpha^\gamma), \tag{8}$$

where the supervolume element reads

$$d\Phi = \frac{(-1)^{N(N-1)/2}}{(2\pi)^{2d} N!} d^{2d}\varphi\, \varepsilon^{i_1 \ldots i_N} d\psi_{i_1} \ldots d\psi_{i_N}\,, \tag{9}$$

and θ_α are $2d$ arbitrary supercentral elements with the Grassmann parity

$$P(\theta_\alpha) = \frac{1}{2}(1-(-1)^N) = P_N\,, \tag{10}$$

and consequently

$$\theta_\alpha\theta_\beta - (-1)^{P_N}\,\theta_\beta\theta_\alpha = 0\,, \tag{11}$$

$$\theta_\alpha f - (-1)^{P(f)P_N}\, f\,\theta_\alpha = 0\,. \tag{12}$$

In other words, for even N there exists a non-trivial central extension by means of $2d$ arbitrary commuting central elements. However, when N is odd there exists a non-trivial supercentral extension by means of $2d$ anti-commuting Grassmann parameters. In the basis (3) the supercentral extension takes the following form,

$$\begin{aligned}
&[\mathcal{F}_{\mathbf{n},i_1\ldots i_k}, \mathcal{F}_{\mathbf{m},j_1\ldots j_l}\}_{\text{CPB}} \\
&= \mathbf{m}\times\mathbf{n}\;\mathcal{F}_{\mathbf{n+m},i_1\ldots i_k j_1\ldots j_l} + kl\;\text{Alt}\,(\delta_{i_k j_1}\mathcal{F}_{\mathbf{n+m},i_1\ldots i_{k-1}j_2\ldots j_l}) \\
&\quad + i\theta_\alpha m^\alpha\,\delta_{\mathbf{n+m},\mathbf{0}}\,\delta_{k+l,N}\,\varepsilon_{i_1\ldots i_k j_1\ldots j_{N-k}}\,.
\end{aligned} \tag{13}$$

To prove this, we first note that the expression

$$(f,g) = \int d\Phi\, fg \tag{14}$$

defines a supersymmetric Poisson-bracket invariant bilinear form on $\mathcal{A}$,

$$(f,g) = (-1)^{P(f)P(g)}(g,f), \tag{15}$$

$$([f,g\}_{\text{PB}}, h) + (-1)^{P(g)P(h)}(f,[h,g\}_{\text{PB}}) = 0\,, \tag{16}$$

$$\int d\Phi[f,g\}_{\text{PB}} = 0\,, \left(C^\alpha(f,g) = \left(f, \frac{\partial}{\partial\varphi_\alpha} g\right) = \omega^{\alpha\beta}(f,[\varphi_\beta, g]_{\text{PB}})\right). \tag{17}$$

The properties (16), (17) are consequences of the simple fact

$$\int d\Phi \frac{\partial}{\partial \varphi_\alpha} f^\alpha = 0\,, \ \forall f^\alpha \in \mathcal{A}\,(\alpha = 1,\ldots,2d). \tag{18}$$

Now we have to check the cocycle condition of $C^\alpha(f, g)$ defined in (8). It is straightforward due to (16) and the Jacobi identities for the Poisson superbrackets:

$$\begin{aligned}
&(-1)^{P(h)P(f)} C^\alpha(f,[g,h\}_{\rm PB}) + (-1)^{P(g)P(f)} C^\alpha(g,[h,f\}_{\rm PB}) \\
&+ (-1)^{P(g)P(f)} C^\alpha(h,[f,g\}_{\rm PB}) \\
&= \omega^{\alpha\beta}(\varphi_\beta,[f,[g,h\}_{\rm PB}\}_{\rm PB})(-1)^{P(h)P(f)} \\
&+ \omega^{\alpha\beta}(\varphi_\beta,[h,[f,g\}_{\rm PB}\}_{\rm PB})(-1)^{P(h)P(g)} \\
&+ \omega^{\alpha\beta}(\varphi_\beta,[g,[h,f\}_{\rm PB}\}_{\rm PB})(-1)^{P(g)P(f)} = 0\,.
\end{aligned} \tag{19}$$

In this way we have in our disposal $2d$-independent cocycles. From Eq. (13), it is easy to see that they are non-trivial, i.e., cannot be canceled out by means of any redefinition of the generators $L_{\mathbf{0},i_1\ldots i_k}$.

Now we are going to quantize the above Poisson bracket construction by introducing the operators $\hat{\varphi}$ and $\hat{\psi}$, instead of the classical phase-space variables,

$$[\hat{\varphi}_\alpha, \hat{\varphi}_\beta] = 2i\hbar\omega_{\alpha\beta}\,, \ \{\hat{\psi}_i, \hat{\psi}_j\} = 2\hbar\delta_{ij}\,. \tag{20}$$

Working with the operators, we decide to fix the Weyl (supersymmetric) ordering and in order to simplify the calculations we use the Weyl symbols of the operators. The associative product of the two symbols of the operators, which correspond, after quantization, to the functions on $T^{2d|N}$, is given by the simple formula[a]

$$f * g = f \exp(\overleftrightarrow{\Delta}) g\,, \tag{21}$$

where

$$\overleftrightarrow{\Delta} = i\hbar \frac{\overleftarrow{\partial}}{\partial \varphi_\alpha}\,\omega_{\alpha\beta}\,\frac{\overrightarrow{\partial}}{\partial \varphi_\beta} + \hbar \frac{\overleftarrow{\partial}_{\rm r}}{\partial \psi_i}\,\frac{\overrightarrow{\partial}_{\rm l}}{\partial \psi^i}\,. \tag{22}$$

The supercommutator is defined as usual,

$$[f,g\}_* = f * g - (-1)^{P(f)P(g)} g * f\,, \tag{23}$$

and for the symbols corresponding to the basis functions (3) we obtain immediately

$$\begin{aligned}
&[\mathcal{F}_{\mathbf{n},i_1\ldots i_k}, \mathcal{F}_{\mathbf{m},j_1\ldots j_l}\}_* \\
&= \sum_{p=0}^{\min(k,l)} \frac{\hbar^p\, k!\, l!}{p!(k-p)!(l-p)!}[(1-(-1)^p)\cos(\hbar\,\mathbf{m}\times\mathbf{n}) \\
&+ i(1+(-1)^p)\sin(\hbar\,\mathbf{m}\times\mathbf{n})]\ \mathrm{Alt}(\delta_{i_k j_1}\ldots\delta_{i_{k-p+1}j_p}\mathcal{F}_{\mathbf{n}+\mathbf{m},i_1\ldots i_{k-p}j_{p+1}\ldots j_l}).
\end{aligned} \tag{24}$$

[a] The theory of symbols of operators for quantum systems with both Bose and Fermi quantities was elaborated in detail by F. A. Berezin[10–12] in continuation of his fundamental studies in superalgebra and superanalysis[13] (see also Refs. 14 and 15).

In the classical limit $\hbar \to 0$ we come back to the Poisson brackets (4) (after an appropriate redefinition of the generators).

The quantum operatorial superalgebra on the supertorus (24), a deformation of the Poisson bracket one, inherits the non-trivial central extension of the latter. A quantum algebra with the supercentral extension is given by the modified supercommutator

$$[f,g\}_{*C}=[f,g\}_{*}+i\hbar\,\theta_{\alpha}C^{\alpha}(f,g), \tag{25}$$

where f and g are the Weyl symbols of the operators f and g.

Now we will specify the general formulae for the particular case $N = 2$ to make them more readable. So, the Poisson bracket superalgebra (with the central term) reads as follows,

$$\begin{aligned}
&[L_{\mathbf{n}},L_{\mathbf{m}}]_{\mathrm{CPB}} = \mathbf{m}\times\mathbf{n}\,L_{\mathbf{n+m}}\,,\\
&[L_{\mathbf{n}},U_{\mathbf{m}}]_{\mathrm{CPB}} = \mathbf{m}\times\mathbf{n}\,U_{\mathbf{n+m}}+i\,\boldsymbol{\theta}\cdot\mathbf{m}\delta_{\mathbf{n+m},\mathbf{0}}\,,\\
&[U_{\mathbf{n}},U_{\mathbf{m}}]_{\mathrm{CPB}} = 0\,,\\
&[L_{\mathbf{n}},Q^{\pm}_{\mathbf{m}}]_{\mathrm{CPB}} = \mathbf{m}\times\mathbf{n}\,Q^{\pm}_{\mathbf{n+m}}\,,\\
&[U_{\mathbf{n}},Q^{\pm}_{\mathbf{m}}]_{\mathrm{CPB}} = \pm\,Q^{\pm}_{\mathbf{n+m}}\,,\\
&\{Q^{\pm}_{\mathbf{n}},Q^{-}_{\mathbf{m}}\}_{\mathrm{CPB}} = \mathbf{m}\times\mathbf{n}\,U_{\mathbf{n+m}}+L_{\mathbf{n+m}}\,,+i\boldsymbol{\theta}\cdot\mathbf{m}\delta_{\mathbf{n+m},\mathbf{0}}\,,
\end{aligned} \tag{26}$$

where $\theta = (\theta_{\alpha})$ are $2d$ arbitrary numerical parameters. We have used the following notations for the $N = 2$ generators,

$$\begin{aligned}
&L_{\mathbf{n}} = e^{i\mathbf{n}\cdot\boldsymbol{\varphi}},\ Q^{\pm}_{\mathbf{n}} = \alpha^{\pm}e^{i\mathbf{n}\cdot\boldsymbol{\varphi}}\,,\\
&U_{\mathbf{n}} = \alpha^{+}\alpha^{-}e^{i\mathbf{n}\cdot\boldsymbol{\varphi}}
\end{aligned} \tag{27}$$

and ($\alpha^{\pm} = 1/\sqrt{2}\,(\psi_1 \pm i\psi_2)$)

$$\{\alpha^{-},\alpha^{+}\}_{\mathrm{PB}} = 1. \tag{28}$$

The quantum deformation of the above algebra has the form (on appropriate normalization of the generators)

$$\begin{aligned}
&[L_{\mathbf{n}},L_{\mathbf{m}}]_{*C} = \sin(\hbar\,\mathbf{m}\times\mathbf{n})\,L_{\mathbf{n+m}}\,,\\
&[L_{\mathbf{n}},U_{\mathbf{m}}]_{*C} = \sin(\hbar\,\mathbf{m}\times\mathbf{n})\,U_{\mathbf{n+m}}+\hbar\boldsymbol{\theta}\cdot\mathbf{m}\delta_{\mathbf{n+m},\mathbf{0}}\,,\\
&[U_{\mathbf{n}},U_{\mathbf{m}}]_{*C} = \hbar^{2}\sin(\hbar\,\mathbf{m}\times\mathbf{n})L_{\mathbf{n+m}}\,,\\
&[L_{\mathbf{n}},Q^{\pm}_{\mathbf{m}}]_{*C} = \sin(\hbar\,\mathbf{m}\times\mathbf{n})Q^{\pm}_{\mathbf{n+m}}\,,\\
&[U_{\mathbf{n}},Q^{\pm}_{\mathbf{m}}]_{*C} = \pm i\hbar\,\cos(\hbar\,\mathbf{m}\times\mathbf{n})Q^{\pm}_{\mathbf{n+m}}\,,\\
&\{Q^{+}_{\mathbf{n}},Q^{-}_{\mathbf{m}}\}_{*C} = \sin(\hbar\,\mathbf{m}\times\mathbf{n})U_{\mathbf{n+m}}\\
&\qquad\qquad + i\hbar\cos(\hbar\,\mathbf{m}\times\mathbf{n})L_{\mathbf{n+m}}+\hbar\,\boldsymbol{\theta}\cdot\mathbf{m}\delta_{\mathbf{n+m},\mathbf{0}}\,.
\end{aligned} \tag{29}$$

The following interesting question is finite-dimensional analogs of the above infinite-dimensional superalgebras. First let $d = 1$ (i.e., $T^{2|N}$ is considered), $\theta = 0$ and $\hbar = 2\pi/M$, where M is an odd integer. Then due to the periodicity of the trigonometric structure constants in (24) after identification of the generators by modulo M,

$$L_{\mathbf{n},i_1\ldots i_k} = L_{\mathbf{n}+M\mathbf{m},i_1\ldots i_k}\,, \quad \forall\ \mathbf{m} \in \mathbb{Z}\times\mathbb{Z}\,, \tag{30}$$

we arrive at the finite-dimensional superalgebra given by (24), where the definition of L_n and the sum $n + m$ on the right-hand side of (24) is supposed to be taken by modulo M. It is easy to verify that the resulting superalgebras are isomorphic to the following classical superalgebras of the Cartan type:

$$A(2^{(N-2)/2}M-1\,|\,2^{(N-2)/2}M-1) \quad \text{for } N \text{ even } (N\neq 0), \tag{31}$$

$$Q(2^{(N-1)/2}M-1) \quad \text{for odd } N. \tag{32}$$

First let $N = 0$. In this purely bosonic case it is known[7] that the corresponding algebra (after factoring out L_0) is isomorphic to $A(M-1)$. Second, let $N=1$ and the superalgebra has the form[7]

$$[L_{\mathbf{n}}, L_{\mathbf{m}}] = \sin\left(\frac{2\pi}{M}\mathbf{m}\times\mathbf{n}\right)L_{\mathbf{n}+\mathbf{m}}\,,$$

$$[L_{\mathbf{n}}, Q_{\mathbf{m}}] = \sin\left(\frac{2\pi}{M}\mathbf{m}\times\mathbf{n}\right)Q_{\mathbf{n}+\mathbf{m}}\,,$$

$$\{Q_{\mathbf{n}}, Q_{\mathbf{m}}\} = \cos\left(\frac{2\pi}{M}\mathbf{m}\times\mathbf{n}\right)L_{\mathbf{n}+\mathbf{m}}\,, \tag{33}$$

($Q_{\mathbf{n}} = 1/2\ \psi e^{i\mathbf{n}\cdot\varphi}$). The generator Q_0 does not appear on the right-hand side of (super)commutators in (33) and can therefore be left out. The generator L_0 forms a one-dimensional center and can be factorized out. As a result we obtain a superalgebra with $A(M-1)$ as its Bose subalgebra, but its Fermi generators are transformed under the adjoint representation ad$A(M-1)$ with respect to the Bose subalgebra. Therefore it is just the classical strange superalgebra $Q(M-1)$.[13,17] Then the Poisson bracket superalgebra[4,8]

$$[L_{\mathbf{n}}, L_{\mathbf{m}}]_{\mathrm{PB}} = \mathbf{m}\times\mathbf{n}\ L_{\mathbf{n}+\mathbf{m}}\,,$$

$$[L_{\mathbf{n}}, Q_{\mathbf{m}}]_{\mathrm{PB}} = \mathbf{m}\times\mathbf{n}\ Q_{\mathbf{n}+\mathbf{m}}\,,$$

$$\{Q_{\mathbf{n}}, Q_{\mathbf{m}}\}_{\mathrm{PB}} = L_{\mathbf{n}+\mathbf{m}}\,, \tag{34}$$

($L_0 = 0$, $Q_0 = 0$) can be viewed as the large-M limit of the series $Q(M)$ and can be denoted as $Q_t(\infty)$ (t means that this limit is obtained in the toroidal basis for $Q(M)$).

Further, let $N = 2$ and we are working with an algebra where $d = 1$, $\theta = 0$, $\hbar = 2\pi/M$, and M is an odd integer. The generator U_0 does not appear on the right-hand side of the commutators and can be left out. L_0 forms a center and can be factorized out. As a result, we arrive at the simple superalgebra with $A(M-1) \oplus A(M-1)$ as its Bose subalgebra (the corresponding sets are $T^{\pm}_{\mathbf{n}} = L_{\mathbf{n}} \pm U_{\mathbf{n}}$). This superalgebra is isomorphic to $A(M-1 \mid M-1)$. Then the Poisson bracket superalgebra (26) (with $d = 1$ and $\theta = 0$) can be called $A_t(\infty \mid \infty)$. The general result (31), (32) is proved in the same way.

Now let us pass to the case of higher-dimensional tori with $d > 1$. Again let $\hbar = 2\pi/M$ with odd M, and $(\omega_{\alpha\beta})$ in (1) is taken to be an arbitrary antisymmetric $2d \times 2d$ matrix such as in Eq. (2) and $(\omega^{\beta\gamma})$ is its inverse:

$$\omega_{\alpha\beta}\, \omega^{\beta\gamma} = \delta^{\gamma}_{\alpha} . \tag{35}$$

Then due to the periodicity of the structure constants the matrix $(\omega^{\alpha\beta})$ defines an automorphism of the quantum algebra (24) (now we consider the case $N = 0$, $\theta = 0$):

$$L_{\mathbf{n}} = L_{\mathbf{n}'}, \quad \mathbf{n}' = (n^{\alpha} + M\omega^{\alpha\gamma} t_{\gamma}), \forall\, t_{\gamma} \in \mathbb{Z} . \tag{36}$$

Taking an identification by modulo the matrix $M\omega^{\alpha\beta}$:

$$L_{\mathbf{n}} = L_{\mathbf{m}} \quad \text{when} \quad \mathbf{n}^{\alpha} - \mathbf{m}^{\alpha} = M\omega^{\alpha\beta} t_{\beta}, \forall\, t_{\beta} \in \mathbb{Z} , \tag{37}$$

we obtain the finite-dimensional algebra isomorphic to $SU(M^d)$. In this way one can see that $SU(M^d)$ can be represented in a form with trigonometric structure constants in many ways. Namely an algebra

$$[L_{\mathbf{n}}, L_{\mathbf{m}}] = \frac{M}{2\pi} \sin\left(\frac{2\pi}{M} m^{\alpha} \omega_{\alpha\beta} n^{\beta} \right) L_{\mathbf{n}+\mathbf{m}} \tag{38}$$

with the basis L_n, labeled by $2d$-dimensional integer-component vectors n, is isomorphic to $SU(M^d)$ when $n + m$ on the right-hand side is supposed to be taken by modulo vectors of the form

$$(M\omega^{\alpha\beta} t_{\beta}) , \ \forall \ t_{\beta} \in \mathbb{Z} .$$

Taking the large-M limit ($M \to \infty$) we arrive at the classical algebra

$$[L_{\mathbf{n}}, L_{\mathbf{m}}] = (m^{\alpha} \omega_{\alpha\beta} n^{\beta}) L_{\mathbf{n}+\mathbf{m}} . \tag{39}$$

Consideration with $N > 0$ and $d > 1$ is quite similar with the superalgebras (31), (32) in the place of $SU(M^d)$.

To conclude, let us summarize the results of the present paper. First, the classical and quantum operatorial superalgebras on the supertorus $T^{2d|N}$ have been constructed manifestly (their structure constants have been calculated). Second, their non-trivial central extensions have been obtained and the manifest expressions for the cocycles are presented. It turns out that for even N $2d$-independent central elements are commuting, but for odd N they are anti-commuting. Third, it has been shown that for special values of the deformation parameters, the infinite-dimensional operatorial quantum superalgebras on the supertorus $T^{2d|N}$ can be reduced to the classical finite-dimensional superalgebras $A(M\,|\,M)$ or $Q(M)$ (for even or odd N respectively, where $N > 0$). These results generalize to the case of arbitrary N with the results of Refs. 1–4, 6, 7 for non-supersymmetric case $N = 0$, and of Refs. 4, 7 and 8 for $N = 1$ supersymmetric case (in these references, mainly the case $d = 1$, two-dimensional torus, has been studied). Note also that concerning the $N = 1$ case, to our knowledge, the isomorphism of (33) to $Q(M)$ was not pointed out earlier and the cocycle formula in terms of integrals was not presented.

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