Modern Physics Letters A, Vol. 5, No. 24 (1990) 1967–1977

INFINITE-DIMENSIONAL GENERALIZATIONS OF SIMPLE LIE ALGEBRAS

E. S. FRADKIN and V. YA. LINETSKY

Theoretical Department, P. N. Lebedev Physical Institute, Leninsky prospect 53, 117924 Moscow, USSR

Received 9 July 1990

Infinite-dimensional algebras associated with simple finite-dimensional Lie algebra g are considered. Higher-spin generalizations of sl(2) are studied in detail. Those of the Virasoro algebra are viewed as their "analytic continuations". Applications in higher-spin theory and in conformal QFT are discussed.

1. Introduction

Among infinite-dimensional (inf-dim.) Lie algebras an outstanding place is occupied by those which contain some given semisimple finite-dimensional (fin-dim.) Lie algebra g as a maximal fin-dim. subalgebra. These algebras play an important (and even determinant) role in two central problems of modern physics: the problem of unification of all interactions and the problem of exact solvability in QFT and statistics. For this reason the problem of construction of a mathematical theory for such algebras (including complete classification similar to the theory of fin-dim. simple Lie algebras) has both physical and mathematical importance.

The algebras containing semisimple g as a maximal fin-dim. subalgebra can be divided into two classes under the natural representation $[g, \cdot\,]$ of g. First-class algebras are decomposed only into a direct sum of fin-dim. irreducible g-modules. Second-class algebras involve also inf-dim. g-modules (irreducible or/and non-decomposable). Examples of first-class algebras are the Kac-Moody algebras and the higher-spin algebras.

The Virasoro algebra gives an example of the second-class algebra (non-decomposable sl_2-module under $[sl_2, \mathrm{Vir}]$, sl_2-$\{L_{\pm 1}, L_0\}$). A more general example is an analytic continuation AC(g) and its central extension $\widetilde{AC}(g)$ of any semisimple fin-dim. g, discovered recently in Ref. 1. In particular, $g = sl_2$, $\widetilde{AC}(sl_2)$ is a Virasoro algebra. We believe that these algebras and their representation theory will play a role in the D-dimensional conformal QFT and statistics similar to the role played by the Virasoro algebra in $D = 2$.

In this paper we shall be concerned mainly with the algebras associated with sl_2. This is the simplest but striking example of the general theory. It is also very important in both higher-spin theory[3–9] and two-dimensional conformal QFT. In particular, the classification of a special subclass of second-class algebras, which are "analytic continuations" of the first-class algebras, will allow one to obtain a classification of the possible Wilson OPE's in $D = 2$ exactly solvable models.

2. The Universal Enveloping Algebra $U(\mathrm{sl}_2)$ and its Factor-Algebras

Let $T_i\,(i = 1, 2, 3)$ be some basis in sl_2 and C be the quadratic sl_2 Casimir element. The universal enveloping algebra[10] $U(\mathrm{sl}_2)$ is an associative algebra with a unit and generating elements T_i obeying the sl_2 commutation relations. $U(\mathrm{sl}_2)$ contains a centre $Z(\mathrm{sl}_2)$ generated by C (the basis is 1 and C^n, where $n = 1, 2,\ldots$). By considering a family of ideals $\mathcal{J}(\lambda\,|n) = (C - \lambda\,1)^n\, U(\mathrm{sl}_2)$ (λ is an arbitrary complex number and $n = 1, 2,\ldots$), one can define a family of factor-algebras,

$$U(\mathrm{sl}_2\,|\,\lambda\,|n) = U(\mathrm{sl}_2)/\mathcal{J}(\lambda\,|n)\,. \tag{1}$$

The commutator AB-BA transforms all the associative algebras into Lie algebras $[U(\mathrm{sl}_2)]$ and $[U(\mathrm{sl}_2\,|\,\lambda\,|n)]$. The algebras $U(\mathrm{sl}_2\,|\,\lambda\,|n)$, with $n > 1$, are not simple.

Let us consider $U(\mathrm{sl}_2\,|\,\lambda) = U(\mathrm{sl}_2\,|\,\lambda\,|\,1)$ and the corresponding Lie algebras in more detail. As was shown in Refs. 11 and 12, these algebras are non-isomorphic to one another at different values λ of the Casimir element. The algebras $U(\lambda)$ and $L(\lambda) = [U(\lambda)]$/(one-dimensional Abelian subalgebra $C1$) are simple for the "general position" values of λ. It is of importance that when $\lambda = -l(l+2)/4$ ($l = 0, 1, 2,\ldots$), i.e., λ coincides with the eigenvalue of C in some fin-dim. sl_2 irrep. $D(l)$ ($\dim D(l) = l + 1$), there exists an ideal χ in $U(-l(l+2)/4)$ such that the following isomorphism takes place,

$$\mathrm{Mat}_{l+1} \simeq U(-\,l(l+2)/4)/\chi\,, \tag{2}$$

and for the Lie algebras

$$\mathrm{gl}_{l+1} \simeq [U(-\,l(l+2)/4)]\,/[\chi]\,, \tag{3}$$

$$\mathrm{sl}_{l+1} \simeq L(-\,l(l+2)/4)\,/[\chi] \tag{4}$$

(Mat_n is a full matrix algebra with $n \times n$ matrices).

The existence of χ can be understood from the following consideration. Let V_λ be some elementary inf-dim. sl_2 module with the eigenvalue of C equals λ (for more details regarding elementary modules or the basic series of representations see e.g. Refs. 10 and 13).

It is well-known that the elementary sl_2 modules are irreducible for "general position" points of λ. However, when λ coincides with some eigenvalue of C in fin-dim. irrep($\lambda = -\,l(l+2)/4$, $l = 0, 1,\ldots$), the elementary module turns out to be non-decomposable. In particular, for each $l = 0, 1,\ldots$ there exists a unique sl_2 elementary module containing an sl_2 irreducible fin-dim. module with dimension $(l + 1)$ as its invariant subspace. An algebra of polynomials on sl_2 generators over this module is isomorphic to $U(\mathrm{sl}_2\,|-\,l(l + 2)/4)$. The restriction on the $(l + 1)$-dimensional irreducible submodule is isomorphic to Mat_{l+1} by the Burnside theorem. The kernel of this restriction is an ideal χ. The same results were obtained in Ref. 14 by calculating the invariant bilinear forms for $U(\mathrm{sl}_2\,|\,\lambda)$. It turned out that for the "exceptional points" $\lambda = -\,l(l + 2)/4$ the bilinear form becomes degenerate and its

0-space is just the ideal χ. The bilinear form is non-degenerate in this case only on the quotient-space Mat_{l+1} in (2).

Now let us consider constructing the sl_2 irreducible basis in U and $U(\lambda)$. Let T_0, T_+, T_- be the Cartan-Weyl basis in sl_2: $[T_0, T_\pm] = \pm T_\pm$, $[T_+, T_-] = -2T_0$ and $C = -T_0^2 + 1/2(T_+T_- + T_-T_+)$. The irreducible basis in $U(sl_2)$ can be constructed in the following way,

$$T_m^{N,s} \sim \Big[\underbrace{T_-, \ldots, [T_-}_{s-m}, (T_+)^s]\ldots\Big](C)^{N-s} , \tag{5}$$

where $N = 0, 1, \ldots$, $s = 0, 1, \ldots, N$, $|m| \leq s$.

Thus U contains all the $sl_2(2s+1)$-dim. reps. $D(2s)$ with infinite multiplicity. The factor-algebras $U(\lambda | n)$ contain all $D(2s)$ with multiplicity n. In particular, in $U(\lambda | 1)$ each irreps. $D(2s)$ enters only once and the corresponding generators have the form T_m^s ($s = 0, 1, \ldots, |m| \leq s$). The manifest expressions for the structure constants for the product $T_m^s T_{m'}^{s'}$ can be obtained as a generalization of the Racah composition law for the matrix algebra. The associative matrix algebra Mat_N in the usual basis E_{ij} has a composition law $E_{ij}E_{i'j'} = \delta_{ji'}E_{ij'}$ ($i, j = 1, \ldots, N$) and the corresponding Lie algebra gl_N has the usual Weyl commutation relations. Racah[15] introduced in Mat_N and gl_N a new basis (see Chap. 7 of Ref. 16),

$$T_m^s = \sqrt{\frac{(2s+1)}{N}} \sum_{m',m''} C^{jsj}_{m'mm''} E_{j-m''+1, j-m'+1} ,$$

$$s \leq 2j, |m| \leq s , \tag{6}$$

where $j = (N-1)/2$ and $C^{\cdots}_{\cdots}$ are the usual Clebsh-Gordan coefficients (all our notations and conventions concerning the angular momentum theory follow Ref. 16). The composition law in Mat_N in the Racah basis has the form (see (7.7.23) in Ref. 16)

$$T_m^s T_{m'}^{s'} = \sum_{s'', m''} (-1)^{2j+s''} \sqrt{(2s+1)(2s'+1)}$$
$$\times \begin{Bmatrix} s & s' & s'' \\ j & j & j \end{Bmatrix} C^{ss's''}_{mm'm''} T^{s''}_{m''} , \tag{7}$$

where $\{\ldots\}$ are the $6j$-symbols. Note that the number of generators (6) is $\sum_{s=0}^{2j}(2s+1) = (2j+1)^2 = N^2$ and equals the dimension of Mat_N. The gl_N commutation relations in the Racah basis take a form like (7) (only the factor $(1 - (-1)^{s+s'+s''})$ appears). The generator T_0^0 forms a one-dimensional centre ($T_0^0 = 1/\sqrt{N}\,\mathbb{1}$) in gl_N and $sl_N = gl_N/CT_0^0$. The generators $T^1_{\pm 1,0}$ form an sl_2-subalgebra in sl_N and the generators $\{T_m^s, |m| \leq s\}$ with fixed s form a basis in spin-s sl_2 irreps. $D(2s)$. In this way sl_N has been decomposed into a direct sum of higher-spin sl_2 irreps. $sl_N = \bigoplus_{s=1}^{N-1} D(2s)$ ($N = 2j+1$), and a maximal spin involved in sl_N is $s_{\max} = 2j$. Owing to the presence of the parity function $p(s + s' + s'' + 1) = 1/2(1 + (-1)^{s+s'+s''+1})$ on the right-hand side of the gl_N commutation relations, the generators with odd spins s form a subalgebra in sl_N which is isomorphic to so_N at odd N, or sp_N at even N (see comments below (7.7.24–26) in Ref. 16).

The Racah generators (6) are normalized as follows,

$$(T^s_m, T^{s'}_{m'}) = (-1)^m \delta^{s,s'} \delta_{m,-m'} , \tag{8}$$

under an invariant symmetric bilinear form, $(A, B) = \mathrm{tr}\,(AB)$, where tr is the usual matrix trace $\mathrm{tr}(\Sigma A^s_m T^s_m) = A^0_0 \sqrt{2j+1}$.

The structure constants of $(\mathrm{sl}_2|\lambda)$ can be obtained from the Racah formula (7) by generalization to the case of arbitrary complex λ. To do this, let us redefine the Racah generators

$$\tilde{T}^s_m = \sqrt{\frac{(N+s)!}{(N-s-1)!}}\, T^s_m \tag{9}$$

and define a new set of $6j$-symbols in such a way that $\begin{Bmatrix} s & s' & s'' \\ j & j & j \end{Bmatrix}$ will become a polynomial functon of j:

$$\begin{bmatrix} s & s' & s'' \\ j & j & j \end{bmatrix} = (-1)^{2j+s''} \left[\frac{(N+s)!\,(N+s')!\,(N-s''-1)!}{(N-s-1)!\,(N-s'-1)!\,(N+s'')!} \right]^{\frac{1}{2}}$$
$$\times \begin{Bmatrix} s & s' & s'' \\ j & j & j \end{Bmatrix} = s!\,s'!\,s''!\,\Delta(s,s',s'') \sum_t (-1)^t \left\{ \prod_{p=1}^{s+s'-s''-t} (N-s''-p) \right\}$$
$$\times \left\{ \prod_{q=1}^{t} (N+s''+q) \right\} \Big[t!\,(t+s''-s)!\,(t+s''-s')!\,(s+s'-s''-t)!$$
$$\times (s-t)!\,(s'-t)! \Big]^{-1} \tag{10}$$

($\Delta(s, s', s'')$ are the triangle coefficients[16]). All dependence on j (or $N = 2j + 1$) in the new $6j$-symbols is a polynomial one.

Now one can define generalized $6j$-symbols $\begin{bmatrix} s & s' & s'' \\ \rho & \rho & \rho \end{bmatrix}$ for arbitrary complex ρ as an analytic continuation in (10) (now $N = 2\rho + 1 \in \mathbb{C}$). It is easy to verify that the generalized $6j$-symbols are actually polynomials on N^2 and do not contain odd degrees on N. Note also that the usual $6j$-symbols $\begin{Bmatrix} s & s' & s'' \\ j & j & j \end{Bmatrix}$ are defined only for $s \le 2j$, $s' \le 2j$, $s'' \le 2j$ (for the other combinations of s, s', s'' they are usually put to be zero). The generalized $6j$-symbols (10) are defined for all combinations of integers s, s', s'' (only the triangle condition $|s - s'| \le s'' \le s + s''$ must be satisfied; if it is not satisfied, the generalized $6j$-symbols are put to be equal to zero). Then the composition law for $U(\mathrm{sl}_2|\lambda)$ $(\lambda = -\rho(\rho+1))$ has the form

$$\tilde{T}^s_m \tilde{T}^{s'}_{m'} = \sum_{s'',m''} \sqrt{(2s+1)(2s'+1)} \begin{bmatrix} s & s' & s'' \\ \rho & \rho & \rho \end{bmatrix}$$
$$\times C^{ss's''}_{mm'm''} \tilde{T}^{s''}_{m''} , \tag{11}$$

where $\rho \in \mathbb{C}$. When $\rho = j = 0, 1, \ldots$ and all the generators T^s_m with $s > 2j$ are equal to zero, we come back to the Racah formula (7). However, when $\rho \in \mathbb{C}$ we have $U(\mathrm{sl}_2|-\rho(\rho+1))$. (Note that the sl_2 Casimir operator $C \sim \begin{bmatrix} 1 & 1 & 0 \\ \rho & \rho & \rho \end{bmatrix} T^0_0$, where

$\begin{bmatrix} 1 & 1 & 0 \\ \rho & \rho & \rho \end{bmatrix} \sim \rho(\rho+1)$, $\tilde{T}_0^0 = \mathbb{1}$ is the unit in U.)

In this way the composition law for $U(\mathrm{sl}_2\,|\,\lambda)$ can be obtained as a straightforward "analytic continuation" of the Racah formula for the Mat_{2j+1} composition law. This "analytic continuation" consists in abolition of the conditions $s \leq 2j$ and in an analytic continuation from $j = 0, 1/2, 1, \ldots$ to $\rho \in \mathrm{C}$.

An invariant symmetric bilinear form for $[U(\mathrm{sl}_2\,|-(N^2-1)/4)]$ can also be easily obtained from (8) for the matrix algebra. The new generators T^s_m (9) have a normalization

$$[\tilde{T}^s_m, \tilde{T}^{s'}_{m'}] = (-1)^m \delta^{s,s'} \delta_{m,-m'} N \prod_{p=1}^{s} (N^2 - p^2)\,. \tag{12}$$

Supposing now $N = 2\rho + 1$ to be a complex number and $s = 0, 1, \ldots$ we obtain a bilinear form for $[U(\mathrm{sl}_2\,|\,\lambda)]$ with arbitrary λ. When N is an integer the bilinear form (12) becomes degenerate: all the generators T^s_m with $s > N$ are its 0-vectors and form a basis in the ideal χ. This is in agreement with the results of Ref. 14 obtained in another way.

The commutation relations for $[U(\mathrm{sl}_2\,|\,\lambda)]$ are immediately obtained from (11) and have the form

$$[\tilde{T}^s_m, \tilde{T}^{s'}_{m'}] = 2 \sum_{s'',m''} (1 - (-1)^{s+s'-s''}) \sqrt{(2s+1)(2s'+1)} \times \begin{bmatrix} s & s' & s'' \\ \rho & \rho & \rho \end{bmatrix} C^{ss's''}_{mm'm''}\, \tilde{T}^{s''}_{m''}\,. \tag{13}$$

To conclude this section, we see that the way of obtaining the structure constants of $U(\mathrm{sl}_2\,|\,\lambda)$ from the Racah basis is practically the same as in Ref. 17 where the structure constants for the algebra $\mathrm{su}(\infty) = \lim_{N\to\infty} \mathrm{su}(N)$ were evaluated.

3. A General Construction for Arbitrary Semisimple Fin-Dim. Lie Algebra

Now let g be some semisimple fin-dim. Lie algebra of rank $(g) = r$ and dim $(g) = n$ and $U(g)$ is its universal enveloping algebra. $U(g)$ contains a centre $Z(g)$ generated by r homogeneous generating elements, r independent symmetrized Casimir operators C_i $(i = 1, \ldots, k)$ of g, by the Chevalley theorem. By considering a family of ideals $\mathcal{J}(\lambda) = \mathcal{J}(\lambda_1, \ldots, \lambda_n)$ generated by a system of equations

$$C_i - \lambda_i \mathbb{1} = 0,\ i = 1, \ldots, r\,, \tag{14}$$

one can define a family of factor-algebras

$$U(g\,|\,\lambda) = U(g)/\mathcal{J}(\lambda)\,. \tag{15}$$

These algebras are simple for "general position" points λ.

However, when $(\lambda_1, \ldots, \lambda_n)$ is a set of eigenvalues of the Casimir operators in some fin-dim. irrep. $D(\lambda)$ of g, the algebra $U(g\,|\,\lambda)$ has an ideal χ such that

$$\mathrm{Mat}_{\dim D(\lambda)} \simeq U(g\,|\,\lambda)/\chi \tag{16}$$

and for the Lie algebras

$$\mathrm{sl}_{\dim D(\lambda)} \simeq L(g\,|\,\lambda)/[\chi]\,. \tag{17}$$

The proof is the same as in the case of sl_2. Again one needs to consider elementary g-modules. There exists an elementary non-decomposable g-module containing the fin-dim. irreducible g-module with the eigenvalues $\{\lambda_i\}$ of the Casimir operators C_i. Again by the Burnside theorem there exists in $U(g\,|\,\lambda)$ an ideal χ such that the isomorphisms (16, 17) take place. Seemingly the ideal χ is a 0-space for a certain symmetric bilinear form on $U(g\,|\,\lambda)$ in the exceptional points.

Algebras $U(g\,|\,\lambda)$ can be interpreted as quantum associative operatorial algebras on the following algebraic manifolds. Let x_A $(A = 1,\ldots, n)$ be the coordinates in the n-dimensional flat space. We associate with g $(\dim g = n)$ a family of algebraic manifolds $M(g\,|\,\lambda)$ $(\dim = n - r)$ which are defined by the equations

$$C_i(x_A) = \lambda_i\,, \quad i = 1,\ldots, r \tag{18}$$

($M(\mathrm{so}_3\,|\,\lambda)$ is a sphere S^2 of radius $\sqrt{\lambda}$). The Poisson bracket is defined by $[f, g]_{\mathrm{PB}} = \partial f/\partial x_A\, f^C_{AB} x_C\, \partial g/\partial x_B$. The quantizaton on $M(g\,|\,\lambda)$ in this case is performed by $x_A \to \hat{x}_A$, where now $[\hat{x}_A, \hat{x}_B] = i\hbar f^C_{AB}\, \hat{x}_C$ and Eq. (18) are satisfied in the operatorial sense. In this case we have an isomorphism $U(g\,|\,\lambda) \simeq \mathrm{aq}(M(g\,|\,\lambda))$ ($\mathrm{aq}(M)$ is an associative operatorial algebra on M). One can perform a transition to the classical limit $\hbar \to 0$ $(\lambda_i \to \infty)$ and

$$\mathcal{A}(M) \simeq \lim_{\lambda_i \to \infty} U(g\,|\,\{\lambda_i\}), \quad \mathrm{PB}(M) \simeq \lim_{\lambda_i \to \infty} [U(g\,|\,\{\lambda_i\})]$$

similarly to the case of sl_2.

We want to point out that the compact six-dim. manifold $M(\mathrm{SU}(3))$ may be of interest for the compactification in $D = 10$ supergravity (and superstrings) to $D = 4$[a]: $\mathbb{R}^{10} \xrightarrow{?} M(\mathrm{SU}(3)) \times \mathbb{R}^4$ (or Ad S^4). $M(\mathrm{SU}(3))$ is a submanifold of seven-dim. sphere S^7 defined by the equations $(x_A, A = 1,\ldots, 8)$ $x_A x_A = R^2$, $d_{ABC}\, x_A x_B x_C = \rho^3$, where the completely symmetric coefficients defining the third order symmetrized Casimir operator of SU(3) are $d_{ABC} = \mathrm{tr}\,(\lambda_{(A}\lambda_B\lambda_{C)})$, where λ_A are the Gell-Mann matrices. In general the compact manifolds $M(g)$ associated with the compact forms of semisimple Lie groups defined by Eq. (18) may have interesting geometrical properties (note that $M(g)$ is an $(n - r)$-dim. submanifold of the sphere S^{n-1}). For example the manifolds $M(\mathrm{SU}_n)$ $(\dim = n^2 - n)$ are defined by the equations

$$\mathrm{tr}\,(X^m) = (R_m)^m, \quad m = 2,\ldots, n, \quad X = x^A \Gamma_A\,, \tag{19}$$

[a] It seems to us that in $D = 10$ superstrings there exists a possible interesting two-stage compactification. The first state reduces R^{10} to Ad $S_4 \times M(\mathrm{SU}(3))$; the cosmological constant of anti-de-Sitter $D = 4$ universe is related with parameters R and ρ defining $M(\mathrm{SU}(3))$. The second stage is a "flat limit" leading to flat Minkowski space and a certain Ricci-flat internal Kähler manifold (Calabi-Yau space). It is actually a phase transition in the string theory.

The manifold $M(\mathrm{SU}(3))$ itself is Kählerian, non-Ricci-flat and has SU(3) as an isometry group. It is an orbit in the adjoint representation Adg of SU(3) in its Lie algebra; we hope to return to this question in a separate publication.

where Γ_A are Hermitian SU_n generators.

4. Analytic Continuation

So far we have considered only first-class higher-spin algebras. The Virasoro algebra (centreless) which is second-class can be obtained from sl_2 by an analytic continuation in the following sense. sl_2 commutation relations $[L_n, L_m] = (m-n) \times L_{n+m}$ (where $\{L_n, |n| \leq 1\}$ are sl_2 generators), after the restriction $|n| \leq 1$ was revoked, would define the Virasoro commutation relations. It is of importance that the Virasoro structure constants are the same as for sl_2; only the regions of definition of the parameters are different. A similar procedure done by us as an analytic continuation has been described in Ref. 1 for the arbitrary semisimple fin-dim. g.

In principle, similar construction of the analytic continuation can be applied to the inf-dim. higher-spin algebras. When the higher-spin algebra is classical (and can be realized by the usual Poisson brackets), this procedure holds without any modification (all the semisimple fin-dim. algebras can be realized by the usual Poisson brackets). To illustrate, let us consider the simplest higher-spin algebra

$$[T_n^s, T_{n'}^{s'}] = (sn' - s'n)\, T_{n+n'}^{s+s'-1}, \tag{20}$$

where $|n| \leq s$, $|n'| \leq s'$. It can be obtained as a classical contraction of (13) when $\tilde{T}_n^s \to q^{-s+1}\tilde{T}_n^s$ and $q \to \infty$ in the new commutation relations. After a revocation of the restrictions $|n| \leq s$ in T_n^s we obtain a classical higher-spin generalization of Vir (see Refs. 7 and 8 for the super-extensions, realizations and BRST formulation for this algebra and[20] for the connection with W_N-algebras[18,19]).

However, in the case of the quantum higher-spin algebras $L(sl_2|\lambda)$ some modifications are necessary. Above all, to revoke the conditions $|m| \leq s$, $|m'| \leq s'$ in (13) we redefine the generators so that the redefined Clebsh-Gordan coefficients in (13) become polynomial functions on n and n':

$$\begin{aligned}
\tilde{C}_{nn'n''}^{ss's''} &= \left[\frac{(s+n)!\,(s-n)!\,(s'+n')!\,(s'-n')!}{(s''+n'')!\,(s''-n'')!}\right]^{\frac{1}{2}} C_{nn'n''}^{ss's''} \\
&= \sqrt{(2s''+1)}\; \Delta(s,s',s'') \sum_{t_1+t_2=s+s'-s''} \frac{(-1)^{t_1}}{t_1!\,t_2!} \\
&\quad \times \left\{\prod_{p_1=0}^{t_1-1} (s-n-p_1)(s'+n'-p_1)\right\} \left\{\prod_{p_2=0}^{t_2-1} (s+n-p_2)(s'-n'-p_2)\right\}, \\
&\prod_{p=0}^{-1} (\ldots) = 1\,.
\end{aligned} \tag{21}$$

However, we cannot revoke the restrictions right now because the Jacobi identities for the new generators T_n^s, $|n| > s$ turn out not to be satisfied. To restore the Jacobi identities for all the generators we need to add on the right-hand side of (13) certain suitable additional terms, an "analytic continuation of zero", which would be equal to zero in the original domain $|n| \leq s$:

$$[T^s_n, T^{s'}_{n'}] = 2\sum_s p(s+s'+s''+1)\begin{bmatrix} s & s' & s'' \\ \rho & \rho & \rho \end{bmatrix}\sqrt{(2s+1)(2s'+1)}$$
$$\times \tilde{C}^{ss's''}_{nn'n''} T^{s''}_{n''} + \left\{\prod_{p=-s}^{s}(n+p)\right\}(\ldots)$$
$$+ \left\{\prod_{q=-s'}^{s'}(n'+q)\right\}(\ldots), \qquad (22)$$

where ... are additional terms which can be calculated from the Jacobi identities. In this way we get an analytic continuation, $AC(L(sl_2|\lambda))$. However, the additional terms strongly change the structure of the algebra. In particular, the generators T^s_n, $n \in \mathbb{Z}$ of $AC(L(sl_2|\lambda))$ are not primary fields under the Virasoro subalgebra T^1_n but primary only under the little conformal algebra so(2, 1). Also at the exceptional points $\lambda = -j(j+1), j = 0, 1/2, 1, \ldots$ it turns out that an analytic continuation of the ideal χ is no longer ideal in $AC(L(sl_2|\lambda))$ and one cannot directly pass to a linear factor-algebra. However, the generators T^s_n with $s \le 2j$ in this case form a nonlinear W_N $(N = 2j + 1)$ algebra (this is our conjecture and we have no proof yet). It is of interest that in the subalgebra of the W_N algebra formed by T^s_n with $|n| \le s$ there exists an ideal χ such that the factor-algebra formed by T^s_n with $s \le N-1$ and $|n| \le s$ is isomorphic to sl_N (with the Racah commutation relations). To illustrate, when $j = 1$ $(N = 3)$ we have some first commutation relations of $L(sl_2|-2)$:

$$[T^1_n, T^s_m] = (m - sn)\, T^s_{n+m},$$
$$[T^2_n, T^2_m] = (m-n)\, T^3_{n+m} + (m-n)(2n^2 + 2m^2 - nm - 8)\, T^1_{n+m} \qquad (23)$$

(here $|n| \le s$ in T^s_n). Setting $T^3_n = 0$, we obtain the Racah commutation relations for sl_3 (8 generators T^1_n, T^2_n). To perform an analytic continuation in (44) to $n \in \mathbb{Z}$ one has to add in $[T^1_n, T^3_m]$ an additional term $n(n^2-1)T^1_{n+m}$, and the generators $T^3_n (m \in \mathbb{Z})$ are not primary under the Virasoro (but primary under so(2,1)). The reader can recognize in (22) (with the additional terms) the commutation relations for W_3.[18] Similarly from $L(sl_2|-j(j+1))$ for other $j = (N-1)/2$ one can obtain the commutation relations for W_N. In the algebras $AC(L(sl_2|\lambda))$ one can pass to a limit $\lambda \to \infty$, which will be a limit $N \to \infty$ for W_N (and SL(N)). In this limit one obtains an algebra, $AC(PB(S^{1,1})/(\text{constant functions}))$. In this way, in our opinion, the limit $N \to \infty$ for W_N is an analytic continuation of the corresponding procedure of Ref. 17 for SU(N). The classical algebra (22) may be viewed as a contraction of $AC(PB(S^{1,1})/R1)$. It should be mentioned that recently the limit $N \to \infty$ in W_N has been considered in several papers.[20] As we mentioned above, in this limit one can obtain either an analytic continuation of the Poisson algebra on the pseudo-sphere or its contraction (22). However, besides these algebras there exists a whole family $AC(L(sl_2|\lambda))$ of higher-spin generalizations of the Virasoro algebra, parametrized by the eigenvalue of the sl_2 Casimir operator λ. With each of these algebras a certain new model can be connected. In the exceptional points $\lambda = -j(j+1)$ these are the Toda field theories. It is interesting that this family of models are analytic continuations of the Toda models for arbitrary real (or complex) λ. The Wilson OPE's for them will be given by $AC(L(sl_2|\lambda))$ and will involve an infinite tower of fields of all spins to infinity. Only in the exceptional points may certain truncation be performed which leads to W_N algebras and the Toda theories.

5. Conclusion

Here we shall discuss briefly some applications of the algebras considered above in the higher-spin theory. It seems natural to choose the universal enveloping algebras $U(\mathrm{so}(3,2))$ and $U(\mathrm{so}(4,2))$ (exactly corresponding Lie algebras and their supersymmetric extensions) as the global higher-spin algebras to construct[3–5] $\mathrm{Ad}S_4$ and conformal[6,9] gauge higher-spin theories generalizing $\mathrm{Ad}S_4$ and conformal supergravities respectively. The spectrum of the gauge fields corresponding to these algebras is very rich. It involves all higher spins with infinite multiplicities as well as a number of auxiliary fields. If such algebras may be localized, i.e., a self-consistent higher-spin dynamics based on them actually exists, then the problem of spontaneous higher-spin symmetry breaking comes into being. In such a theory it may proceed in several stages. The first one may reduce the symmetry algebra $L(\mathrm{so}(3,2))$ (here we shall discuss the $\mathrm{Ad}S_4$ case) to its factor-algebra $L(\mathrm{so}(3,2)\,|\,\lambda_1,\lambda_2)$ defined by the equations $C_1=\lambda_1$, $C_2=\lambda_2$, where C_1 and C_2 are the second and third order Casimir operators for so(3, 2). The physical interpretation of such a symmetry breaking might be the following. In the theory a family of so(3, 2)-invariant vacua parametrized by the values λ_1, λ_2 of the so(3, 2) Casimir operators may exist (each vacuum is a certain so(3, 2) representation characterized by λ_1, λ_2); then over each vacuum the symmetry algebra is a representation of $L(\mathrm{so}(3,2))$ with the fixed values of C_1, C_2, i.e., the factor-algebra $U(\mathrm{so}(3,2)\,|\,\lambda_1,\lambda_2)$. The gauge fields corresponding to the ideal $\mathcal{I}(\mathrm{so}(3,2)\,|\,\lambda_1,\lambda_2)$ should become massive in this picture.

The coupling constants for the interactions of massless higher-spin fields defined by $L(\mathrm{so}(3,2)\,|\,\lambda_1,\lambda_2)$ structure constants will manifestly depend on the parameters λ_1, λ_2 characterizing the vacuum. However, in theories with the global algebra $L(\mathrm{so}(3,2)\,|\,\lambda_1,\lambda_2)$ there is still an infinite tower of massless higher-spin fields. The second stage of symmetry breaking should reduce the symmetry algebras to fin-dim. gravity algebra, and only spins 2, 3/2,... (in the supersymmetric case) should remain massless. But there is an interesting intermediate possibility. As we discussed above, the algebras $L(\mathrm{so}(3,2)\,|\,\lambda_1,\lambda_2)$ for the exceptional points of (λ_1,λ_2) (which are eigenvalues of C_1, C_2 in the fin-dim. so(3, 2) irreps.) are not simple; there exists an ideal χ such that one can pass to a fin-dim. factor-algebra (35), (36). The remaining algebras sl_N ($N=\dim D(\lambda_1,\lambda_2)$) may be interpreted as fin-dim. higher-spin algebras in this context. They contain so(3, 2) as a subalgebra and also involve higher-spin so(3, 2) representations. It looks very attractive that in the exceptional points there is a possibility of such spontaneous symmetry breaking so that in the resulting theory only the fields corresponding to these fin-dim. factor-algebras remain massless. The highest massless spin $S_{\max}$ would be defined by $N(\lambda_1,\lambda_2)=\dim D(\lambda_1,\lambda_2)$. In this way, along with the theories with massless fields with spins 2, 3/2,... and massive higher spins and the theories with an infinite tower of massless higher spins (in $\mathrm{Ad}S_4$) might exist also a family of intermediate theories with massless spins 3, 5/2, 2,..., with 4, 7/2, 3,... and so on (apart from the fin-dim. massless sector there is also an infinite tower of massive fields). The massless truncation of these theories (when massive fields are not taken into account) might be considered as step-by-step approximations to the complete theory. It looks natural that such approximations would be finite up to n loops, where n depends linearly on $S_{\max}$. For example, the AdS_4 supergravity finite up to two loops would be the first step, the theory with spins 3, 5/2, 2,... finite (possibly) up to four loops would be the second step and so on.

It should also be mentioned that in these hypothetic theories the upper limit on the degree of supersymmetry is increased in comparison with supergravity ($N \leq 8$, or $N \leq 4$ in the conformal case). In this way, in principle the theory with massless spins 3, 5/2, 2,... (in AdS_4 background or conformally-invariant) might include the standard model group (which is impossible within the boundaries of supergravity).

Note also that the possibility of constructing gauge theories in lower dimensions ($D = 3$ Chern-Simons theories) involving a finite number of higher spins in a similar context was discussed in Ref. 14.

Appendix

$U(\mathrm{sl}_2 | \lambda)$ has an ideal χ when the eigenvalue λ of the Casimir element coincides with the eigenvalue $-j(j+1)$ $(2j = 0, 1, 2, \ldots)$ in some fin-dim. irrep. $D(2j)$. Here we demonstrate in a straightforward way from the structure constants of $U(\mathrm{sl}_2 | \lambda)$ in Eq. (11) that the generators $\tilde{T}^s_m$ with $s \geq 2j + 1$ ($2j$ is an integer) form an ideal χ and one can pass to a factor-algebra (2) formed by $\tilde{T}^s_m$ with $s \leq 2j$. The property χ, being an ideal $\chi U \subset \chi$, is equivalent to the following condition for the generalized $6j$-symbols involved in the structure constants,

$$\begin{bmatrix} s & s' & s'' \\ j & j & j \end{bmatrix} = 0 \tag{A1}$$

for

$$s \geq 2j + 1, \quad \forall\, s', \text{ and } s'' \leq 2j \tag{A2}$$

(generalized $6j$-symbols are symmetric under the interchange of the first and second columns with s and s'). It means that in the decomposition of the product $\tilde{T}^s_m \tilde{T}^{s'}_{m'}$ where spin $s \geq 2j + 1$ there appear only generators with spins $s'' \geq 2j + 1$ also. The property (A1), (A2) follows directly from the expression (10). Indeed, the factor $[t!\ldots]^{-1}$ is different from zero if and only if the following conditions are satisfied,

$$s - s' \leq t \leq s, \quad s' - s'' \leq t \leq s',$$
$$t \geq 0, \quad t \leq s + s' - s'' \tag{A3}$$

or

$$t \leq s + s' - 2j - 1\,.$$

The second factor $\prod_{p=1}^{f+s'-s''-t} (2j + 1 - s'' - p)$ is equal to zero if and only if $s + s' - s'' - t \geq 2j + 1 - s'' \geq 1$.

However, the last condition follows from $t \leq s'$ (A3) and $s \geq 2j + 1$ (A2). In this way each term in the sum in the definition of generalized $6j$-symbols is (10) equal to zero for s, s' and s'' lying in the domain (A2), and hence (A1), (A2) takes place.

References

1. E. S. Fradkin and V. Ya. Linetsky, *Mod. Phys. Lett.* **A5** (1990) 1167; talks at the *XVIII Colloq. on Group Theoretical Methods in Physics*, June, 1990, Moscow.
2. E. S. Fradkin and M. Ya. Palchik, preprint Helsinki HU-TFT-90-11.
3. E. S. Fradkin and M. A. Vasiliev, *Ann. Phys. (N. Y.)* **117** (1987) 63.
4. E. S. Fradkin and M. A. Vasiliev, *Phys. Lett.* **B189** (1987) 89; *Nucl. Phys.* **B291** (1987) 141; *Int. J. Mod. Phys.* **A3** (1988) 2983; M. A. Vasiliev, *Ann. Phys. (N. Y.)* **190** (1989) 59; *Nucl. Phys.* **B324** (1989) 503.
5. M. A. Vasiliev, *Fortschr. Phys.* **35** (1987) 741.
6. E. S. Fradkin and V. Ya. Linetsky, *Ann. Phys. (N. Y.)* **198** (1990) 252; *Dok. Akad. Nauk SSSR* **331** (1990) 84.
7. E. S. Fradkin and V. Ya. Linetsky, *Mod. Phys. Lett.* **A4** (1989) 2649 (preprint Tokyo INS-Rep-783 (Oct., 89)).
8. E. S. Fradkin and V. Ya. Linetsky, *Higher spin symmetry in lower dimensions*, preprint ICTP/90/51, *Proc. Trieste Conf. on Supermembranes and Physics in 2 + 1 Dimensions* (World Scientific, 1990).
9. E. S. Fradkin and V. Ya. Linetsky, *Phys. Lett.* **B231** (1989) 97; *Superconformal higher spin theory in the cubic approximation*, Göteborg preprint 90–9, submitted to *Nucl. Phys.*; *Mod. Phys. Lett.* **A4** (1989) 2363; 2635; *Ann. Phys. (N. Y.)* **198** (1990) 293.
10. J. Dixmier, *Enveloping Algebras* (North-Holland, 1977).
11. M. Bordemann, J. Hoppe, and P. Schaller, *Phys. Lett.* **B232** (1989) 199.
12. E. Bergshoeff, M. P. Blencowe, and K. Stelle, Imperial College preprint/TH/88-89/9.
13. A. U. Klimyk, *Matrix Elements and the Clebsh-Gordan Coefficients of the Group Representations* (Kiev, 1979, in Russian); V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov, *Lect. Notes in Phys.* **63** (1977).
14. M. A. Vasiliev, *Pis'ma Zh. Eksp. Teor. Fiz.* **50** (1989) 344; preprint FIAN 214, Moscow, 1989.
15. G. Racah, *Phys. Rev.* **61** (1942) 186; **62** (1942) 438; **63** (1943) 438; **76** (1949) 1352.
16. L. C. Biedencharn and J. D. Louck, *Angular Momentum in Quantum Physics* (Addison-Wesley, 1981).
17. J. Hoppe, Ph. D. Thesis MIT (1982); *Florence '86 Workshop on Constraints Theory and Relativistic Dynamics*, eds. G. Longhi and L. Lusanna (World Scientific, 1987).
18. A. B. Zamolodchikov, *Theor. Math. Phys.* **65** (1986) 1205; V. A. Fateev and A. B. Zamolodchikov, *Nucl. Phys.* **B280** [FS18] (1987) 644.
19. A. Bilal, preprints CERN-TH.5567/89, 5493/89; E. Gremer and J. -L. Gervais, preprint LPTENS 89/19; A. Bilal and J. -L. Gervais, *Nucl. Phys.* **B318** (1989) 579; K. Schoutens, A. Sevrin, and P. van Nieuwenhuizen, preprint THU-88/44, ITP-SB-88-70.
20. I. Bakas, *Phys. Lett.* **B228** (1989) 57; A. Bilal, *Phys. Lett.* **B227** (1989) 406.