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A Superconformal Theory of Massless Higher Spin Fields in $D=2+1$

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We construct a superconformal theory of higher spin fields in a space-time of dimension $D=2+1$. The construction relies on the infinite-dimensional superalgebra shsc$(N|3)$ with the superconformal algebra osp$(N|4)$ as a maximal finite-dimensional subalgebra.

The invariant Chern–Simons action for the higher spin superconformal theory is an extension of the usual conformal supergravity action for particles with maximal spin two. The quantization was carried out and the generating functional of the Green functions was obtained.

1. INTRODUCTION

In recent works [1–8], new infinite-dimensional Lie superalgebras have been constructed which extended the usual supergravity superalgebra for the anti-de Sitter space adS_4, and as such can allow one to construct interacting higher spin fields in adS_4.

The most transparent way to construct these superalgebras is achieved when one uses an operator realization thereof in terms of arbitrary order polynomials in the Heisenberg operators viewed as generating elements [4]. This is an extension of the usual presentation [9] for the finite-dimensional superalgebra in terms of the polynomials of order $\leqslant 2$. The $D=4$ action can be written down as a generalized MacDowell–Mansouri functional [10]. The interaction of higher spin gauge fields is non-analytical in the cosmological constant which plays the role of an independent dimensionful parameter of the theory, thus allowing the inclusion of new terms, of higher order in fields and their derivatives, into the action and the field transformation laws.

This may lead to an infinite rank theory, in the terminology of Refs. [11, 12]. The proof of the existence of a consistent interaction to all orders appears as a highly non-trivial problem. Consistency of the cubic interactions was shown in Refs. [5, 6]. The essential non-analyticity precludes one from a transition to the flat limit, which rules out the possibility of constructing an analogous theory in a flat

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background. The situation is quite different, however, whithin the conformal-invariant approach to the higher spin theory.

In the conformal invariant theory of spin s, the kinetic terms are $\phi_s \Box^s P_s \phi_s$ (Bose) and $\psi_s \Box^{s-1/2} \partial\!\!\!/ P_s \psi_s$ (Fermi) [13], involving higher order derivatives. The dimensions of the fields differ from those in the usual Poincaré-invariant theory. Conformal invariance implies the absence of dimensionful constants, which considerably restricts the possible form of the interaction. Let us note that the existence of higher spin conformal invariant theory is not ruled out by the results of Refs. [14–17]. In these works the incompatibility was shown of the higher spin gauge invariance with the interaction with gravity. The mean reasoning invoked the appearance in the gauge variation of the higher spin action of the terms proportional to the Weyl tensor. These terms cannot be cancelled by adding new terms into the metric transformation law. For the conformal theory, the terms involving the Weyl tensors can be eliminated by corresponding alterations of the metric transformation law, thus by-passing the no-go statements of the above-mentioned works. To construct a complete theory, one is first of all to construct a superalgebra which would be a higher spin generalization of the conformal superalgebra SU(2, 2 | 1). Such a superalgebra, shsc$^\infty$(4 | 1), has been obtained by us, and its form in four space-time dimensions will be published elsewhere [26]. In the present work we construct the infinite-dimensional superalgebras shsc(N | 3) (super higher spin conformal) which extends the superconformal algebra in $D=2+1$. Conformal superalgebras in $D=2+1$ and the adS$_4$-superalgebras are well known to be isomorphic. Analogous to that, it turns out that shsc(N | 3) is isomorphic to shs(N | 4), and our generators and those of Ref. [4] differ by a choice of basis in the spaces of irreducible representations of so(3, 2).

Briefly, the program of constructing the infinite-dimensional conformal superalgebras and the gauge theory for higher-spins is as follows:

(1) A suitable operational realization of the finite-dimensional conformal superalgebra is to be choosen.

(2) An infinite-dimensional associative algebra of all orders polynomials of the generating elements choosen in step 1, is to be constructed and the associative multiplication in conformal basis is calculated.

This will be the basis of constructing the conformal infinite-dimensional Lie superalgebra and its localization.

The aim of this article is to realize this program of constructing the global superalgebra shsc(N | 3) and its localization to obtain the gauge theory of conformal higher-spins in $D=2+1$.

The action invariant under shsc(N | 3) is written as the Chern–Simons functional, and the equations of motion have the form $R=0$. They generalize the equations of motion of the usual conformal supergravity in $D=2+1$. It is noteworthy that our superconformal theory for higher spins is constructed with allowance made for all spins[1] $s \geq 1$, and only in the case of spins does not exceed 2 is the finite-dimensional version realized (the usual conformal supergravity).

The three-dimensional conformal theory of higher spins may be of interest both from the point of view of the preparation of a four-dimensional conformal theory and from the point of view of the theory of higher spins in adS_4.

Another possible application of the proposed theory is in the spin membrane model [18]. In the paper [18] it is shown that a (Minkowski) space-time Lorentz covariant spinning membrance action cannot be constructed withing the framework of the three-dimensional super-Poincaré tensor calculus. The $\mathrm{shsc}(N|3)$-theory may prove to be helpful for constructing such a formulation. The higher-spin models in $D=2+1$ may be useful for covariant quantization of the super membrane models.

The paper is organized as follows.

In Section 2, an operator realization of the $D=2+1$ conformal superalgebra $\mathrm{osp}(N|4)$ is given in terms of first- and second-order polynomials by the Heisenberg generating elements.

In Section 3, the N-extended $D=2+1$ conformal supergravity in the two-component spinor notations is considered.

In Section 4, operator realizations of higher spin representations of so(3, 2) are obtained in terms of the Weyl symbols.

In Section 5, conformal higher spin superalgebras $\mathrm{shsc}(N|3)$ are constructed. The gauge fields of $\mathrm{shsc}(N|3)$ are introduced and the curvatures of $\mathrm{shsc}(N|3)$ are calculated.

In Section 6, the action of the $\mathrm{shsc}(N|3)$-invariant gauge theory and the equations of motion for the conformal higher spin fields are discussed.

In Section 7, quantization of the conformal higher spin theory in $D=2+1$ in the phase space and the configuration space is performed. Some appendixes are given at the end of the paper.

Appendix 1 includes the basic notations. Appendix 2 presents the necessary information on the algebra $aq(2; \mathbb{C})$ and superalgebra $\mathrm{shs}(1|2)$. The multiplication formulae in the weight (relative to $\mathrm{so}(2,1) \simeq \mathrm{sp}(2,\mathbb{R})$) and spinor bases are given. Appendix 3 includes all necessary formulae and relations with the Clebsch–Gordan coefficients and $9j$-coefficients in the weight and spinor basis. Appendix 4 presents several relations with Winger d-functions. Appendix 5 tabulates the multiplets of conformal higher-spin fields. Appendix 6 includes the analysis of the linearized conformal higher-spin equations.

2. Superconformal Algebra in $D=2+1$

The aim of the present paper is the generalization of the ordinary superconformal algebra and its localization in $D=2+1$ to the case of the higher spins and this will

[1] Possibly, there exist finite-dimensional "approximations." For example, $\lim_{M\to\infty} \mathrm{osp}(N|M) = \mathrm{osp}(N|\infty)$. It should be mentioned that the similar limits for classical Lie groups (SU, SO, Sp) are considered in homotopic topology in the context of the Bott periodicity theorems. See also for $\mathrm{SU}(\infty)$ [28].

be carried out using the method of operator realization. In this connection we consider it reasonable to first demonstrate the efficiency of this method on the example of the ordinary conformal algebra, the structure of which is described by the following non-vanishing commutators:

$$[M_{ab}, M_{cd}] = \eta_{ac} M_{bd} - \eta_{bc} M_{ad} + \eta_{ad} M_{cb} - \eta_{bd} M_{ca}, \tag{2.1a}$$

$$[M_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a, \tag{2.1b}$$

$$[M_{ab}, K_c] = \eta_{ac} K_b - \eta_{bc} K_a, \tag{2.1c}$$

$$[D, P_a] = -P_a, \; [D, K_a] = K_a, \tag{2.1d}$$

$$[P_a, K_b] = 2(\eta_{ab} D + M_{ab}), \tag{2.1e}$$

where (M, P, K, D) are generators of conformal transformations (Lorentz transformations, translations, special conformal transformations, dilatations).

To extend the conformal algebra to the higher-spin case, it is convenient to go over to the two-component so(2, 1)-spinors:

$$P_a = \sigma_a{}^{\alpha(2)} P_{\alpha(2)}, \qquad K_a = \sigma_a{}^{\alpha(2)} K_{\alpha(2)}, \qquad M_{ab} = \tfrac{1}{2} \sigma_{ab}^{\alpha(2)} M_{\alpha(2)}, \tag{2.2}$$

where the $\sigma_a^{\alpha(2)} = (I, \sigma_1, \sigma_3)$, $\sigma_{1,3}$ are Pauli matrices and

$$\sigma_{ab}{}^{\alpha(2)} = \sigma_a{}^{\alpha}{}_{\gamma} \sigma_b{}^{\alpha\gamma}. \tag{2.3}$$

The matrices σ satisfy the relations

$$\sigma_a{}^{\alpha(2)} \sigma^b{}_{\alpha(2)} = 2\delta^b_a, \qquad \sigma_a{}^{\alpha(2)} \sigma^a{}_{\gamma(2)} = 2\delta^{\alpha(2)}_{\gamma(2)}, \tag{2.4}$$

$$\tfrac{1}{2} \sigma_{[a}{}^{\alpha(2)} \sigma_{b]}{}^{\gamma(2)} = \mathscr{E}^{\alpha\gamma} \sigma_{ab}{}^{\alpha\gamma} \qquad ([ab] = ab - ba) \tag{2.5}$$

(for the notation and the rules of handling the indices see Appendix 1).

In the representation for the so(2, 1) $\simeq$ sp(2; $\mathbb{R}$)-spinors the relations (2.1a–e) become[2]

$$[M_{\alpha(2)}, M_{\beta(2)}] = 2\mathscr{E}_{\alpha\beta} M_{\alpha\beta}, \tag{2.6a}$$

$$[M_{\alpha(2)}, P_{\beta(2)}] = 2\mathscr{E}_{\alpha\beta} P_{\alpha\beta}, \tag{2.6b}$$

$$[M_{\alpha(2)}, K_{\beta(2)}] = 2\mathscr{E}_{\alpha\beta} K_{\alpha\beta}, \tag{2.6c}$$

$$[D, P_{\alpha(2)}] = -P_{\alpha(2)}, \qquad [D, K_{\alpha(2)}] = K_{\alpha(2)}, \tag{2.6d}$$

$$[P_{\alpha(2)}, K_{\beta(2)}] = \mathscr{E}_{\alpha\beta} M_{\alpha\beta} + \mathscr{E}_{\alpha\beta} \mathscr{E}_{\alpha\beta} D. \tag{2.6e}$$

The algebra so(3, 2) $\simeq$ sp(4; $\mathbb{R}$) (2.6a–e) admits a simple operator realization.

Let a_α, b_α form the Heisenberg algebra

$$[a_\alpha, b_\beta] = 2i\mathscr{E}_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = [b_\alpha, b_\beta] = 0, \tag{2.7}$$

$$a^+_\alpha = a_\alpha, \qquad b^+_\alpha = b_\alpha. \tag{2.8}$$

[2] In our notations (see Appendix 1) $\mathscr{E}_{\alpha\beta} \mathscr{E}_{\alpha\beta} = \frac{1}{2}(\mathscr{E}_{\alpha_1\beta_1} \mathscr{E}_{\alpha_2\beta_2} + \mathscr{E}_{\alpha_2\beta_1} \mathscr{E}_{\alpha_1\beta_2})$.

Then one can easily verify that the operators so(3, 2) are

$$M_{\alpha(2)}=\frac{1}{4i}(a_\alpha b_\alpha+b_\alpha a_\alpha), \tag{2.9a}$$

$$P_{\alpha(2)}=\frac{1}{4i}a_\alpha a_\alpha, \qquad K_{\alpha(2)}=\frac{1}{4i}b_\alpha b_\alpha, \tag{2.9b}$$

$$D=\frac{1}{8i}(a_\alpha b^\alpha+b^a a_\alpha). \tag{2.9c}$$

As is known, the polynomials quadratic in the generating elements of the Heisenberg algebra form a symplectic algebra. The relations (2.9a–c) give the usual operator realization of sp(4; ℝ) in the basis convenient for our purposes. Consider the operator realization of the supersymmetric extension of the so(3, 2)-superalgebra osp(N|4). To realize osp(1|4), one should add to the operators (2.9a–c) the operators

$$Q_\alpha=\tfrac{1}{2}a_\alpha, \qquad S_\alpha=\tfrac{1}{2}b_\alpha. \tag{2.10}$$

The commutation relations osp(1|4) include, along with (2.6a–c), also

$$\{Q_\alpha, Q_\gamma\}=2iP_{\alpha\gamma}, \qquad \{S_\alpha, S_\gamma\}=2iK_{\alpha\gamma}, \tag{2.11a}$$

$$\{Q_\alpha, S_\gamma\}=i(M_{\alpha\gamma}+\mathscr{E}_{\alpha\gamma}D), \tag{2.11b}$$

$$[D, Q_\alpha]=-\tfrac{1}{2}Q_\alpha, \qquad [D, S_\alpha]=\tfrac{1}{2}S_\alpha, \tag{2.11c}$$

$$[M_{\alpha(2)}, Q_\gamma]=\mathscr{E}_{\alpha\gamma}Q_\alpha, \qquad [M_{\alpha(2)}, S_\gamma]=\mathscr{E}_{\alpha\gamma}S_\alpha, \tag{2.11d}$$

$$[P_{\alpha(2)}, S_\gamma]=\mathscr{E}_{\alpha\gamma}Q_\alpha, \qquad [K_{\alpha(2)}, Q_\gamma]=\mathscr{E}_{\alpha\gamma}S_\alpha. \tag{2.11e}$$

Here Q_α are generators of the supersymmetry, S_α are generators of the special conformal supersymmetry.

To obtain an extended supersymmetry osp(N|4), one should add the Clifford algebra generating elements to the (2.8–2.7):

$$\{\psi_i, \psi_j\}=2\delta_{ij}, \qquad [\psi, a]=[\psi, b]=0, \tag{2.12}$$

$$\psi_i^+=\psi_i, \qquad i, j=1, ..., N. \tag{2.13}$$

Then the generators

$$Q_{i\alpha}=\tfrac{1}{2}\psi_i a_\alpha, \qquad S_{i\alpha}=\tfrac{1}{2}\psi_i b_\alpha, \qquad T_{i(2)}=\tfrac{1}{4}\psi_i\psi_i \tag{2.14}$$

along with (2.9a–c) form a osp(N|4):

$$\{Q_{i\alpha}, Q_{j\beta}\}=2i\delta_{ij}P_{\alpha\beta}, \tag{2.15a}$$

$$\{S_{i\alpha}, S_{j\beta}\}=2i\delta_{ij}K_{\alpha\beta}, \tag{2.15b}$$

$$\{Q_{i\alpha}, S_{j\beta}\} = i\delta_{ij}(M_{\alpha\beta} + \mathscr{E}_{\alpha\beta}D) + 2i\mathscr{E}_{\alpha\beta}T_{ij}, \tag{2.15c}$$

$$[T_{i_1i_2}, Q_{j\beta}] = \tfrac{1}{2}(\delta_{i_2j}Q_{i_1\beta} - \delta_{i_1j}Q_{i_2\beta}), \tag{2.15d}$$

$$[T_{i_2i_2}, S_{j\alpha}] = \tfrac{1}{2}(\delta_{i_2j}S_{i_1\alpha} - \delta_{i_1j}S_{i_2\alpha}), \tag{2.15e}$$

$$[T_{i_1i_2}, T_{j_1j_2}] = \tfrac{1}{2}(\delta_{i_2j_1}T_{i_1j_2} + \delta_{i_1j_2}T_{i_2j_1} - \delta_{i_1j_1}T_{i_2j_2} - \delta_{i_2j_2}T_{i_1j_1}). \tag{2.15f}$$

The relations (2.11a–e) for $N > 1$ differ only by the appearance of the internal symmetry index in Q and S. The operators $T_{i(2)}$ form the basis in so(N). The formulas presented demonstrate the advantage of the spinor formalism, namely, the absence of γ-matrices in (2.15a–c) and a simple oscillator realization.

All the Bose generators are anti-Hermitian, whereas the Fermi generators are Hermitian (the super-anti-Hermitian basis [9])

$$A^+ = (-1)^{\pi(A)+1} A, \tag{2.16}$$

where $\pi(A)$ is the Grassmann parity of A.

3. Conformal Supergravity in $D = 2 + 1$

We present here the results of localization of a superconformal algebra (conformal supergravity) in a three-dimensional space-time. The gauge fields ω_μ of conformal supergravity are

$$\omega_\mu = e_\mu^{\alpha(2)}P_{\alpha(2)} + f_\mu^{\alpha(2)}K_{\alpha(2)} + \omega_\mu{}^{\alpha(2)}M_{\alpha(2)} + b_\mu D + A_\mu^{i(2)}T_{i(2)} + \psi_\mu^{i\alpha}Q_{i\alpha} + \phi_\mu^{i\alpha}S_{i\alpha}, \tag{3.1}$$

where (P, K, M, D, T, Q, S) are generators of the conformal group (2.9a–c, 2.14) and it is assumed that

$$\omega T = (-1)^{\pi(\omega)\pi(T)}\, T\omega \tag{3.2}$$

(the Grassmann shell of second class, see [19]). The fields ($e, f, \omega, b, A, \phi, \psi$) with usual statistic are, respectively, the dreibein, the connection for the conformal boosts, the Lorentz connection, the dilatation connection, the so(N) Yang-Mills field, the connection for the conformal supersymmetries, and the gravitino. These furnish the adjoint representation of osp($N|4$). All the fields are Hermitian, and

$$\omega_\mu^+ = -\omega_\mu \tag{3.3}$$

and the curvatures have the form:

$$R_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]. \tag{3.4}$$

In conformal supergravity in $D=2+1$ (according to (2.6a–e), (2.11a–e) and (2.15a–f)) the curvatures become

$$R_{\mu\nu\alpha(2)}(P)=\mathscr{D}_{[\mu}e_{\nu]\alpha(2)}-2i\psi_{\mu i\alpha}\psi_{\nu}{}^{i}{}_{\alpha}, \tag{3.5a}$$

$$R_{\mu\nu\alpha(2)}(K)=\mathscr{D}_{[\mu}f_{\nu]\alpha(2)}-2i\phi_{\mu i\alpha}\phi_{\nu}{}^{i}{}_{\alpha}, \tag{3.5b}$$

$$R_{\mu\nu\alpha(2)}(M)=\partial_{[\mu}\omega_{\nu]\alpha(2)}+2\omega_{\mu\alpha\gamma}\omega_{\nu\alpha}{}^{\gamma}+e_{[\mu\alpha\gamma}f_{\nu]\alpha}{}^{\gamma}-i\psi_{[\mu i\alpha}\phi_{\nu]}{}^{i}{}_{\alpha}, \tag{3.5c}$$

$$R_{\mu\nu}(D)=\partial_{[\mu}b_{\nu]}+e_{[\mu\alpha(2)}f_{\nu]}{}^{\alpha(2)}-i\psi_{[\mu i\alpha}\phi_{\nu]}{}^{i\alpha}, \tag{3.5d}$$

$$R_{\mu\nu i(2)}(T)=\partial_{[\mu}A_{\nu]i(2)}+2A_{\mu ik}A_{\nu}{}^{k}{}_{i}-2i\psi_{[\mu i\alpha}\phi_{\nu]i}{}^{\alpha}, \tag{3.5e}$$

$$R_{\mu\nu i\alpha}(Q)=\mathscr{D}_{[\mu}\psi_{\nu]i\alpha}+e_{[\mu\alpha\gamma}\phi_{\nu]i}{}^{\gamma}+A_{[\mu ik}\psi_{\nu]}{}^{k}{}_{i}, \tag{3.5f}$$

$$R_{\mu\nu i\alpha}(S)=\mathscr{D}_{[\mu}\phi_{\nu]i\alpha}+f_{[\mu\alpha\gamma}\psi_{\nu]i}{}^{\gamma}+A_{[\mu ik}\phi_{\nu]}{}^{k}{}_{i}, \tag{3.5g}$$

$$\mathscr{D}_{\mu}\omega^{c}_{\alpha(n)}=\partial_{\mu}\omega^{c}_{\alpha(n)}+n\omega_{\mu\alpha\gamma}\omega^{c}_{\alpha(n-1)}{}^{\gamma}+cb_{\mu}\omega^{c}_{\alpha(n)}, \tag{3.6}$$

where the antisymmetrization $[\cdot\cdot]$ is supposed only by μ, ν and c is the conformal weight of the generator T^{c},

$$[D,T^{c}]=cT^{c}, \tag{3.7}$$

that corresponds to the field ω^{c}. The conformal weights of (P, M, D, K, Q, S, T) are $(-1, 0, 0, 1, -\frac{1}{2}, \frac{1}{2}, 0)$, respectively. The action of conformal supergravity in $D=2+1$ can be written as the Chern–Simons functional

$$S=\int \operatorname{tr}(\omega\wedge d\omega+\tfrac{2}{3}\,\omega\wedge\omega\wedge\omega), \tag{3.8}$$

where $\omega=\omega_{\mu}dx^{\mu}$. The equations of motion of the action (3.8) are as follows

$$R_{\mu\nu\alpha(2)}(P)=0, \tag{3.9a}$$

$$R_{\mu\nu\alpha(2)}(K)=0, \tag{3.9b}$$

$$R_{\mu\nu\alpha(2)}(M)=0, \tag{3.9c}$$

$$R_{\mu\nu}(D)=0, \tag{3.9d}$$

$$R_{\mu\nu i(2)}(T)=0, \tag{3.9e}$$

$$R_{\mu\nu i\alpha}(Q)=0, \tag{3.9f}$$

$$R_{\mu\nu i\alpha}(S)=0. \tag{3.9g}$$

By virtue of the curvatures homogeneous transformation law, Eqs. (3.9) are invariant under the gauge transformations

$$\delta\omega_{\mu}{}^{A}=\partial_{\mu}\mathscr{E}^{A}+[\omega_{\mu},\mathscr{E}]^{A}. \tag{3.10}$$

The fields f, ω, and ϕ can be expressed through the independent fields e, b, ψ, and

A with the help of Eqs. (3.9a, c, f). The b field can be, as usual, put to zero by fixing a gauge which spoils the K-symmetry.

Let us calculate now the total number of degrees of freedom off the mass shell: n_{total} = (number of independent field components) − (dimension of the gauge group), to which the Fermi degrees of freedom contribute with the minus sign

$$\begin{aligned} n_{\text{total}} &= n(e) + n(b) + n(A) - n(\psi) - n(\mathscr{E}(P)) \\ &\quad - n(\mathscr{E}(M)) - n(\mathscr{E}(K)) - n(\mathscr{E}(D)) - n(\mathscr{E}(T)) \\ &\quad + n(\mathscr{E}(Q)) + n(\mathscr{E}(S)) = N^2 - 3N + 2, \end{aligned} \tag{3.11}$$

where $n(\cdot)$ denotes the number of components of the argument.

When $N=1$ and $N=2$ the total number of degrees of freedom is zero, that is, a necessary condition for the closure of the supersymmetry algebra is fulfilled.

4. Higher Spin Representations of the $D=2+1$ Conformal Superalgebra

In Section 2 we considered the realization of the conformal algebra by polynomials of second order in the Heisenberg algebra generating elements. The corresponding gauge fields were of spins $\leqslant 2$. Now we generalize this construction to the case of polynomials of arbitrary order in the generating elements (2.7, 12). Consider first the case with only a and b generating elements present. Note that our generating elements a and b are related to the generating elements q and r of Ref. [4] as

$$q_\alpha = \frac{1}{\sqrt{2}} (e^{i(\pi/4)} a_\alpha + e^{-i(\pi/4)} b_\alpha), \tag{4.1}$$

$$r_{\dot\alpha} = \frac{1}{\sqrt{2}} (e^{-i(\pi/4)} a_\alpha + e^{i(\pi/4)} b_\alpha), \tag{4.2}$$

$$q_\alpha^+ = r_{\dot\alpha}, \; [q_\alpha, q_\gamma] = 2i\mathscr{E}_{\alpha\gamma}, \; [r_{\dot\alpha}, r_{\dot\beta}] = 2i\mathscr{E}_{\dot\alpha\dot\beta}. \tag{4.3}$$

The order $2s$ polynomials in a and b (with $s = 0, \frac{1}{2}, 1, \ldots$) span the spaces of irreducible representations of so(3, 2). In these spaces we choose the conformal basis which is related to the reduction from the algebra to a subalgebra,

$$\text{so}(3,2) \to \text{so}(2,1) \oplus \text{so}(1,1). \tag{4.4}$$

This basis is suitable for constructing a conformal theory. The fields transformed according to the corresponding representations so(3, 2) have a manifest Lorentz structure in the three-dimensional space-time (so(2, 1)) and for each field its Weyl (conformal) weight (so(1, 1)) is determined.

Let now[3] a_α, b_α be Weyl symbols of the corresponding operators. The basis in a real linear space V of polynomials in a and b may be chosen in the form

$$T_{\alpha(2l),\beta(2l')} = \frac{1}{\sqrt{(2l)!\,(2l')!}}\, a_{\alpha(2l)} b_{\alpha(2l')}, \qquad l, l' = 0, \frac{1}{2}, 1, \ldots \tag{4.5}$$

The product of the Weyl symbols is given by the formula

$$A * B = A \exp(\overleftrightarrow{\Delta})\, B, \tag{4.6a}$$

where $\overleftrightarrow{\Delta}$ is a differential operator, acting simultaneously to the left and to the right, and

$$\overleftrightarrow{\Delta} = i\left(\frac{\overleftarrow{\partial}}{\partial a^\alpha}\frac{\overrightarrow{\partial}}{\partial b_\alpha} + \frac{\overleftarrow{\partial}}{\partial b^\alpha}\frac{\overrightarrow{\partial}}{\partial a_\alpha}\right). \tag{4.6b}$$

The complex algebra of polynomials in a and b with the multiplication (4.6) coincides with the algebra $aq(0|4; \mathbb{C})$ [4].

In the space V there acts the representation of the $D = 2+1$ conformal algebra (2.6a–e). The action of the generators in this representation is determined by

$$\tilde{T}(A) = T * A - A * T = [T, A]_*, \tag{4.7}$$

where T are symbols of the operators so(3, 2) (2.9a–c) and $A \in V$, i.e., $\tilde{T}(A)$ is the Weyl symbol of the commutator $[T, A]$.

One can easily verify that these generators of so(3, 2), in terms of the symbols are

$$\tilde{P}_{\alpha(2)} = -a_\alpha \frac{\partial}{\partial b^\alpha}, \tag{4.8a}$$

$$\tilde{K}_{\alpha(2)} = -b_\alpha \frac{\partial}{\partial a^\alpha}, \tag{4.8b}$$

$$\tilde{M}_{\alpha(2)} = -\left(a_\alpha \frac{\partial}{\partial a^\alpha} + b_\alpha \frac{\partial}{\partial b^\alpha}\right), \tag{4.8c}$$

$$\tilde{D} = \frac{1}{2}\left(b_\alpha \frac{\partial}{\partial b_\alpha} - a_\alpha \frac{\partial}{\partial a_\alpha}\right). \tag{4.8d}$$

In particular, it is seen directly that the operator $\tilde{D}$ is diagonal in the basis (4.4),

$$\tilde{D}(T_{\alpha(2l),\beta(2l')}) = (l' - l)\, T_{\alpha(2l),\beta(2l')}. \tag{4.9}$$

[3] Both the operators and their symbols are denoted by the same letters. This does not lead to misunderstanding since we henceforth work only with symbols.

The operators $\tilde{P}_{\alpha(2)}$ and $\tilde{K}_{\alpha(2)}$, respectively lower and raise the Weyl weight by unity. The basis (4.4) is not irreducible relative to the so(2, 1) subalgebra generated by the operators $\tilde{M}_{\alpha(2)}$.

To construct the conformal basis so(3, 2) → so(2, 1) ⊕ so(1, 1), it is necessary to expand (4.4) into irreducible multispinors. To this end, we shall use the formulae of Appendix 3.

The irreducible components (4.4) have the form

$$\hat{T}^{(l,l')}_{\alpha(2l'')}(b, a) = \bar{C}_{\alpha(2l''),}{}^{\beta(2l),\gamma(2l')} T_{\beta(2l),\gamma(2l')}(-1)^{l+l'-l''}, \tag{4.10}$$

where $\bar{C}$ are Clebsch–Gordan (C–G) coefficients from Appendix 3.

The conformal weight of the new basis element is

$$\tilde{D}(\hat{T}^{(l,l')}_{\alpha(2l'')}) = (l'-l)\, \hat{T}^{(l,l')}_{\alpha(2l'')}. \tag{4.11}$$

As a consequence of the triangle relation for the C–G coefficients

$$|l-l'| \leqslant l'' \leqslant l+l' \tag{4.12}$$

we have that the conformal weight $c = l'-l$ may run the values

$$c = -l'', \; -l''+1, ..., l''. \tag{4.13}$$

Let us write an explicit form of the basis elements in terms of a and b:

$$T^{(s,c)}_{\alpha(2l'')}(b, a) = \sqrt{\frac{(2l''+1)!}{(s-l'')!\,(l''+c)!\,(l''-c)!\,(s+l''+1)!}} \times a_{\alpha(l''-c)} b_{\alpha(l''+c)} (a^{\alpha} b_{\alpha})^{s-l''}, \tag{4.14}$$

where we have denoted $l+l' = s$, $l'-l = c$, and employed formulae (A.3.22, .23). The highest weight vectors relative to $\tilde{D}$ are

$$T^{(s,s)}_{\alpha(2l)}(b, a) = \frac{1}{\sqrt{(2s)!}}\, b_{\alpha(2s)}, \tag{4.15}$$

$$\tilde{D}(T^{(s,s)}_{\alpha(2s)}) = s T^{(s,s)}_{\alpha(2s)}. \tag{4.16}$$

Applying the operators $\tilde{P}_{\alpha(2)}$ to $T^{(s,s)}_{\alpha(2s)}(b, a)$, we obtain the complete basis of the irreducible representation space (irreps) so(3, 2) with dimension

$$d(s) = \frac{(2s+3)(2s+1)(s+1)}{3} \tag{4.17}$$

(dimension of the adjoint rep. is $d(1)=10$ and spinorial rep. is $d(\frac{1}{2})=4$). The generators

$$\{T^{(s,c)}_{\alpha(2l)};\ c=-s,\ -s+1,\ \ldots,\ s;\ l=|c|,\ \ldots,\ s\} \tag{4.18}$$

form the basis in this irreps.

Let us calculate the Weyl product of the two symbols (4.14). This calculation consists of two stages. We first make a convenient change of generators and then apply formulae of Appendix 3.

(1) Let us make an invertible change of generating elements

$$q_\alpha=\frac{1}{\sqrt{2}}(a_\alpha+b_\alpha), \tag{4.19a}$$

$$z_\alpha=\frac{1}{\sqrt{2}}(a_\alpha-b_\alpha) \tag{4.19b}$$

and inversely

$$a_\alpha=\frac{1}{\sqrt{2}}(q_\alpha+z_\alpha), \tag{4.20a}$$

$$b_\alpha=\frac{1}{\sqrt{2}}(q_\alpha-z_\alpha). \tag{4.20b}$$

The new generating elements satisfy the relations

$$q^+_\alpha=q_\alpha, \qquad z^+_\alpha=z_\alpha, \qquad [q_\alpha, q_\beta]_*=2i\mathscr{E}_{\alpha\beta}, \tag{4.21a}$$

$$[z_\alpha, z_\beta]_*=-2i\mathscr{E}_{\alpha\beta}, \qquad [q_\alpha, z_\beta]_*=0 \tag{4.21b}$$

(attention should be paid to opposite signs in the commutation relations for q and z). The change of generating elements (4.19a, b) gives rise to an invertible change of the basis in each of the so(3, 2) irrepses

$$T^{(s,c)}_{\alpha(2l)}(b,a)=\sum_{k=-l}^{l} d^l_{c,k}\left(\frac{\pi}{2}\right)T^{(s,k)}_{\alpha(2l)}(q,z), \tag{4.22}$$

$$T^{(s,k)}_{\alpha(2l)}(q,z)=\sum_{c=-l}^{l} d^l_{k,c}\left(-\frac{\pi}{2}\right)T^{(s,c)}_{\alpha(2l)}(b,a), \tag{4.23}$$

where $d^l_{c,k}(\pm\pi/2)$ are the particular values of the Wigner functions (see Appendix 4).

To prove (4.22), (4.23), it suffices to note that the change (4.19, 4.20) is a unitary transformation from the group SU(2)

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \pm 1\\ \pm 1 & 1\end{pmatrix}=\begin{pmatrix}\cos(\beta/2) & -\sin(\beta/2)\\ \sin(\beta/2) & \cos(\beta/2)\end{pmatrix}\Bigg|_{\beta=\pm\pi/2} \tag{4.24}$$

and the coefficients of the transformations (4.22, 23) are matrix elements of the representation of this matrix in $(2l+1)$-dimensional space $\{T^{(s,c)}_{\alpha(2l)}, c=-l, ..., l\}$ (s and l are fixed), i.e., $d^l_{c,k}$ is the Wigner function at the points $\beta=\pm\pi/2$. This, however, may also be obtained by explicit calculation by substituting (4.20a–b) into (4.14).

(2) The next step consists in calculating the product of the elements of the new basis $\{T^{(s,k)}_{\alpha(2l)}(q,z)\}$. The new basis has been made from (4.14) by the change $a\to z$, $b\to q$. This basis is connected with the decomposition $\mathrm{so}(3,2)\to\mathrm{so}(2,1)\oplus\mathrm{so}(2,1)\to\mathrm{so}(2,1)$. Indeed, the elements $T^{(s,k)}_{\alpha(2l)}(q,z)$ form the representation $l_1=\frac{1}{2}(s+k)$ of the algebra so(2, 1) in terms of q (the corresponding generators are $\tilde{M}^q{}_{\alpha(2)}=-q_\alpha(\partial/\partial q^\alpha))$ and a representation $l_2=\frac{1}{2}(s-k)$ in terms of z (the correspondent generators are $\tilde{M}^z_{\alpha(2)}=-z_\alpha(\partial/\partial z^\alpha))$. Thus the elements $T^{(s,k)}_{\alpha(2l)}$ form the representation (l) of so(2, 1) in terms of both variables (q and z) with the generator as a sum of M^q and M^z

$$\tilde{M}_{\alpha(2)}=-\left(q_\alpha\frac{\partial}{\partial q^\alpha}+z_\alpha\frac{\partial}{\partial z^\alpha}\right). \tag{4.25}$$

Applying the multiplication formula (A.2.4), intertwining formula (A.3.18), and taking into account (4.14) (where $a, b\to z, q$), we come to

$$\begin{aligned}
&T^{(l_1+l_2,\,l_1-l_2)}_{\alpha(2l)}(q,z) * T^{(l'_1+l'_2,\,l'_1-l'_2)}_{\beta(2l')}(q,z)\\
&=\sum_{l'',l''_1,l''_2}\frac{(-1)^{l_2+l'_2-l''_2}\,i^{l_1+l_2+l'_1+l'_2-l''_1-l''_2}}{\sqrt{\Delta(l_1,l'_1,l''_1)\,\Delta(l_2,l'_2,l''_2)(2l''_1+1)(2l''_2+1)}}\\
&\quad\times\begin{bmatrix} l_1 & l_2 & l\\ l'_1 & l'_2 & l'\\ l''_1 & l''_2 & l''\end{bmatrix} C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')}T^{(l''_1+l''_2,\,l''_1-l''_2)}_{\gamma(2l'')}(q,z),
\end{aligned} \tag{4.26}$$

where $[\cdots]$ are related to the 9j-symbols by (A.3.8). The sign $(-1)^{l_2+l'_2-l''_2}$ has appeared in the account of signs in (4.21a, b).

The last step in the calculation consists in going over in (4.26) to the initial basis $\mathrm{so}(3,2)\to\mathrm{so}(2,1)\oplus\mathrm{so}(1,1)$ by formulae (4.22, 23). We finally arrive at

$$\begin{aligned}
T^{(s,c)}_{\alpha(2l)} * T^{(s',c')}_{\beta(2l')} = &\sum_{(s'',c'',l'',u,v,t)} i^{s+s'-s''}\delta(2u-l-l'+l'')\,\delta(2v-l+l'-l'')\\
&\times\delta(2t-l'+l-l'')\begin{pmatrix} s & s' & s''\\ c & c' & c''\\ l & l' & l''\end{pmatrix}\mathscr{E}_{\alpha(2u),\beta(2u)}T^{(s'',c'')}_{\alpha(2v)\beta(2t)},
\end{aligned} \tag{4.27}$$

$s'', l'', u, v, t = 0, \frac{1}{2}, 1, ...,$

where the coefficients have the form

$$\begin{pmatrix} s & s' & s'' \\ c & c' & c'' \\ l & l' & l'' \end{pmatrix} = \delta(c+c'-c'')\,\mathscr{E}(s,s',s'')\,\mathscr{E}(l,l',l'')$$
$$\times\left[\frac{(2l+1)!\,(2l'+1)!\,(2l''+1)!}{(l+l'-l'')!\,(l-l'+l'')!\,(l'-l+l'')!\,(l+l'+l''+1)!}\right]^{1/2}$$
$$\times\sum_{k,k',k''}(-1)^{(s+s'-s''-k-k'+k'')/2}$$
$$\times\frac{d^{l}_{c,k}\left(\frac{\pi}{2}\right)d^{l'}_{c',k'}\left(\frac{\pi}{2}\right)d^{l''}_{k'',c''}\left(-\frac{\pi}{2}\right)}{\sqrt{\Delta\left(\frac{s+k}{2},\frac{s'+k'}{2},\frac{s''+k''}{2}\right)\Delta\left(\frac{s-k}{2},\frac{s'-k'}{2},\frac{s''-k''}{2}\right)}}$$
$$\times\begin{Bmatrix} \frac{s+k}{2}, & \frac{s-k}{2}, & l \\ \frac{s'+k'}{2}, & \frac{s'-k'}{2}, & l' \\ \frac{s''+k''}{2}, & \frac{s''-k''}{2}, & l'' \end{Bmatrix},$$
$$\mathscr{E}(a,b,c)=1(0)\qquad\text{for}$$
$$c\in[|a-b|,a+b],\ (c\notin[|a-b|,a+b]),$$
$$k=-l,\dots,l,\quad k'=-l',\dots,l',\quad k''=-l'',\dots,l''. \tag{4.28}$$

We have expressed the structural coefficients of the $*$-product in the conformal basis in terms of the $9j$-symbols and particular values of Wigner d-functions.

The explicit form of the numerical coefficients (4.28) is rather combersome, but of practical importance for us is the following simple symmetry property

$$\begin{pmatrix} s & s' & s'' \\ c & c' & c'' \\ l & l' & l'' \end{pmatrix} = (-1)^{s+s'+s''+l+l'+l''}\begin{pmatrix} s' & s & s'' \\ c' & c & c'' \\ l' & l & l'' \end{pmatrix}, \tag{4.29}$$

which follows from the symmetry property of the $9j$-coefficients.

When anti-commuting generating elements ψ_i are also present, the generators

$$T^{(s,c)}_{i(k),\alpha(2l)}=\frac{1}{k!}\psi_{i_1}\cdots\psi_{i_k}T^{(s,c)}_{\alpha(2l)} \tag{4.30}$$

make up a superconformal basis which is related to the reduction

$$\mathrm{osp}(N|4) \to \mathrm{so}(3,2) \oplus \mathrm{so}(N) \to \mathrm{so}(2,1) \oplus \mathrm{so}(1,1) \oplus \mathrm{so}(N) \tag{4.31}$$

in the irrepses of the conformal superalgebra $\mathrm{osp}(N|4)$.

The Hermitian conjugation (2.8, 2.13) acts on the generators (4.30) according to

$$T^{+(s,c)}_{i(k),\alpha(2l)} = (-1)^{k(k-1)/2}\, T^{(s,c)}_{i(k),\alpha(2l)}, \tag{4.32}$$

where the sign $(-1)^{k(k-1)/2}$ emerges due to the anti-commutativity of ψ_i.

The generators (4.19) furnish an associative algebra $aq(N|4;\mathbb{C})$ [4] under the multiplication of symbols (4.6) (where now $\vec{\Delta}$ involves also the $(\bar{\partial}_r/\partial\psi_i)(\vec{\partial}_l/\partial\psi^i)$ term along with the RHS of (4.6b)). It will be convenient below to limit ourselves to the subalgebra $aq^E(N|4;\mathbb{C})$ spanned by the generators (4.30) with even $(k+2s)$.

5. Conformal Higher Spin Superalgebra shsc($N|3$): Gauge Fields and Curvatures

To construct a Lie superalgebra by a given associative algebra, it is necessary:

(1) to fix the Grassmann parity of the generators;

(2) to choose the class of the Grassmann shell (GS) (this is of particular importance for the case of $N>1$, see [19, 4]);

(3) to define the supercommutator.

The Grassmann parity of the generators (4.30) is defined as

$$\pi(T^{(s,c)}) = 0(1) \qquad \text{for } s \text{ integer(half integer)}. \tag{5.1}$$

Following the arguments of [4], it is convenient to work with the second-class GS and consider the algebra $aq^E(N|4;\mathbb{C})$, specified by even $(k+2s)$. The gauge fields have the form[4]

$$\omega_\mu = \sum \tfrac{1}{2}\, i^{-|2s|_2-1}\, \omega_{\mu,}^{(s,c)i(k)}{}_{,}^{\alpha(2l)}\, T^{(s,c)}_{i(k),\alpha(2l)},$$
$$s=0, \tfrac{1}{2}, 1, \ldots; \quad k=0,1,\ldots,N \qquad (k+2s \text{ is even});$$
$$c=-s, -s+1, \ldots, s; \quad l=|c|, |c|+1, \ldots, s. \tag{5.2}$$

The field statistic coincides with the generator statistic. By the property of the second class GS we have

$$\pi(\omega^s_\mu) = \pi(T^s), \qquad \omega_\mu{}^s T^{s'} = (-1)^{4ss'}\, T^{s'} \omega_\mu{}^s. \tag{5.3}$$

[4] The multiplets of the gauge fields are given in Appendix 5.

The factor $i^{-|2s|_2}$ in (5.2) is introduced for convenience in the case of second class GS [4].

Hermitian conjugation will be determined as follows:

$$\omega_\mu^+ = -\omega_\mu, \qquad \omega^{+(s,c)}_{\mu, i(k), \alpha(2l)} = (-1)^{k(k-1)/2}\, \omega^{(s,c)}_{\mu, i(k), \alpha(2l)} \tag{5.4}$$

(see (4.21)).

The superalgebra (5.1–5.4) with the supercommutator

$$[A, B\}_* = A * B - (-1)^{\pi(A)\pi(B)}\, B * A \tag{5.5}$$

is an extension of the $D = 2+1$ conformal superalgebra osp$(N|4)$ with a maximum spin two to all conformal higher spins. We shall denote it as shsc$(N|3)$ (c-conformal). By construction, it is isomorphic to the algebra shs$(N|4)$. The difference consists in the choice of another basis in the irreducible representation spaces so(3, 2). The curvatures of the algebra shsc$(N|3)$ we determine according to the general formula

$$R^A_{\mu\nu} = \partial_\mu \omega^A_\nu - \partial_\nu \omega^A_\mu + f^A_{\ BC}\, \omega^B_\mu \omega^C_\nu, \tag{5.6}$$

$$\begin{aligned} R^{(s,c)}_{\mu\nu, i(k), \alpha(2l)} &= \partial_\mu \omega^{(s,c)}_{\nu, i(k), \alpha(2l)} - (\mu \leftrightarrow \nu) \\ &\quad + \sum i^{s'+s''-s+z-|z|_2-1} \frac{k!}{u!\, v!\, z!}\, \delta(k-u-v) \\ &\quad \times \delta(2p - l' - l'' + l)\, \delta(2q - l' + l'' - l)\, \delta(2t - l'' + l' - l) \\ &\quad \times \delta(|4s's'' + s' + s'' - s + uv + z(u+v) + 1|_2) \\ &\quad \times \begin{pmatrix} s' & s'' & s \\ c' & c'' & c \\ l' & l'' & l \end{pmatrix} \omega^{(s',c')}_{\mu, i(u)\, j(z), \alpha(2q)\gamma(2p)}\, \omega^{(s'',c'')\, j(z)}_{\nu, i(v)\qquad\ , \alpha(2t)}{}^{\gamma(2p)}. \end{aligned} \tag{5.7}$$

In this formula, summation parameters $s', c', l', u+z$ and $s'', c'', l'', v+z$ take their values as in (5.2).

The appearance of factor

$$\delta(|4s's'' + s' + s'' - s + uv + (u+v)z + 1|_2) \tag{5.8}$$

is due to the simple fact that

$$1 - (-1)^n = 2\delta(|n+1|_2). \tag{5.9}$$

The property $\chi_\alpha \varphi^\alpha = -\chi^\alpha \varphi_\alpha$ gives in (5.8) the term $(s' + s'' - s)$, the Grassmann parity of the fields gives $(4s's'')$, and the antisymmetry with respect to the internal indices gives $(uv + (u+v)z)$. The expression (5.8) is equal to $\delta(|s' + s'' - s + z + 1|_2)$ by force of restriction that $k + 2s$ is even.

As usual, the curvatures (5.7) are homogeneously transformed under gauge transformations shsc$(N|3)$

$$\delta\omega_\mu{}^A = \partial_\mu \mathscr{E}^A + f^A_{BC}\omega_\mu{}^B \mathscr{E}^C. \tag{5.10}$$

The superalgebra shsc$(N|3)$ possesses a symmetric invariant bilinear form

$$(A, B) = \mathrm{tr}(A * B), \tag{5.11}$$

where a "trace" is defined as

$$\mathrm{tr}(A(Z)) = A(0), \qquad Z = (a, b, \psi). \tag{5.12}$$

In terms of the expansion coefficients of the type of Eq. (5.2), the bilinear form reads

$$(A, B) = \sum_{s,c,l,k} \frac{i^{2l+k-|k|_2}}{k!} A^{(s,c)}_{i(k),\alpha(2l)} B^{(s,-c)i(k)\,\alpha(2l)}. \tag{5.13}$$

Similarly, the trace (5.12) is used to define multi-linear invariant forms $\mathrm{tr}(A_1 * \cdots * A_n)$. Note that the form (5.11) satisfies the Hermiticity condition

$$(A, B)^+ = (A, B). \tag{5.14}$$

6. A Chern–Simons Action and the Equations of Motion of the Superconformal Theory of Higher-Spins in $D = 2+1$

In three dimensions, the action invariant under gauge transformations of the algebra shsc$(N|3)$ has the form[5]

$$S = \int \mathrm{tr}(\omega \wedge d\omega + \tfrac{2}{3}\,\omega \wedge \omega \wedge \omega), \tag{6.1}$$

where $\omega = \omega_\mu dx^\mu$ is the 1-form of the gauge connection (5.2). The equations of motion of the fields of conformal higher spins in $D = 2+1$ are

$$R^{(s,c)}_{\mu\nu,\,i(k),\,\alpha(2l)} = 0. \tag{6.2}$$

These equations are invariant under the gauge transformations of shsc$(N|3)$ (5.10) by virtue of the homogeneity of the curvature transformation law

$$\delta R_{\mu\nu}{}^A = f^A{}_{BC} R_{\mu\nu}{}^B \mathscr{E}^C. \tag{6.3}$$

[5] $\omega \wedge \omega = \frac{1}{2}(\omega_\mu * \omega_\nu - \omega_\nu * \omega_\mu)\, dx^\mu \wedge dx^\nu$.

Let us show now, that within the sector of fields with spins $s=1, \frac{3}{2}, 2$, the above equations generate the usual osp$(N|4)$-invariant conformal supergravity Eqs. (3.9).

It is not difficult to note that T generators are related to the generators in (3.1) by the transformation (see (4.14))

$$\left(\frac{1}{2\sqrt{2}\,i} T^{(1,-1)}_{\alpha(2)}, \frac{1}{2\sqrt{2}\,i} T^{(1,1)}_{\alpha(2)}, \frac{1}{2i} T^{(1,0)}_{\alpha(2)}, \frac{i}{2\sqrt{2}} T^{(1,0)}, \frac{1}{2} T^{(0,0)}_{i(2)}, \right.$$
$$\left. \frac{1}{2} T^{(1/2,-1/2)}_{i,\alpha}, \frac{1}{2} T^{(1/2,1/2)}_{i,\alpha}\right) = (P_{\alpha(2)}, K_{\alpha(2)}, M_{\alpha(2)}, D, T_{i(2)}, Q_{i\alpha}, S_{i\alpha}). \quad (6.4)$$

The conformal supergravity curvatures (3.5) follow from the general formula (6.14) (when all fields with $2s+k>2$ are equal to zero) by the identification

$$\begin{aligned}(e^{\alpha(2)}_{\mu}, f^{\alpha(2)}_{\mu}, \omega^{\alpha(2)}_{\mu}, b_{\mu}, A_{\mu i(2)}, Q_{\mu i\alpha}, \psi_{\mu i\alpha})\\ = (\sqrt{2}\,\omega^{(1,-1)\alpha(2)}_{\mu,}, \sqrt{2}\,\omega^{(1,1)\alpha(2)}_{\mu,}, \omega^{(1,0)\alpha(2)}_{\mu,}, -\sqrt{2}\,\omega^{(1,0)}_{\mu},\\ -i\omega^{(0,0)}_{\mu,i(2)}, \omega^{(1/2,1/2)}_{\mu,i,\alpha}, \omega^{(1/2,-1/2)}_{\mu,i,\alpha})\end{aligned} \quad (6.5)$$

and similarly for the curvatures.

7. Generating Functional of Higher Spin Theory in $D=2+1$

In this section we shall consider the quantization of the superconformal theory of higher spins in $D=2+1$. To this end we consider the quantization of the theory with the Chern–Simons Lagrangian of the general form

$$L(t) = \int d^2x \mathscr{E}^{\mu\nu\sigma}(\omega^A_{\mu}\partial_{\nu}\omega^B_{\sigma} + \tfrac{1}{3} f^B_{CD}\omega^A_{\mu}\omega^C_{\nu}\omega^D_{\sigma})Q_{AB}. \quad (7.1)$$

Here f^A_{BC} are the structure constants of the Lie superalgebra G and Q_{AB} is the invariant bilinear form on G with the following symmetry properties

$$\begin{aligned} &f^C_{AB}Q_{CD} + f^C_{DB}Q_{CA}(-1)^{n_A n_D} = 0, \qquad Q_{AB}Q^{BC} = \delta^C_A, \\ &Q_{AB} = (-1)^{n_A n_B} Q_{BA}, \qquad Q_{AB} = 0 \qquad \text{for} \quad n_A = n_B + 1, \end{aligned} \quad (7.2)$$

n_A is the Grassmann parity of T_A. The Lagrangian (8.1) is useful to rewrite in the form[6]

$$L(t) = \int d^2x \mathscr{E}^{0ab}(\omega^A_0 R^B_{ab} - \omega^A_a \partial_0 \omega^B_b)\, Q_{AB}. \quad (7.3)$$

[6] In this section $a, b = 1, 2$.

The standard procedure of canonical quantization gives the following first class ($T_A(x)$) and second class ($\theta^a_A(x)$) constraints corresponding to (7.3)

$$\theta^a_A = P^a_A - \mathscr{E}^{ab}\omega^B_b Q_{BA}, \qquad \mathscr{E}^{ab} = \mathscr{E}^{0ab}, \tag{7.4}$$

$$T_A(x) = -\mathscr{E}^{ab} R^B_{ab} Q_{BA}|_{\theta=0}, \tag{7.5}$$

where P^a_A is the canonical momenta for ω^A_a. Using the standard definition of the Poisson bracket

$$\{A, B\}_{PB} = \int d^2x \left(\frac{\delta_r A}{\delta\omega^C_a}\frac{\delta_l B}{\delta P^a_C} - (-1)^{n_A n_B}\frac{\delta_r B}{\delta\omega^A_a}\frac{\delta_l A}{\delta P^a_A}\right), \tag{7.6}$$

we find for the second class constraints

$$\{\theta^a_A(x), \theta^b_B(x')\}_{PB} = -2\mathscr{E}^{ab} Q_{BA}\delta^2(x-x'). \tag{7.7}$$

Then the Dirac bracket takes the form

$$\{A, B\}_D = \{A, B\}_{PB} - \tfrac{1}{2}\int d^2x \{A, \theta^a_A(x)\}_{PB}\,\mathscr{E}_{ab} Q^{BA}\{\theta^b_B(x), B\}_{PB} \tag{7.8}$$

Following Dirac we thus get for the (super) commutator of the operators $\hat{A}$ and $\hat{B}$

$$[\hat{A}, \hat{B}] = i\{A, B\}_D. \tag{7.9}$$

While in general Eq. (7.9) is true only in the semiclassical approximation (see [27]). In the Chern–Simons theory, where the second class constraints are linear functions of the canonical variables, Eq. (7.9) for the commutator of the canonical variables remains true also in the exact operator sense. The involution of the first class constraints takes the form

$$\{T_A(x), T_B(x')\}_D = f^C_{AB} T_C(x)\,\delta^2(x-x'). \tag{7.10}$$

Then the hamiltonian is equal to zero on the constraints. Thus, we have seen that the Chern–Simons theory of the general type is a system with constraints of the first and second class, and since the structure functions f^C_{AB} are constants, this theory is quantized following Ref. [11] as a theory of rank one using the method of the generalized canonical quantization. Then the generating functional in the generalized phase space (which consists of the initial phase variables ω^A_a and P^a_A, Lagrange multipliers ($\lambda^A = \omega^A_0$) to the first class constraints and Lagrange multipliers (π_A) to the gauge conditions for the first class constraints and corresponding ghost variables C^A and $\bar{C}_A$ has the form [11]

$$Z = \int D\omega^A_a DP^a_A D\lambda^A D\pi_A DC^A D\bar{C}_A \delta(\theta^a_A)$$

$$\times (\mathrm{Det}\{\theta, \theta\}_{PB})^{1/2} \exp i \int d^3x (P^a_A \partial_0 \omega^A_a - \mathscr{H}_{\text{eff}}). \tag{7.11}$$

Here the effective hamiltonian is

$$\mathcal{H}_{\text{eff}}=T_A\lambda^A-\pi_A\frac{\delta_l\Psi}{\delta\bar{C}_A}-\{\Psi,T\}_D-\frac{\delta_r\Psi}{\delta C^A}u^A-\frac{\delta_r\Psi}{\delta\lambda^A}\left(\dot{C}^A-\frac{\delta_r u^A}{\partial C^B}\lambda^B\right). \tag{7.12}$$

The gauge functional is

$$\Psi=\int d^3x\bar{C}_A\Psi^A(\omega,P,\lambda,\dot{\lambda},C,\bar{C}), \tag{7.13a}$$

and we use the following notations

$$T=\int d^2xC^AT_A(x), \qquad u^A(x)=\tfrac{1}{2}f^A_{BD}C^DC^B(-1)^{n_B+1}. \tag{7.13b}$$

Carrying out the integration over the momenta P_A in (7.11) (substituting $\lambda^A=\omega_0{}^A$ and assuming that the gauge condition (7.13) does not depend on P), we get the following expression for the generating functional in the configuration space

$$Z=\int D\omega^A_\mu D\pi_A DC^A D\bar{C}_A(\text{Det}\{\theta,\theta\}_{PB})^{1/2}\exp iS_{\text{eff}}, \tag{7.14}$$

where

$$S_{\text{eff}}=S+\int d^3x\left(\pi_A\frac{\delta_l\Psi}{\delta\bar{C}_A}+\frac{\delta_r\Psi}{\delta\omega^A_\mu}\mathcal{D}_\mu C^A-\frac{1}{2}\frac{\delta_r\Psi}{\delta C^A}f^A_{BD}C^DC^B(-1)^{n_B}\right). \tag{7.15}$$

Here S is the original Chern–Simons action. It is not difficult to see, that the same expression for the effective action can be obtained using the lagrangian method of Refs. [24, 25].

In the case of the theory of higher spins in $D=2+1$ the effective action in AdS_3 and in superconformal theory can be rewritten in the following compact form

$$S_{\text{eff}}=S+\int d^3x\,\text{tr}\left(\pi*\frac{\delta_l\Psi}{\delta\bar{C}}+\frac{\delta_r\Psi}{\delta\omega_\mu}*\mathcal{D}_\mu C+\frac{\delta_r\Psi}{\delta C}*C*C\right), \tag{7.16}$$

where

$$C=C^AT_A, \qquad \bar{C}=\bar{C}_AT^A, \qquad \Psi=\int d^3x\bar{C}_A\Psi^A,$$

$$\frac{\delta_{r,l}\Psi}{\delta\varphi}=\frac{\delta_{r,l}\Psi}{\delta\varphi_A}T_A, \qquad \varphi=(\omega,C,\bar{C}).$$

Here $T_A(T^A)$ are the generators of the superalgebra of higher spins.

In conclusion, let us note that the elementary count of the physical degrees of freedom in the phase space shows that in $D=2+1$ their total number is equal to zero. In fact, $2\times$(number of physical degrees of freedom)$=2\times$(number of fields)$-$(number of second class constraints)$-2\times$(number of first class constraints)$=0$ for gauge fields of each spin $s\geqslant 1$. However, the higher-spin Chern–Simons theory will be interesting from a topological point of view.

APPENDIX 1: NOTATIONS AND CONVENTIONS

We follow the conventions of Refs. [1–8]. The two-component spinorial indices are raised and lowered by means of $\mathscr{E}_{\alpha\beta} = -\mathscr{E}_{\beta\alpha}$, $\mathscr{E}^{\alpha\beta}$, $\mathscr{E}_{12} = \mathscr{E}^{12} = 1$, as $A^\alpha = \mathscr{E}^{\alpha\beta} A_\beta$, $A_\beta = \mathscr{E}_{\alpha\beta} A^\alpha$. The internal so($N$) indices ($i$, j, k, ...) are raised and lowered by δ^{ij}, δ_{ij}.

A symmetrization (anti-symmetrization) is implied for any set of upper or lower spinorial (internal) indices denoted by alike letters. When this symmetrization is carried out, the maximal possible number of upper and lower indices denoted by the same letter should be contracted. We use notations such as

$$A_{\underbrace{\alpha\cdots\alpha}_{n}} = A_{\alpha(n)}, \qquad A_{\underbrace{i\cdots i}_{n}} = A_{i(n)}, \qquad \underbrace{\mathscr{E}_{\alpha\beta}\cdots\mathscr{E}_{\alpha\beta}}_{n} = \mathscr{E}_{\alpha(n),\,\beta(n)}, \tag{A.1}$$
$$\underbrace{\delta^\gamma_\alpha\cdots\delta^\gamma_\alpha}_{n} = \delta^{\gamma(n)}_{\alpha(n)}, \qquad \underbrace{q_\alpha\cdots q_\alpha}_{n} = q_{\alpha(n)}, \quad \text{etc.}$$

The three-dimensional world indices $\mu, \nu, \sigma = 0, 1, 2$. The flat metric $\eta_{\mu\nu}$ has the signature $(+, -, -)$.

We often use the notations $\delta(n) = 1(0)$, $n = 0$ $(n \neq 0)$;

$$|n|_2 = 0 \ (1) \qquad \text{at } n \text{ even (odd)}. \tag{A.2}$$

APPENDIX 2: THE shs(2|1) SUPERALGEBRA

The associative complex algebra aq(2; $\mathbb{C}$) (associative quantum) [4] is formed by the generating elements $\hat{q}_\alpha$ ($\alpha = 1, 2$ are sp(2) spinorial indices) with commutation relations

$$[\hat{q}_\alpha, \hat{q}_\beta] = 2i\mathscr{E}_{\alpha\beta}. \tag{A.2.1}$$

In terms of Weyl symbols $A(q)$ of the operators $\hat{A}(\hat{q})$ the associative product of two symbols is given by the formula

$$A * B = A \exp\left(i \frac{\overleftarrow{\partial}}{\partial q^\alpha} \frac{\overrightarrow{\partial}}{\partial q_\alpha}\right) B. \tag{A.2.2}$$

The basis in aq(2; $\mathbb{C}$) is

$$T_{\alpha(2l)} = (1/i\sqrt{(2l)!})\, q_{\alpha(2l)}, \; l = 0, \tfrac{1}{2}, 1, \ldots. \tag{A.2.3}$$

The multiplication (A.2.2) in this basis has the form

$$T_{\alpha(2l_1)} * T_{\beta(2l_2)} = \sum_{l_3} \frac{i^{l_1+l_2-l_3-1}}{\sqrt{\Delta(l_1, l_2, l_3)(2l_3+1)}} C_{\alpha(2l_1),\beta(2l_2),}{}^{\gamma(2l_3)} T_{\gamma(2l_3)}, \qquad (A.2.4)$$

where C are the spinorial C–G coefficients.

An arbitrary element from aq(2; ℂ) has the form

$$A = \sum_l A^{\alpha(2l)} T_{\alpha(2l)}, \qquad (A.2.5)$$

where only a finite number of the coefficients $A^{\alpha(2l)}$ is nonzero.

Define in aq(2; ℂ) a Grassmann parity of the generators and coefficients, by

$$\pi(T_{\alpha(2l)}) = \pi(A^{\alpha(2l)}) = 0(1) \qquad \text{at } l \text{ integer(half integer)}. \qquad (A.2.6)$$

The supercommutator

$$[A, B\}_* = A * B - (-1)^{\pi(A)\pi(B)} B * A \qquad (A.2.7)$$

makes aq(2; ℂ) into a Lie superalgebra shs(1|2; ℂ) [3, 4]. The supercommutator of two generators (A.2.3) has the form

$$[T_{\alpha(2l_1)}, T_{\beta(2l_2)}\}_* = 2 \sum_{l_3} \frac{i^{l_1+l_2-l_3-1}}{\sqrt{\Delta(l_1, l_2, l_3)(2l_3+1)}} \times \delta(|4l_1 l_2 + l_1 + l_2 - l_3 + 1|_2)\, C_{\alpha(2l_1),\beta(2l_2),}{}^{\gamma(2l_3)} T_{\gamma(2l_3)}, \qquad (A.2.8)$$

where $\delta(|4l_1 l_2 + l_1 + l_2 - l_3 + 1|_2)$ appears due to the symmetry property of the C–G coefficients (A.3.15) and the definition (A.2.6) of the Grassmann parity.

The superalgebra shs(1|2; ℂ) possesses an invariant symmetric bilinear form

$$(A, B) = \mathrm{tr}(A * B), \qquad (A.2.9)$$

$$\mathrm{tr}(A(q)) = A(0). \qquad (A.2.10)$$

The Hermitian conjugation

$$q_\alpha^+ = q_\alpha, \qquad A^+ = -A \qquad (A.2.11)$$

gives rise to a real form shs(1|2) in shs(1|2; ℂ).

In conclusion, note that it is possible to introduce a new basis in aq(2; ℂ) by

$$T^l_m = \frac{(q_1)^{l+m} (q_2)^{l-m}}{i\sqrt{(l+m)!\,(l-m)!}} \qquad (q_\alpha = q_1, q_2). \qquad (A.2.12)$$

In this basis, the multiplication formulae (A.2.4) and (A.2.8) take the form

$$T^{l_1}_{m_1} * T^{l_2}_{m_2} = \sum_{l_3, m_3} \frac{i^{l_1+l_2-l_3-1}}{\sqrt{\Delta(l_1, l_2, l_3)(2l_3+1)}} \times C^{l_1 l_2 l_3}_{m_1 m_2 m_3} T^{l_3}_{m_3},$$

$$[T^{l_1}_{m_1}, T^{l_2}_{m_2}\}_* = 2 \sum_{l_3, m_3} \frac{i^{l_1+l_2-l_3-1}}{\sqrt{\Delta(l_1, l_2, l_3)(2l_3+1)}} \times C^{l_1 l_2 l_3}_{m_1 m_2 m_3} \delta(|4l_1 l_2 + l_1 + l_2 - l_3 + 1|_2)\, T^{l_3}_{m_3} \tag{A.2.13}$$

with C being the usual C–G coefficients.

APPENDIX 3:

A. *The Clebsch–Gordan Coefficients* [22]

According to the Raca formula, we have

$$C^{ll'l''}_{mm'm''} = \delta(m+m'-m'')\sqrt{\Delta(l, l', l'')(2l''+1)} \times \sum_t \frac{(-1)^t \sqrt{\begin{pmatrix}(l+m)!\,(l-m)!\,(l'+m')! \\ \times (l'-m')!\,(l''+m'')!\,(l''-m'')!\end{pmatrix}}}{\begin{pmatrix}t!\,(l+l'-l''-t)!\,(l-m-t)!\,(l'+m'-t)! \\ \times (l''-l'+m+t)!\,(l''-l-m'+t)!\end{pmatrix}} \tag{A.3.1}$$

with the following symmetry properties

$$C^{ll'l''}_{mm'm''} = (-1)^{l+l'-l''} C^{ll'l''}_{-m-m'-m''}, \tag{A.3.2}$$

$$C^{ll'l''}_{mm'm''} = (-1)^{l+l'-l''} C^{l'll''}_{m'mm''}. \tag{A.3.3}$$

We define $C^{ll'l''}_{mm'm''} = 0$ if $l'' \notin [|l-l'|, l+l']$.

The orthogonality relations are

$$\sum_{m_1, m_2} C^{l_1 l_2 l}_{m_1 m_2 m} C^{l_1 l_2 l'}_{m_1 m_2 m'} = \delta^{l,l'} \delta_{m,m'}, \tag{A.3.4}$$

$$\sum_{l,m} C^{l_1 l_2 l}_{m_1 m_2 m} C^{l_1 l_2 l}_{m_1' m_2' m} = \delta_{m_1, m_1'} \delta_{m_2, m_2'}. \tag{A.2.5}$$

The triangle coefficient is

$$\Delta(l, l', l'') = \frac{(l+l'-l'')!\,(l-l'+l'')!\,(l'-l+l'')!}{(l+l'+l''+1)!} \tag{A.3.6}$$

The Wigner 9j-symbols

$$\begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix}$$

are determined by the relations

$$\sum_{(\mu_1\mu_2 m_1 m_2 m'_1 m'_2)} C^{j'_1 j'_2 j'}_{m'_1 m'_2 m'} C^{j_1 j_2 j}_{m_1 m_2 m} C^{k_1 k_2 k}_{\mu_1 \mu_2 \mu} C^{j_1 k_1 j'_1}_{m_1 \mu_1 m'_1} C^{j_2 k_2 j'_2}_{m_2 \mu_2 m'_2} = \begin{bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \\ j'_1 & j'_2 & j' \end{bmatrix} C^{jkj'}_{m\mu m'}, \tag{A.3.7}$$

$$\begin{bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \\ j'_1 & j'_2 & j' \end{bmatrix} = \sqrt{(2j'_1+1)(2j'_2+1)(2j+1)(2k+1)} \begin{Bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \\ j'_1 & j'_2 & j' \end{Bmatrix} \tag{A.3.8}$$

By permuting any two rows or two columns, 9*j*-symbols are multiplied by the sign

$$(-1)^{\sum_{k,l=1}^{3} j_{kl}}. \tag{A.3.9}$$

By the definition, the 9*j*-symbols are equal to zero if at least in one row or one column the triangle condition is not fulfilled.

B. *The Spinor Clebsch–Gordan Coefficients*

In the representations so(2, 1) one can introduce a spinor basis $T_{\alpha(2l)}$, $\alpha = 1, 2$ such that

$$T^l_m = T_{\underbrace{1\cdots1}_{l-m}\ \underbrace{2\cdots2}_{l+m}}. \tag{A.3.10}$$

Then to the formulas in the weight basis

$$T^{ll'}_{mm'} = \sum_{l'',m''} C^{ll'l''}_{mm'm''} T^{(l,l')l''}_{m''}, \tag{A.3.11}$$

$$T^{(l,l')l''}_{m''} = \sum_{m,m'} C^{ll'l''}_{mm'm''} T^{ll'}_{mm'} \tag{A.3.12}$$

there correspond the formulas in the spinor basis

$$T_{\alpha(2l),\beta(2l')} = \sum_{l''} C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} T^{(l,l')}_{\gamma(2l'')}, \tag{A.3.13}$$

$$T^{(l,l')}_{\gamma(2l'')} = \bar{C}_{\gamma(2l''),}{}^{\alpha(2l),\beta(2l')} T_{\alpha(2l),\beta(2l')}. \tag{A.3.14}$$

The relations (A.3.3–A.3.7) in the spinor basis have the form

$$C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} = (-1)^{l+l'-l''} C_{\beta(2l'),\alpha(2l),}{}^{\gamma(2l'')}, \tag{A.3.15}$$

$$\sum_{l''} C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} \bar{C}_{\gamma(2l''),}{}^{\delta(2l),\rho(2l')} = \delta^{\delta(2l)}_{\alpha(2l)} \delta^{\rho(2l')}_{\beta(2l')}, \tag{A.3.16}$$

$$\bar{C}_{\gamma(2l''),}{}^{\alpha(2l),\beta(2l')} C_{\alpha(2l),\beta(2l'),}{}^{\rho(2l'')} = \delta^{\rho(2l'')}_{\gamma(2l''),} \tag{A.3.17}$$

$$\bar{C}_{\alpha(2j),}{}^{\lambda(2j_1),\rho(2j_2)} \bar{C}_{\beta(2k),}{}^{\delta(2k_1),\mathscr{E}(2k_2)} C_{\lambda(2j_1),\delta(2k_1),}{}^{\xi(2j'_1)}$$
$$\times C_{\rho(2j_2),\mathscr{E}(2k_2),}{}^{\zeta(2j'_2)} C_{\xi(2j'_1),\zeta(2j'_2),}{}^{\gamma(2j')} = \begin{bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \\ j'_1 & j'_2 & j' \end{bmatrix} C_{\alpha(2j),\beta(2k),}{}^{\gamma(2j')}. \tag{A.3.18}$$

One can readily obtain the explicit form of the spinor C–G coefficients if one note that (A.3.13) is an expansion of the multispinor $T_{\alpha(2l),\beta(2l')}$ into symmetric multispinors. As is known [23] the formula of expansion into symmetric spinors is

$$T_{\alpha(2l),\beta(2l')} = \sum \frac{(2l)!\,(2l')!\,(2l''+1)!}{(2k)!\,(2s)!\,(2t)!\,(l+l'+l''+1)!}$$
$$\times \delta(2u-l-l'+l'')\,\delta(2s-l+l'-l'')$$
$$\times \delta(2t+l-l'-l'')\,\mathscr{E}_{\alpha(2u),\beta(2u)}\,T^{(l,l')}_{(\alpha(2s)\beta(2t))}, \tag{A.3.19}$$

$$T^{(l,l')}_{(\alpha(2s)\beta(2t))} = T_{(\alpha(2s)\gamma(l+l'-s-t),\beta(2t))_{\alpha,\beta}}{}^{\gamma(l+l'-s-t)}, \tag{A.3.20}$$

$(\alpha\cdots\beta\cdots)_{\alpha,\beta}$ implies complete symmetrization with respect to α and β. Denoting the coefficients in (A.3.19) by $(c(l,l',l''))^2$, one can write (A.3.19) in the form (A.3.13) introducing the coefficients

$$C_{\alpha(2l),\beta(2l'),}{}^{\gamma(2l'')} = c(l,l',l'')\,\mathscr{E}_{\alpha(2u),\beta(2u)}\,\delta^{\gamma(2l'')}_{\alpha(2s)\beta(2t)}, \tag{A.3.21}$$
$$2u = l+l'-l'',\quad 2s = l+l''-l',\quad 2t = l'+l''-l,$$

$$\bar{C}_{\gamma(2l''),}{}^{\alpha(2l),\beta(2l')} = c(l,l',l'')\,\mathscr{E}^{\alpha(2u),\beta(2u)}\delta^{\alpha(2s)\beta(2t)}_{\gamma(2l'')}, \tag{A.3.22}$$

$$c(l,l',l'') = \left[\frac{(2l)!\,(2l')!\,(2l''+1)!}{(l+l'-l'')!\,(l+l''-l')!\,(l'+l''-l)!\,(l+l'+l''+1)!}\right]^{1/2}. \tag{A.3.23}$$

Note that all the formulae (A.3.12, A.3.11, A.4, A.5, A.7) and (A.3.14, A.3.13, A.3.16–A.3.18) are obtained from one another through a transformation of the type (A.3.10).

APPENDIX 4: THE WIGNER FUNCTIONS [22]

$$d^l_{m,m'}\left(\frac{\pi}{2}\right) = \frac{1}{2^l}\sum_p \frac{(-1)^p\sqrt{(l+m)!\,(l-m)!\,(l+m')!\,(l-m')!}}{p!\,(l-m'-p)!\,(l+m-p)!\,(m'-m+p)!},$$

$$(d^l_{m,m'}(\beta))^{-1} = (d^l_{m,m'}(-\beta)),\ d^l_{m,m'}(-\beta) = d^l_{m',m}(\beta),$$

$$\sum_{n=-l}^{l} d^l_{m,n}(\beta)\,d^l_{n,m'}(-\beta) = \delta_{m,m'},\ d^l_{m,-m'}\left(\frac{\pi}{2}\right) = (-1)^{l+m} d^l_{m,m'}\left(\frac{\pi}{2}\right).$$

APPENDIX 5: THE REPRESENTATIONS OF THE $D=2+1$ SUPERCONFORMAL ALGEBRA osp(N|4) FOR ANY SPIN

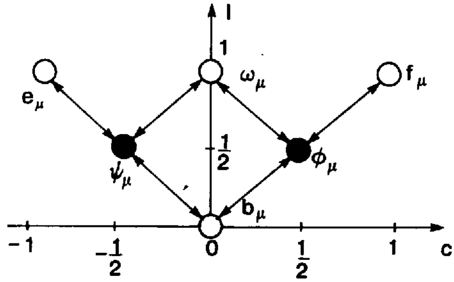

FIG. 1. The adjoint representation of osp(1 | 4).

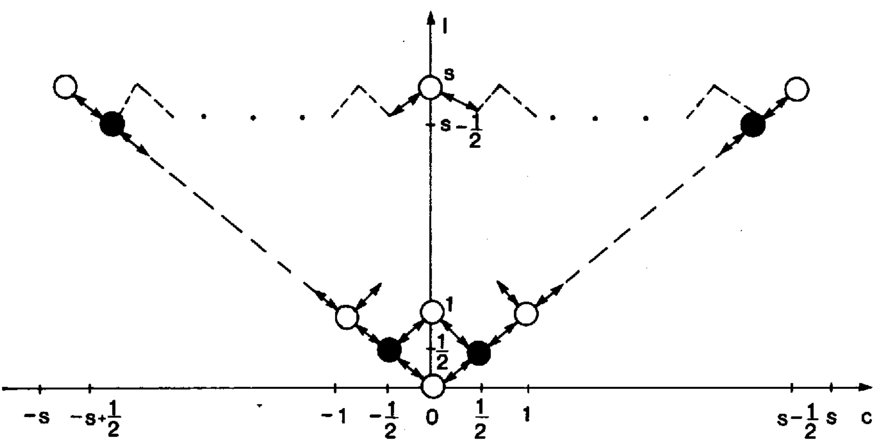

FIG. 2. The representation $(s, s) \oplus (s-\frac{1}{2}, s-\frac{1}{2})$ of so(3, 2), c: the conformal weight of the generator; l: the signature of the so(2, 1) representation; ○: the Bose generators; ●: the Fermi generators; the operators S_α and Q_α act along the arrows $\rightarrow$ and $\leftarrow$, i.e., rise and lower the conformal weight, respectively; $(e_\mu, f_\mu, \omega_\mu, b_\mu, \phi_\mu, \psi_\mu)$—the set of fields corresponding to the ordinary conformal supergravity generators.

APPENDIX 6: LINEARIZED EQUATIONS OF MOTION FOR THE CONFORMAL HIGHER SPIN FIELDS IN $D=2+1$

The linearized equations of motion (6.2) have the form[7]

$$\begin{aligned} R^{L(s,c)}_{\mu\nu,\alpha(2l)} = {} & \partial_\mu \omega^{(s,c)}_{\nu,\alpha(2l)} + A(s,c,l)\,\sigma_{\mu\alpha\gamma}\,\omega^{(s,c+1)\gamma}_{\nu,\alpha(2l-1)} \\ & - B(s,c,l)\,\sigma_{\mu\alpha(2)}\,\omega^{(s,c+1)}_{\nu,\alpha(2l-2)} \\ & - C(s,c,l)\,\sigma_{\mu\gamma(2)}\,\omega^{(s,c+1)\gamma(2)}_{\nu,\alpha(2l)} - (\mu\leftrightarrow\nu) = 0, \end{aligned} \tag{A.6.1}$$

where real coefficients $A(s,c,l)$, $B(s,c,l)$, and $C(s,c,l)$ are nonzero if and only if the corresponding arguments s, c, and l lie in the domain (4.18).

The equations $R^l(s,c,l)=0$, except for $R^L(s,s,s)=0$, contain the fields of conformal weights $c+1$ only algebraically (below we sometimes use short-hand notations such as $\omega(s,c,l)$, $R(s,c,l)$ and $\mathscr{E}(s,c,l)$). These equations can be solved recurrently in $c=-s, -s+1, ..., s$. As a result, all the $\omega(s,c,l)$ fields get expressed through the "physical" field $\omega(s,-s,s)$ up to a gauge transformation. At the first step $c=-s$, the field $\omega(s,-s+1,s)$ and $\omega(s,-s+1,s-1)$ enter in the equation $R^L(s,-s,s)=0$ only algebraically. The field $\omega(s,-s+1,s-1)$ transforms under the gauge transformations with parameters $\mathscr{E}(s,-s+2,l)(l=s-2,s-1,s)$. It can be put to zero by fixing a gauge which spoils the symmetry under these transformations (this is analogous to the $b_\mu=0$ gauge which breaks the K-symmetry in conformal gravity). The field $\omega(s,-s+1,s)$ can now be expressed in terms of the derivatives of $\omega(s,-s,s)$ alone. It is not difficult to show that in the general case as well (for the fixed s), the fields of conformal weight $c+1$ can be expressed in terms of the derivatives of weight—c fields up to gauge transformations. It would suffice to show that the number (n_1) of independent equations in (A.6.1) with fixed s and c is equal to the number of $\omega(s,c+1,l)$ field components $(l=|c+1|,...,s)$ diminished by the number (n_2) of gauge parameters $\mathscr{E}(s,c+2,l)$, $(l=|c+2|,...,s)$:

$$\begin{aligned} n_1-n_2 = {} & \left(3\sum_{l=|c|}^{s}(2l+1) - \sum_{l=|c-1|}^{s}(2l+1)\right) \\ & - \left(3\sum_{l=|c+1|}^{s}(2l+1) - \sum_{l=|c+2|}^{s}(2l+1)\right) = 0 \end{aligned} \tag{A.6.2}$$

(the second sum inside the first bracket is subtracted in accordance with the linearized Bianchi identities satisfied by the curvatures present in (A.6.1)).

Thus it follows that all the fields are expressed through the derivatives of $\omega(s,-s,s)$. The equations of motion for $\omega(s,-s,s)$ are order $-(2s+1)$ differential equations of the form

$$R^{L(s,s)}_{\mu\nu,\alpha(2s)} = 0. \tag{A.6.3}$$

[7] In the zeroth order the flat dreibein $\omega^{0(1,-1)}_{\mu,\alpha(2)} = \sigma_{\mu\alpha(2)}$, while all other fields are equal to zero. When considering the free case, we drop out the internal indices.

Calculate now the number of the degrees of freedom off the mass-shell for the conformal field of spin $s+1$ ($s=0, \frac{1}{2}, 1, ...$)

$$\begin{aligned} n_s^{\text{off shell}} &= n(\omega(s, -s, s)) - n(\mathscr{E}(s, -s, s)) \\ &\quad - n(\mathscr{E}(s, -s+1, s)) - n(\mathscr{E}(s, -s+1, s-1)) \\ &= 3(2s+1)-(2s+1)-(2s+1)-(2s-1)=2. \end{aligned} \tag{A.6.4}$$

We have shown that the set of fields $\omega(s, c, l)$ describes "conformal pure spin states" with spin $s+1$ and with two degrees of freedom off the mass-shell.

Calculate now the total number of the degrees of freedom off the mass-shell in the N-extended superconformal higher spin theory in $D=2+1$. The generators (4.30) with a fixed degree of homogeneity $k+2s=2s_{\text{max}}$ transform under an irrep of the superalgebra osp$(N|4)$. The corresponding gauge fields furnish an osp$(N|4)$-supermultiplet with maximal spin $s_{\text{max}}+1$. The number of off-shell degrees of freedom for this supermultiplet is

$$B(s_{\text{max}}) - F(s_{\text{max}}) = 2 \sum_{k=0}^{2s_{\text{max}}} (-1)^k C_N^k = 2C_{N-1}^{2s_{\text{max}}}, \quad C_p^q = \frac{p!}{q!\,(p-q)!}. \tag{A.6.5}$$

For $N=1$ and $N=2$, $C_{N-1}^{2s_{\text{max}}}=0$ for all $s_{\text{max}}=1, 2, ...$, so the necessary condition for the supersymmetry algebra to close is satisfied for the $N=1$ and 2 conformal supermultiplets.

When $N>2$ this necessary condition is broken for only a finite number of supermultiplets with $2s_{\text{max}} \leqslant N-1$.

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