ELSEVIER

Nuclear Physics B 444 (1995) 577–601

NUCLEAR PHYSICS B

BFV quantization on hermitian symmetric spaces

E.S. Fradkin, V.Ya. Linetsky [1]

International Center for Advanced Science and Technology (ICAST), P.O. Box 4239, Ann Arbor, MI 48106, USA
and P.N. Lebedev Physical Institute, Russian Academy of Sciences, Leninsky Prospect 53, Moscow 117924, Russian Federation

Received 15 March 1995; accepted 18 April 1995

Abstract

Gauge-invariant BFV approach to geometric quantization is applied to the case of hermitian symmetric spaces G/H. In particular, gauge invariant quantization on the Lobachevski plane and sphere is carried out. Due to the presence of symmetry, master equations for the first-class constraints, quantum observables and physical quantum states are exactly solvable. BFV-BRST operator defines a flat G-connection in the Fock bundle over G/H. Physical quantum states are covariantly constant sections with respect to this connection and are shown to coincide with the generalized coherent states for the group G. Vacuum expectation values of the quantum observables commuting with the quantum first-class constraints reduce to the covariant symbols of Berezin. The gauge-invariant approach to quantization on symplectic manifolds synthesizes geometric, deformation and Berezin quantization approaches.

1. Introduction

The present paper is a sequel to our paper [1] where a *gauge-invariant approach to geometric quantization* was developed. It yields a quantum description for dynamical systems with non-trivial geometry and topology of the phase space and, in particular, allows for quantization of the entire algebra of quantum observables. The approach of Ref. [1] is a global version of the gauge-invariant quantization method for second-class constraints developed by Batalin, Fradkin and Fradkina (BFF) [2–5]. The approach of Refs. [2–5] is a natural extension of the general BFV quantization method [6–11] to dynamical systems with second-class constraints and curved phase space. The central

[1] E-mail address: yavlin@engin.umich.edu.

SSDI 0550-3213(95)00187-5

element of the BFF approach is a *BFF conversion procedure* of converting second-class constraints into the first-class ones by introducing extra gauge degrees of freedom [2,3]. Then the resulting system with the first-class constraints can be quantized according to the standard BFV quantization method. In Ref. [4] BFF conversion procedure was applied to dynamical systems with curved phase space. That is, a curved phase space is described by second-class constraints in an enlarged phase space, these second-class constraints are converted into the first-class ones according to the BFF conversion procedure, and then the resulting gauge-invariant system is quantized.

In Ref. [1] this program was implemented for an arbitrary symplectic manifold $(\mathcal{M}, \omega)$. In particular, this approach allowed us to quantize an entire algebra of classical observables. It was demonstrated that the resulting algebra of Weyl symbols of quantum observables is isomorphic to the quantum deformation of the algebra of classical observables known in deformation quantization [12–14]. Moreover, when the quantization condition with metaplectic anomaly is satisfied, it yields a construction of the Hilbert space of quantum states.

In the present paper we will apply the approach of Ref. [1] to the quantization on hermitian symmetric spaces and, in particular, on the Lobachevski plane and two-dimensional sphere. In this case, the presence of global symmetry G allows us to exactly solve the master equations for the quantum first-class constraints, quantum observables and physical quantum states. When $\mathcal{M}$ is a symmetric space G/H, the quantum first-class constraints define a flat G-connection in the Fock bundle over $\mathcal{M}$. Then physical quantum states are covariantly constant sections of the Fock bundle over $\mathcal{M}$ with respect to this connection, and they coincide with the generalized G/H coherent states [15].

The present paper is organized as follows. In Section 2 the main results of Ref. [1] are summarized. In Subsection 3.1 classical hamiltonian mechanics on the Lobachevski plane is considered. Darboux coordinates are introduced and it is shown that the Darboux transformation leads to the classical version of the Holstein–Primakoff realization of SU(1,1). In Subsections 3.2 and 3.3 classical BFF conversion is carried out in the Darboux and covariant gauges. As a result, dynamical systems on the Lobachevski plane are described by the physically equivalent gauge-invariant systems with first-class constraints. Due to the manifest SU(1,1) symmetry, the master equations for the first-class constraints are exactly solvable. In Subsection 4.1 direct quantization in Darboux coordinates is carried out. It leads to the Holstein–Primakoff realization of SU(1,1) and non-linear commutation relation between the operators $\hat{z}$ and $\hat{\bar{z}}$ corresponding to the classical coordinates on the Lobachevski plane. In Subsections 4.2–4.4 BFV quantization of the BFF converted system with first-class constraints is carried out. In Subsection 4.2 quantum first-class constraints are explicitly found. Due to the SU(1,1) symmetry, the quantum master equations turn out to be exactly solvable. These quantum first-class constraints define a flat SU(1,1) connection in the Fock bundle over the Lobachevski plane with the standard Fock spaces as fibers. In Subsection 4.3 the master equations for physical quantum states are exactly solved. It is shown that physical quantum states, that are covariantly constant sections of the Fock bundle with the flat SU(1,1) connection, coincide with the generalized SU(1,1) coherent states [15]. In Subsection 4.4 the master equations for quantum observables are solved. It is shown that the vacuum expectation values of quantum observables of our gauge-invariant system with first-class constraints coincide with the covariant symbols of Berezin [16]. In particular, Berezin's multiplication formula for covariant symbols is obtained as a vacuum expectation value $\langle 0|\hat{A}_1\hat{A}_2|0\rangle$. In Section 5 gauge-invariant quantization on arbitrary hermitian symmetric spaces is considered.

2. Summary of the BFV approach to geometric quantization

This section contains a brief summary of the main results obtained in Ref. [1].

Consider a symplectic manifold $\mathcal{M}$ with non-degenerate symplectic structure ω. In a generic coordinate system x^μ, $\mu = 1, 2, \dots, 2N$, a dynamical system with the hamiltonian $H = H(x)$ is described by the hamiltonian equations of motion

$$\dot{x}^\mu = [x^\mu, H]^\omega_{\rm PB} , \tag{2.1}$$

where the Poisson brackets are defined by the inverse of the symplectic form $\omega_{\mu\nu}$, $\omega_{\mu\nu}\omega^{\nu\rho} = \delta^\rho_\mu$,

$$[x^\mu, x^\nu]^\omega_{\rm PB} = \omega^{\mu\nu}(x) . \tag{2.2}$$

The BFV approach to quantization of the system (2.1), (2.2) consists of three steps [2–5,1]. First, one re-formulates it as a system with second-class constraints and canonical Poisson brackets. Next, the second-class constraints are converted into the first-class ones according to the BFF conversion procedure by introducing extra gauge degrees of freedom. Then, the resulting gauge system is quantized according to the standard BFV quantization method.

First, a new set of variables p_μ, $\mu = 1, 2, \dots, 2N$, is introduced and a new symplectic structure on the enlarged phase space of both variables x^μ and p_μ is defined

$$[x^\mu, p_\nu]_{\rm PB} = \delta^\mu_\nu , \tag{2.3}$$

i.e. the enlarged phase space is the cotangent bundle $T^*\mathcal{M}$. To reduce the number of physical degrees of freedom, the new variables are subjected to the second-class constraints [4]

$$\theta_\mu = p_\mu - V_\mu \approx 0 , \tag{2.4}$$

where $V_\mu = V_\mu(x)$ is a symplectic potential

$$\partial_\mu V_\nu - \partial_\nu V_\mu = \omega_{\mu\nu}(x), \tag{2.5}$$

so that

$$[\theta_\mu, \theta_\nu] = \omega_{\mu\nu}(x) . \tag{2.6}$$

If ω is not exact, V is globally defined as a connection in the Kostant–Souriau line bundle L over $\mathcal{M}$ with the curvature $(-i\omega/\hbar)$. Thus, under the substitution $p_\mu \to$

$-i\hbar\partial_\mu$, the second-class constraints (2.4) can be interpreted as covariant derivatives $\nabla_\mu = \partial_\mu - (i/\hbar)V_\mu$ acting on the sections of L.

To convert the second-class constraints into the first-class ones, extra gauge degrees of freedom are introduced [2–5] that are described by the canonical variables ϕ_a, $a = 1,2,\dots,2N$, with the Poisson brackets

$$[\phi_a,\phi_b] = -\Lambda_{ab}, \tag{2.7}$$

where $\Lambda_{ab} = -\Lambda_{ba}$ is a flat symplectic metric with the inverse Λ^{ab}, $\Lambda_{ab}\Lambda^{bc} = \delta^c_a$.

According to the BFF conversion method [2–5], abelian first-class constraints are sought for in the form

$$\mathcal{T}_\mu = p_\mu - W_\mu(x,\phi) \approx 0 , \tag{2.8}$$

$$[\mathcal{T}_\mu,\mathcal{T}_\nu]_{\rm PB} = 0 , \tag{2.9}$$

with the initial conditions

$$\mathcal{T}_\mu|_{\phi=0} = \theta_\mu, \text{ or } W_\mu|_{\phi=0} = V_\mu . \tag{2.10}$$

Abelian Poisson brackets can be regarded as zero-curvature equations ($\partial^a_\phi = \partial/\partial\phi_a$)

$$\mathcal{R}_{\mu\nu} := \partial_\mu W_\nu - \partial_\nu W_\mu - \partial^a_\phi W_\mu \Lambda_{ab}\partial^b_\phi W_\nu = 0 , \tag{2.11}$$

with the initial condition (2.10).

For every classical observable $A = A(x)$ of the original dynamical system on $\mathcal{M}$, a BFF-extended classical observable $\mathcal{A} = \mathcal{A}(x,\phi)$ is constructed so that it commutes with the first-class constraints

$$[\mathcal{T}_\mu,\mathcal{A}]_{\rm PB} = 0 , \tag{2.12a}$$

i.e.

$$\mathcal{D}_\mu\mathcal{A} = 0 , \tag{2.12b}$$

$$\mathcal{D}_\mu := \partial_\mu - \partial^a_\phi W_\mu \Lambda_{ab}\partial^b_\phi , \tag{2.12c}$$

and satisfies the initial condition

$$\mathcal{A}(x,\phi)|_{\phi=0} = A(x) . \tag{2.13}$$

Eqs. (2.11) and (2.12) with the initial conditions (2.10), (2.13) can be solved perturbatively by expanding in the powers of ϕ and solutions are given by [1]

$$\mathcal{T}_\mu = p_\mu - V_\mu - h^a_\mu\phi_a + \frac{1}{2}\Delta^{ab}_\mu\phi_a\phi_b - \frac{3}{4}\Lambda^{a_1 b}h^\nu_b R^{a_2 a_3}_{\mu\nu}\phi_{a_1}\phi_{a_2}\phi_{a_3} + \sum_{k=4}^{\infty}\frac{1}{k!}W^{a_1\dots a_k}_\mu\phi_{a_1}\dots\phi_{a_k} \approx 0 , \tag{2.14}$$

and

$$\mathcal{A} = A + \Phi^{\mu}\partial_{\mu}A + \frac{1}{2}\Phi^{\mu_1}\Phi^{\mu_2}\nabla_{\mu_1\mu_2}A$$
$$+\frac{1}{6}\Phi^{\mu_1}\Phi^{\mu_2}\Phi^{\mu_3}(\nabla_{\mu_1\mu_2\mu_3}A - \frac{1}{4}R_{\nu\mu_1\mu_2}{}^{\sigma}\omega_{\sigma\mu_3}\omega^{\nu\rho}\partial_{\rho}A)$$
$$+\sum_{k=4}^{\infty}\frac{1}{k!}\mathcal{A}^{a_1\ldots a_k}\phi_{a_1}\ldots\phi_{a_k} , \tag{2.15}$$

where h^a_μ is a symplectic frame on $\mathcal{M}$ with the inverse h^μ_a, $\Delta^{ab}_\mu = \Delta^{ba}_\mu$ – torsion-free $\mathrm{Sp}(2N;\mathbb{R})$ symplectic connection, $R^{ab}_{\mu\nu}$-curvature of this connection, $W^{a_1\ldots a_k}_\mu$ – symplectic "higher spin fields" that are expressed through the higher orders of the symplectic curvature $R^{ab}_{\mu\nu}$, $\Phi^\mu := \phi_a \Lambda^{ab} h^\mu_b$, ∇_μ is a covariant derivative on $\mathcal{M}$ with respect to the torsion-free linear symplectic connection $\Gamma_{\mu,\nu}{}^{\rho}$ defined by $\mathrm{Sp}(2N;\mathbb{R})$ connection Δ^{ab}_μ according to the relation

$$\partial_\mu h^a_\nu + \Delta^{ab}_\mu \Lambda_{bc} h^c_\nu - \Gamma_{\mu\nu}{}^{\rho} h^a_\rho = 0 , \tag{2.16}$$

and $R^{\sigma}_{\mu\nu\rho}$ is the corresponding curvature tensor, $R^{ab}_{\mu\nu}(\Delta) = R_{\mu\nu\rho}{}^{\sigma}(\Gamma) h^a_\sigma h^\rho_c \Lambda^{cb}$.

Geometrically, the BFF conversion procedure can be interpreted as follows. Consider a bundle $\mathcal{E}$ over $\mathcal{M}$ with the fibers $F \simeq C^\infty(\mathbb{R}^{2N})$ (where $(\mathbb{R}^{2N}, \Lambda)$ is a flat phase space of the variables ϕ_a) and the structure group $\mathcal{G} \simeq \mathrm{Symp}(\mathbb{R}^{2N})$ of symplectomorphisms (canonical transformations) of $(\mathbb{R}^{2N}, \Lambda)$. Then the first-class constraints $\mathcal{T}_\mu$ define a flat connection $\mathcal{D}$ on $\mathcal{E}$ and classical observables commuting with the constraints are covariantly constant sections of $\mathcal{E}$. Thus, to every classical observable A on $\mathcal{M}$, the BFF conversion procedure associates a covariantly constant section $\mathcal{A}$ of the $\mathrm{Symp}(\mathbb{R}^{2N})$-bundle $\mathcal{E}$ endowed with a flat connection defined by the abelian first-class constraints (2.15).

Now one can carry out the quantization directly

$$[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\delta^\mu_\nu,\ \hat{x}^\mu = x^\mu, \hat{p}_\mu = -i\hbar\partial_\mu, \tag{2.17}$$
$$[\hat{\phi}_a, \hat{\phi}_b] = -i\hbar\Lambda_{ab} . \tag{2.18}$$

The quantum first-class constraints are commuting operators [1]

$$\hat{\mathcal{D}}_\mu := \frac{i}{\hbar}\hat{\mathcal{T}}_\mu = \partial_\mu - \frac{i}{\hbar}[V_\mu + (h^a_\mu + \hbar^2 \overset{(2)}{W}{}^a_\mu)\hat{\phi}_a$$
$$-\frac{1}{2}\Delta^{ab}_\mu\hat{\phi}_a\hat{\phi}_b + \frac{3}{4}\Lambda^{ad}h^\nu_d R^{bc}_{\mu\nu}\hat{\phi}_a\hat{\phi}_b\hat{\phi}_c + \ldots] , \tag{2.19}$$

$$[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu] = 0 , \tag{2.20}$$

where

$$\overset{(2)}{W}{}^a_\mu = -\frac{1}{2}\Lambda^{ab}h^\nu_b\,\overset{(2)}{\omega}_{\mu\nu}, \tag{2.21a}$$

$$\overset{(2)}{\omega}_{\mu\nu} = \frac{1}{128}(R_{\mu\rho\lambda}{}^{\delta}R_{\nu\sigma\delta}{}^{\lambda}\omega^{\rho\sigma} + 2R_{\mu\lambda\rho}{}^{\delta}R_{\nu\delta\sigma}{}^{\lambda}\omega^{\rho\sigma}), \tag{2.21b}$$

and dots designate the terms of higher order in $\hat{\phi}$ and $\hbar$. All the quantum corrections are found from the commutativity conditions (2.20). (In Eq. (2.19), Weyl (symmetric) ordering of the operators $\hat{\phi}$ is used.)

BFF-extended quantum observables $\hat{\mathcal{A}}$ commute with the constraints operators $\hat{\mathcal{D}}_\mu$,

$$[\hat{\mathcal{D}}_\mu, \hat{\mathcal{A}}] = 0, \tag{2.22}$$

and are given by [1]

$$\hat{\mathcal{A}} = A + \hat{\Phi}^\mu (\partial_\mu A + \hbar^2 \overset{(2)}{\mathcal{A}_\mu}) + \frac{1}{2}\hat{\Phi}^{\mu_1}\hat{\Phi}^{\mu_2}\nabla_{\mu_1\mu_2}A$$
$$+\frac{1}{6}\hat{\Phi}^{\mu_1}\hat{\Phi}^{\mu_2}\hat{\Phi}^{\mu_3}(\nabla_{\mu_1\mu_2\mu_3}A - \frac{1}{4}R_{\nu\mu_1\mu_2}{}^{\sigma}\omega_{\sigma\mu_3}\omega^{\nu\rho}\partial_\rho A) + \dots , \tag{2.23}$$

where $\hat{\Phi}^\mu := \hat{\phi}_a \Lambda^{ab} h^\mu_b$, $\nabla_{\mu_1 \dots \mu_n} = \frac{1}{n!}\nabla_{(\mu_1} \dots \nabla_{\mu_n)}$, $(\mu_1 \dots \mu_n)$ denotes (unweighted) symmetrization, the quantum correction $\overset{(2)}{\mathcal{A}_\mu}$ is given by

$$\overset{(2)}{\mathcal{A}_\mu} = \frac{1}{32}R_{\mu\nu\sigma}{}^{\rho_1}\omega^{\nu\rho_2}\omega^{\sigma\rho_3}\nabla_{\rho_1\rho_2\rho_3}A + \frac{5}{6}\overset{(2)}{\omega_{\mu\nu}}\omega^{\nu\rho}\partial_\rho A \ , \tag{2.24}$$

and the dots in (2.23) denote the terms of higher orders in $\hat{\phi}$ and $\hbar$. The quantum corrections are found from the commutativity condition (2.22).

From expression (2.23) for quantum observables, the star product $*_\Gamma$ on the symplectic manifold $\mathcal{M}$ with symplectic connection Γ is obtained in the "unitary" gauge $\phi_a = 0$ [1]

$$A *_\Gamma B = (\mathcal{A} * \mathcal{B})|_{\phi=0} = AB + \frac{1}{2}i\hbar\partial_\mu A\ \omega^{\mu\nu}\partial_\nu B$$
$$-\frac{1}{8}\hbar^2\nabla_{\mu_1\mu_2}A\ \omega^{\mu_1\mu_2,\nu_1\nu_2}\nabla_{\nu_1\nu_2}B - \frac{1}{48}i\hbar^3\mathcal{L}_{\mu_1\mu_2\mu_3}A\ \omega^{\mu_1\mu_2\mu_3,\nu_1\nu_2\nu_3}\mathcal{L}_{\nu_1\nu_2\nu_3}B + \dots , \tag{2.25}$$

where

$$\mathcal{L}_{\mu_1\mu_2\mu_3} = \nabla_{\mu_1\mu_2\mu_3} + \frac{1}{6}R_{\rho(\mu_1\mu_2}{}^{\lambda}\omega_{\mu_3)\lambda}\omega^{\rho\sigma}\partial_\sigma \ . \tag{2.26}$$

and $\omega^{\mu_1 \dots \mu_n, \nu_1 \dots \nu_n} = \omega^{\mu_1\nu_1} \dots \omega^{\mu_n\nu_n}$. This star product coincides with the star product known in deformation quantization [12–14]. Indeed, in Darboux coordinates, operator $\mathcal{L}$ can be rewritten in the form

$$\mathcal{L}_{\mu_1\mu_2\mu_3} = \partial_{\mu_1\mu_2\mu_3} - 3\Gamma^\rho_{(\mu_1\mu_2}\partial_{\mu_3)\rho} + \partial_\nu\Gamma^\sigma_{(\mu_1\mu_2}\omega_{\mu_3)\sigma}\omega^{\nu\rho}\partial_\rho, \tag{2.27}$$

and the third order term in Eq. (2.25) coincides with the Chevalley cocycle S^3_Γ of Refs. [12–14].

Next, the Hilbert space representation of the algebra of quantum observables is constructed. The operators $\hat{\phi}_a$ are separated into the creation and annihilation operators (In Ref. [1] we have selected the gauge degrees of freedom so that their creation operators generate states with negative norm, Eq. (6.1) of Ref. [1], similar to the temporal component of the electromagnetic potential. Then to obtain the physical Hilbert space one has to use the quartet mechanism, as in electrodynamics. Here we select the gauge degrees of freedom so that they generate positive norm states.)

$$[\hat{a}^i, \hat{a}_j^\dagger] = \hbar\delta^i_j, \qquad (\hat{a}^i)^\dagger = \hat{a}_i^\dagger, \quad i,j = 1,2,\ldots,N . \tag{2.28}$$

(Comparing to electrodynamics, if one chooses a complex polarization on $\mathcal{M}, x^\mu \to (z^\alpha, \bar{z}_\beta)$, then ϕ is a counterpart of A_0, $\bar{z}$ is a counterpart of A_3 and z is a counterpart of the physical degrees of freedom.) The quantum ghosts and their momenta are introduced (we consider the minimal sector only)

$$\hat{C}^\mu = C^\mu, \qquad \hat{\bar{\mathcal{P}}}_\mu = i\hbar\frac{\partial_\ell}{\partial C^\mu} , \tag{2.29}$$

and the Fock vacuum is selected

$$\hat{a}^i|0\rangle = 0, \quad i = 1,2,\ldots,N . \tag{2.30}$$

The minimal BFV-BRST operator $\hat{\Omega}$ is given by

$$\hat{\Omega} = C^\mu \hat{\mathcal{T}}_\mu, \qquad \hat{\mathcal{T}}_\mu = -i\hbar\left(\partial_\mu - \frac{i}{\hbar}\hat{W}_\mu\right) , \tag{2.31}$$

with the quantum constraints $\hat{\mathcal{T}}_\mu$, where the "gauge field" $\hat{W} = W(x, \hat{a}^\dagger, \hat{a})$ is now a Wick ordered operator.

Physical quantum states $|\psi_{\rm phys}\rangle$ must satisfy the master equation

$$\hat{\Omega}|\psi_{\rm phys}\rangle = 0 . \tag{2.32}$$

Locally, in the coordinate neighborhood Σ of the point x_0, the general solution for physical quantum states is given by

$$|\psi_{\rm phys}\rangle = \hat{\mathcal{U}}(x, x_0)|\psi_{\rm phys,0}\rangle , \tag{2.33}$$

where the operator of parallel transport from x_0 to $x \in \Sigma$ is defined by

$$\hat{\mathcal{U}}(x, x_0) = P\exp\left(\frac{i}{\hbar}\int_{x^0}^{x}\hat{W}\right), \qquad \hat{W} = {\rm d}x^\mu \hat{W}_\mu , \tag{2.34}$$

and $|\psi_{\rm phys,0}\rangle = \psi_{\rm phys,0}(\hat{a}^\dagger)|0\rangle$ is an initial condition independent of x.

Similarly, for the quantum observables which are now hermitian operators commuting with $\hat{\Omega}$, a general solution is given in the form

$$\hat{\mathcal{A}} = \hat{\mathcal{U}}\hat{\mathcal{A}}_0\,\hat{\mathcal{U}}^\dagger , \tag{2.35}$$

with the x-independent initial condition $\hat{\mathcal{A}}_0 = \mathcal{A}_0(\hat{a}^\dagger, \hat{a})$. Globally, $|\psi_{\rm phys}\rangle$ are covariantly constant sections of the Fock bundle.

Let $\mathcal{R}_0 = \{\hat{a}^\dagger_{i_1}\ldots\hat{a}^\dagger_{i_k}|0\rangle,\ k = 0,1,\ldots\}$ be the standard Fock space, $\mathcal{G}$- the group of unitary operators in $\mathcal{R}_0$ and $\mathcal{F}_0$ – the associative algebra of hermitian operators in $\mathcal{R}_0$.

Consider a Fock bundle $\mathcal{RM}$ over $\mathcal{M}$ with fibers $F \simeq \mathcal{R}_0$ and the structure group $\mathcal{G}$. Then the BFV-BRST operator $\hat{\Omega}$ defines a flat connection in $\mathcal{RM}$, and physical states are covariantly constant sections of $\mathcal{RM}$ with respect to this connection. Further, the adjoint operator $ad\hat{\Omega}$ defines a flat connection on the associated bundle $\mathcal{FM}$ of hermitian operators over $\mathcal{M}$ with fibers $F \simeq \mathcal{F}_0$ and the structure group $\mathcal{G}/\mathrm{U}(1)$ acting on the fibers by conjugations $\hat{\mathcal{U}}\hat{A}\hat{\mathcal{U}}^\dagger$. Then the quantum observables $\hat{A}$ are covariantly constant sections of $\mathcal{FM}$ with respect to this connection. In Ref. [1] it was shown that, in the case when the infinite-dimensional structure group $\mathcal{G}$ can be reduced to its maximal finite-dimensional subgroup isomorphic to $[\mathrm{Sp}(2\mathrm{N};\mathbb{R}) \otimes \mathrm{U}(1)]/\mathbb{Z}_2$, the necessary and sufficient condition for the existence of the Fock bundle $\mathcal{RM}$ over a symplectic manifold $\mathcal{M}$ coincides with the corrected quantization condition with metaplectic anomaly known in geometric quantization [17–21], i.e. the first Chern class $c_1(LK^{1/2}) = c_1(L) + \frac{1}{2}c_1(K)$ of the line bundle $LK^{1/2}$, where L is the Kostant-Souriau bundle and $K^{1/2}$ is the square root of the determinant bundle K, must be integer

$$\frac{1}{2\pi\hbar}\int_{\Sigma}\omega + \frac{1}{4\pi}\int_{\Sigma} R^i_i \in \mathbb{Z}\,, \tag{2.36}$$

where R^i_j is the Ricci curvature of $\mathrm{U}(N)$-connection in the bundle of unitary frames over $\mathcal{M}$. The $LK^{1/2}$ is a vacuum (sub)bundle of the Fock bundle with sections $\psi(x)|0\rangle$ independent of $\hat{a}^\dagger_i$ and a connection $V + \frac{1}{2}\hbar\Delta^i_i$, where V is the symplectic potential and Δ^i_j is the $\mathrm{U}(N) \subset \mathrm{Sp}(2n;\mathbb{R})$ connection. It is also called the bundle of pure symplectic spinors [18]. As we have stressed in Ref. [1], to quantize the entire algebra of quantum observables it is essential to consider an entire infinite-dimensional Fock bundle with the flat connection $\hat{\Omega}$, rather than only the line bundle $LK^{1/2}$ which generally does not allow a flat connection.

3. Classical mechanics on the Lobachevski plane and classical BFF conversion

3.1. Darboux coordinates and the classical Holstein–Primakoff realization

Let us consider a Poincaré model of the Lobachevski plane realized as a unit disc on the complex plane $D_1 = \{z : |z| < 1\}$ with the Poisson brackets

$$[z, \bar{z}]^{\omega}_{\mathrm{PB}} = i(1 - z\bar{z})^2\,. \tag{3.1}$$

The SU(1,1) symmetry acts on D_1 by fractional-linear transformations

$$z \to z_g = \frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\,, \quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad g \in \mathrm{SU}(1,1), \tag{3.2}$$

and D_1 can be represented as a coset $D_1 \approx \mathrm{SU}(1,1)/\mathrm{U}(1)$. At the infinitesimal level, generators of this symmetry are hamiltonian vector fields with the hamiltonian functions

$$L_0 = \frac{i}{2}\left(\frac{1+z\bar{z}}{1-z\bar{z}}\right) , \tag{3.3a}$$

$$L_+ = \frac{iz}{1-z\bar{z}} , \tag{3.3b}$$

$$L_- = \frac{i\bar{z}}{1-z\bar{z}} . \tag{3.3c}$$

It is easy to check that the functions (3.4) do indeed satisfy the su(1,1) commutation relations with respect to the Poisson brackets (3.2)

$$[L_0, L_\pm]^{\omega}_{\rm PB} = \pm L_\pm , \tag{3.4a}$$

$$[L_+, L_-]^{\omega}_{\rm PB} = -2L_0 . \tag{3.4b}$$

One can introduce Darboux coordinates on the Lobachevski plane as follows:

$$\xi = (1-z\bar{z})^{-1/2}z, \qquad \bar{\xi} = (1-z\bar{z})^{-1/2}\bar{z} , \tag{3.5}$$

and the inverse transformation is

$$z = (1+\xi\bar{\xi})^{-1/2}\xi, \qquad \bar{z} = (1+\xi\bar{\xi})^{-1/2}\bar{\xi} . \tag{3.6}$$

In the new coordinates the Poisson brackets (3.1) acquire canonical form

$$[\xi, \bar{\xi}]^{\omega}_{\rm PB} = i , \tag{3.7}$$

and the su(1,1) generators (3,4) take the form

$$L_0 = \frac{i}{2}(2\xi\bar{\xi}+1) , \tag{3.8a}$$

$$L_+ = i(1+\xi\bar{\xi})^{1/2}\xi, \tag{3.8b}$$

$$L_- = i(1+\xi\bar{\xi})^{1/2}\bar{\xi} . \tag{3.8c}$$

We will call this Poisson bracket realization of su(1,1) classical Holstein–Primakoff realization since it is a classical form of the oscillator realization of su(1,1) discovered in Ref. [22] (see also Ref. [23]).

Let us compare it with the more frequently used realizations

$$L_0 = \frac{i}{2}\xi\bar{\xi}, \qquad L_+ = \frac{i}{2}\xi^2, \qquad L_- = \frac{i}{2}\bar{\xi}^2 , \tag{3.9}$$

and

$$L_0 = \frac{i}{2}(2\xi\bar{\xi}+1), \qquad L_+ = i\xi(\xi\bar{\xi}+1), \qquad L_- = i\bar{\xi}. \tag{3.10}$$

Similar to the realization (3.10), after quantization the Holstein–Primakoff realization yields the entire series of representations of su(1,1) with the lowest weights $1/(2\hbar)$. At the same time, similar to the realization (3.9), $\hat{L}_+$ and $\hat{L}_-$ are hermitian conjugate under the natural hermitian conjugation $(\hat{\xi})^\dagger = \hat{\bar{\xi}}$. Thus the realization (3.8) combines the desired properties of both realizations (3.9) and (3.10). The quantum Holstein–Primakoff realization was first used in Ref. [22] as a means to diagonalize certain hamiltonians. As we have seen, it can be interpreted as a quantization of the Darboux coordinates on the Lobachevski plane.

3.2. Classical BFF conversion in the Darboux gauge

Our dynamical system with the Poisson brackets (3.1) and the hamiltonian $H = H(z,\bar{z})$ can be represented by a physically equivalent system with the flat phase space and first-class constraints. First, let us introduce canonical momenta p and $\bar{p}$ to z and $\bar{z}$ and a new symplectic structure with the non-zero Poisson brackets

$$[z,p]_{\rm PB} = 1 \,, \tag{3.11a}$$

$$[\bar{z},\bar{p}]_{\rm PB} = 1 \tag{3.11b}$$

(the non-linear Poisson brackets (3.1) are distinguished from the flat brackets (3.11) by the index ω, as in $[\ ,\ ]^{\omega}_{\rm PB}$). Then the second-class constraints that reduce the number of degrees of freedom can be chosen in the form

$$\theta = p + \frac{i\bar{z}}{2(1-z\bar{z})} \approx 0, \qquad \bar{\theta} = \bar{p} - \frac{iz}{2(1-z\bar{z})} \approx 0 \,, \tag{3.12}$$

and their Poisson bracket is

$$[\theta,\bar{\theta}]_{\rm PB} = i(1-z\bar{z})^{-2} \,. \tag{3.13}$$

Now it is easy to check that the original non-linear Poisson brackets (3.1) are recovered as Dirac brackets with respect to the second-class constraints θ and $\bar{\theta}$.

Our next task is to convert the second-class constraints into the first-class ones. Following the receipt of Ref. [2-5,1], let us introduce additional canonical variables ϕ and $\bar{\phi}$

$$[\phi,\bar{\phi}]_{\rm PB} = -i \,. \tag{3.14}$$

According to the BFF conversion procedure, the abelian first-class constraints are sought for in the form

$$\mathcal{T} = p + iW(z,\bar{z};\phi,\bar{\phi}) \approx 0, \qquad \mathcal{T}|_{\phi=0} = \theta, \tag{3.15a}$$

$$\bar{\mathcal{T}} = \bar{p} - i\bar{W}(z,\bar{z};\phi,\bar{\phi}) \approx 0, \qquad \bar{\mathcal{T}}|_{\phi=0} = \bar{\theta} \,, \tag{3.15b}$$

and must satisfy the abelian Poisson brackets relations

$$[\mathcal{T},\bar{\mathcal{T}}]_{\rm PB} = 0 \tag{3.16}$$

($W = W(z,\bar{z};\phi,\bar{\phi})$ is a function of $z,\bar{z},\phi$ and $\bar{\phi}$ to be found from Eq. (3.16)). As we have discussed in [1], there is a gauge freedom in solving the Eq. (3.16). We will consider two most interesting gauges, one is related to the Darboux coordinates (3.5) and the other is manifestly covariant under the SU(1,1) symmetry on the Lobachevski plane (3.2).

Let us start with the Darboux gauge. First, canonical momenta to the Darboux coordinates (3.5) are given by

$$\pi = \frac{1}{2}(1 - z\bar{z})^{1/2}\{(2 - z\bar{z})p - \bar{z}^2\bar{p}\} , \tag{3.17a}$$

$$\bar{\pi} = \frac{1}{2}(1 - z\bar{z})^{1/2}\{(2 - z\bar{z})\bar{p} - z^2 p\} , \tag{3.17b}$$

with the inverse

$$p = \frac{1}{2}(1 + \xi\bar{\xi})^{1/2}\{(2 + \xi\bar{\xi})\pi + \bar{\xi}^2\bar{\pi}\} , \tag{3.18a}$$

$$\bar{p} = \frac{1}{2}(1 + \xi\bar{\xi})^{1/2}\{(2 + \xi\bar{\xi})\bar{\pi} + \xi^2\pi\} , \tag{3.18b}$$

and

$$[\xi, \pi]_{\text{PB}} = 1, \qquad [\bar{\xi}, \bar{\pi}]_{\text{PB}} = 1 . \tag{3.19}$$

Then the second-class constraints θ and $\bar{\theta}$ (3.12) can be reduced to a new set of second-class constraints θ_D and $\bar{\theta}_D$ (D stands for Darboux)

$$\theta = \frac{1}{2}(1 + \xi\bar{\xi})^{1/2}\{(2 + \xi\bar{\xi})\theta_D + \bar{\xi}^2\bar{\theta}_D\} , \tag{3.20a}$$

$$\bar{\theta} = \frac{1}{2}(1 + \xi\bar{\xi})^{1/2}\{(2 + \xi\bar{\xi})\bar{\theta}_D + \xi^2\theta_D\} , \tag{3.20b}$$

with the inverse

$$\theta_D = \frac{1}{2}(1 - z\bar{z})^{1/2}\{(2 - z\bar{z})\theta - \bar{z}^2\bar{\theta}\} , \tag{3.21a}$$

$$\bar{\theta}_D = \frac{1}{2}(1 - z\bar{z})^{1/2}\{(2 - z\bar{z})\bar{\theta} - z^2\theta\} , \tag{3.21b}$$

where

$$\theta_D = \pi + \frac{1}{2}i\bar{\xi} , \tag{3.22a}$$

$$\bar{\theta}_D = \bar{\pi} - \frac{1}{2}i\xi , \tag{3.22b}$$

and

$$[\theta_D, \bar{\theta}_D]_{\text{PB}} = i . \tag{3.23}$$

Now, the Darboux second-class constraints can be converted into the first-class ones

$$\mathcal{T}_D = \theta_D + i\phi = \pi + \frac{1}{2}i\bar{\xi} + i\phi \approx 0 , \tag{3.24a}$$

$$\bar{\mathcal{T}}_D = \bar{\theta}_D - i\bar{\phi} = \bar{\pi} - \frac{1}{2}i\xi - i\bar{\phi} \approx 0 . \tag{3.24b}$$

Further, the original first-class constraints $\mathcal{T}$ and $\bar{\mathcal{T}}$ can be found by using the Darboux transformations (3.6a) and (3.20)

$$\mathcal{T} = p + \frac{i\bar{z}}{2(1-z\bar{z})} + \frac{i}{2}(1-z\bar{z})^{-3/2}\{(2-z\bar{z})\phi - \bar{z}^2\bar{\phi}\} \approx 0\,, \tag{3.25a}$$

$$\bar{\mathcal{T}} = \bar{p} - \frac{iz}{2(1-z\bar{z})} - \frac{i}{2}(1-z\bar{z})^{-3/2}\{(2-z\bar{z})\bar{\phi} - z^2\phi\} \approx 0\,. \tag{3.25b}$$

Indeed, it is easy to see that the Poisson bracket (3.16) does vanish.

BFF-extended classical observables that commute with the first-class constraints are now obtained by substituting BFF-extended coordinates $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ for the original coordinates z and $\bar{z}$ in the original expressions for classical observables. Given the hamiltonian $H = H(z,\bar{z})$, the BFF-extended hamiltonian is $\mathcal{H} = \mathcal{H}(\mathcal{Z},\bar{\mathcal{Z}})$,

$$\mathcal{Z} = (z + (1-z\bar{z})^{1/2}\bar{\phi}) \tag{3.26a}$$
$$\times[1 - z\bar{z} + (z + (1-z\bar{z})^{1/2}\bar{\phi})(\bar{z} + (1-z\bar{z})^{1/2}\phi)]^{-1/2}\,,$$

$$\bar{\mathcal{Z}} = (\bar{\mathcal{Z}})\,. \tag{3.26b}$$

To obtain the expression (3.26) one first has to pass to the Darboux coordinates (3.6a), make the shift $\xi + \bar{\phi}$ and $\bar{\xi} + \phi$ and then return back to the original coordinates z and $\bar{z}$ using the inverse Darboux transformation (3.6).

3.3. *Classical BFF conversion in the covariant gauge*

In this section we will give a solution to the classical zero-curvature equations directly in the manifestly su(1,1) covariant gauge. We will start with the following ansatz for the first-class constraints (3.15)

$$\mathcal{T} = \theta + \frac{i}{1-z\bar{z}}\{\bar{z}\phi\bar{\phi} + \phi f(\phi\bar{\phi})\} \approx 0\,, \tag{3.27a}$$

$$\bar{\mathcal{T}} = \bar{\theta} - \frac{i}{1-z\bar{z}}\{z\phi\bar{\phi} + \bar{\phi} f(\phi\bar{\phi})\} \approx 0\,, \tag{3.27b}$$

where $f(\phi\bar{\phi})$ is a real analytic function of $\phi\bar{\phi}$ to be found from the zero-curvature equations subject to the initial condition $f(0) = \varepsilon$, $\varepsilon = \pm 1$.

As we have seen in Section 2, coefficients in front of the terms quadratic in ϕ and $\bar{\phi}$ are nothing but the symplectic connection Δ^{ab}_{μ}. The symplectic connection is not uniquely defined. For the Lobachevski plane considered as a symplectic manifold, it can be set to zero, as we have done in the previous section. That corresponds to the choice of the Darboux gauge. However, this Darboux choice is incompatible with the Kähler structure of the Lobachevski plane and, therefore, is not SU(1,1) covariant. If we require our symplectic connection to be compatible with the natural Kähler structure of the Lobachevski plane, the symplectic connection is uniquely fixed and its coefficients coincide with the Christoffel symbols of the riemannian metric on the Lobachevski plane. We will call this choice covariant gauge. Then the coefficients in front of the quadratic term $\phi\bar{\phi}$ in the curly brackets in Eq. (3.27) are just the Christoffel symbols. Now let us consider the term $\phi f(\phi\bar{\phi})$, which can be expanded in a series $\varepsilon\phi(1 + f_1\phi\bar{\phi} + f_2(\phi\bar{\phi})^2 + \ldots)$. Coefficients in front of the linear term are just the components of the SU(1,1)-covariant frame on the Lobachevski plane (two choices of the initial condition in (3.27), $\varepsilon = 1$ and -1, correspond to the two alternate frame orientations). The coefficient $f_1/(1 - z\bar{z})$ in front of the third order term $\phi^2\bar{\phi}$ is just a normalized riemannian curvature tensor $R_{z\bar{z}z\bar{z}}$ of the Lobachevski plane. Generally, the coefficients $f_1, f_2, \ldots$ would have to be found step by step. However, in the case under investigation, because of the SU(1,1) symmetry, zero-curvature equations are exactly solvable. Indeed, calculating the Poisson brackets one finds

$$[\mathcal{T}, \bar{\mathcal{T}}]_{\text{PB}} = \frac{i}{(1 - z\bar{z})^2}(1 + 2\phi\bar{\phi}) + \frac{1}{(1 - z\bar{z})^2}[\phi f, \bar{\phi} f]_{\text{PB}} = 0\,, \tag{3.28}$$

and the equation to be solved takes the form

$$f^2 + 2xff' = 1 + 2x, \qquad x = \phi\bar{\phi}, \qquad f(0) = \varepsilon\,. \tag{3.29}$$

It can be immediately integrated to yield

$$f = \varepsilon(1 + \phi\bar{\phi})^{1/2}\,. \tag{3.30}$$

The arbitrariness in the sign ($\varepsilon = 1$ or -1) is all that is left from the gauge arbitrariness after fixing the SU(1,1)-covariant gauge. Geometrically, these two sign choices correspond to the two alternate frame orientations.

The first-class constraints we have found admit an elegant representation in terms of the su(1,1) generators. Indeed, they can be re-written in the form

$$\mathcal{T} = p + \frac{1}{1 - z\bar{z}}\{\bar{z}L_0 + \varepsilon L_-\} \approx 0\,, \tag{3.31a}$$

$$\bar{\mathcal{T}} = \bar{p} - \frac{1}{1 - z\bar{z}}\{zL_0 + \varepsilon L_+\} \approx 0\,, \tag{3.31b}$$

where L_0 and $L_\pm$ are classical Holstein–Primakoff generators of SU(1,1)

$$L_0 = \frac{i}{2}(2\phi\bar{\phi} + 1)\,, \tag{3.32a}$$

$$L_+ = i(1 + \phi\bar{\phi})^{1/2}\bar{\phi}\,, \tag{3.32b}$$

$$L_- = i(1 + \phi\bar{\phi})^{1/2}\phi\,. \tag{3.32c}$$

Thus, we have found that the first-class constraints are linear in SU(1,1) generators. The arbitrariness in sign ε corresponds to the automorphism of su(1,1), $L_0 \to L_0$, $L_\pm \to -L_\pm$.

4. Quantization

4.1. Direct quantization in the Darboux gauge and Holstein–Primakoff realization

Before we proceed with quantization of our dynamical system with the first-class constraints (3.31), let us consider direct quantization of the system (3.7), (3.8) in the Darboux coordinates. Let us introduce creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ for the classical variables $\bar{\xi}$ and ξ

$$[\hat{a}, \hat{a}^\dagger] = \hbar \ . \tag{4.1}$$

Then the operators corresponding to the classical SU(1,1) generators (3.8) and to the original coordinates (3.5) can be written as follows:

$$\hat{L}_0 = \frac{1}{\hbar}(\hat{a}^\dagger \hat{a} + \frac{1}{2}), \qquad (\hat{L}_0)^\dagger = \hat{L}_0 \ , \tag{4.2a}$$

$$\hat{L}_- = \frac{1}{\hbar}(1 + \hat{a}^\dagger \hat{a})^{1/2} \hat{a} \ , \tag{4.2b}$$

$$\hat{L}_+ = \frac{1}{\hbar}\hat{a}^\dagger (1 + \hat{a}^\dagger \hat{a})^{1/2} \ , \qquad (\hat{L}_-)^\dagger = \hat{L}_+ \ , \tag{4.2c}$$

$$\hat{z} = \hat{a}^\dagger (1 + \hat{a}^\dagger \hat{a})^{-1/2}, \tag{4.2d}$$

$$\hat{\bar{z}} = (1 + \hat{a}^\dagger \hat{a})^{-1/2} \hat{a}, \qquad \hat{z}^\dagger = \hat{\bar{z}} \ . \tag{4.2e}$$

Note how the operator ordering is performed in Eqs. (4.2). The combination $\hat{a}^\dagger \hat{a}$ is treated as a whole operator (the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$) and functions thereof are treated as such (i.e. if we have a classical function $f(\xi\bar{\xi}) = f_0 + f_1 \xi\bar{\xi} + f_2 (\xi\bar{\xi})^2 + \ldots$, the corresponding operator is $\hat{f} = f_0 + f_1 \hat{a}^\dagger \hat{a} + f_2 (\hat{a}^\dagger \hat{a})^2 + \ldots$, and no normal ordering is performed inside the monomials $(\hat{a}^\dagger \hat{a})^n$). Any excess creation or annihilation operators are put left or right, correspondingly (as in Wick ordering).

The operators (4.2a)–(4.2c) satisfy the su(1,1) commutation relations

$$[\hat{L}_0, \hat{L}_\pm] = \pm \hat{L}_\pm \ , \tag{4.3a}$$

$$[\hat{L}_+, \hat{L}_-] = -2\hat{L}_0 \ , \tag{4.3b}$$

and for the operators $\hat{z}$ and $\hat{\bar{z}}$ corresponding to the initial coordinates z and $\bar{z}$ one has

$$[\hat{z}, \hat{\bar{z}}] = -\frac{\hbar}{1-\hbar}(1 - \hat{z}\hat{\bar{z}})(1 - \hat{\bar{z}}\hat{z}) \ . \tag{4.4}$$

To calculate the commutation relations the following properties of functions of the number operator $\hat{N}$ are used

$$f(\hat{N})\hat{a}^\dagger = \hat{a}^\dagger f(\hat{N} + \hbar) \ , \tag{4.5a}$$

$$f(\hat{N})\hat{a} = \hat{a} f(\hat{N} - \hbar) \ . \tag{4.5b}$$

The operators $\hat{L}_0$, $\hat{L}_\pm$ are just the quantum Holstein–Primakoff generators of su(1,1). Using Eqs. (4.2) and the properties (4.5), one can calculate the su(1,1) Casimir operator

$$\hat{C} = -\hat{L}_0^2 + \frac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+) = -\frac{1}{2\hbar}\left(\frac{1}{2\hbar} - 1\right) . \tag{4.6}$$

Thus, $\ell = 1/2\hbar$ plays a role of the su(1,1) spin. Indeed, if $|0\rangle$ is a Fock vacuum vector,

$$\hat{a}|0\rangle = 0 , \tag{4.7}$$

then

$$\hat{L}_0|0\rangle = \frac{1}{2\hbar}|0\rangle, \qquad \hat{L}_-|0\rangle = 0, \qquad \hat{L}_+|0\rangle = \frac{1}{\hbar}\hat{a}^\dagger|0\rangle , \tag{4.8}$$

and $|0\rangle$ is an su(1,1) lowest weight vector with the weight $1/2\hbar$. Generally, the lowest weight is equal to $1/(2\alpha)$, where dimensionless constant α is a product of the Planck constant and the square of the radius of the Lobachevski plane, $\alpha = \hbar r^2$. For the sake of simplicity we have set $r = 1$.

The operators $\hat{z}$ and $\hat{\bar{z}}$ corresponding to the original coordinates of the Lobachevski plane obey an interesting non-linear commutation relation (4.4) with a renormalized Planck constant $\hbar/(1-\hbar)$. To obtain it, one just has to notice that

$$\hat{z}\hat{\bar{z}} = \hat{N}(1 - \hbar + \hat{N})^{-1} , \tag{4.9}$$

$$\hat{\bar{z}}\hat{z} = (\hat{N} + \hbar)(1 + \hat{N})^{-1} . \tag{4.10}$$

4.2. Quantum first-class constraints: A flat SU(1,1) connection in the Fock bundle

Let us now proceed with the quantization of the first-class constraints (3.27). All the phase space variables (z,p), $(\bar{z},\bar{p})$ and $(\phi,\bar{\phi})$ are canonical, so we can proceed directly

$$\hat{z} = z, \qquad \hat{p} = -i\hbar\frac{\partial}{\partial z} , \tag{4.11a}$$

$$\hat{\bar{z}} = \bar{z}, \qquad \hat{\bar{p}} = -i\hbar\frac{\partial}{\partial \bar{z}} , \tag{4.11b}$$

$$\hat{a}^\dagger := \hat{\bar{\phi}} , \qquad \hat{a} := \hat{\phi}, \qquad [\hat{a},\hat{a}^\dagger] = \hbar . \tag{4.11c}$$

Then the quantum first-class constraints take the form

$$\mathcal{D} := \frac{i}{\hbar}\hat{\mathcal{T}} = \frac{\partial}{\partial z} - \frac{1}{1 - z\bar{z}}\{\bar{z}\hat{L}_0 + \varepsilon\hat{L}_-\} , \tag{4.12a}$$

$$\bar{\mathcal{D}} := \frac{i}{\hbar}\hat{\bar{\mathcal{T}}} = \frac{\partial}{\partial \bar{z}} + \frac{1}{1 - z\bar{z}}\{z\hat{L}_0 + \varepsilon\hat{L}_+\} , \tag{4.12b}$$

$$[\mathcal{D},\bar{\mathcal{D}}] = 0 , \tag{4.13}$$

where $\hat{L}_0$ and $\hat{L}_\pm$ are su(1,1) generators (4.2).

Geometrically, the Hilbert space $\mathcal{R}$ of our system with first-class constraints is a Fock bundle over the Lobachevski plane with copies of the standard Fock space $\mathcal{R}_0 = \{(\hat{a}^\dagger)^n|0\rangle,\ n = 0, 1, 2, \ldots\}$ as fibers. Moreover, the quantum first-class constraints define a flat SU(1,1) connection (4.12).

4.3. Physical quantum states: Generalized coherent states as solution of the master equations $\hat{\Omega}|\psi\rangle = 0$

Subspace of physical quantum states $\mathcal{R}_{\text{phys}}$ is defined as a space of covariantly constant sections of the Fock bundle

$$\mathcal{D}|\psi_{\text{phys}}\rangle = 0, \qquad \bar{\mathcal{D}}|\psi_{\text{phys}}\rangle = 0\ , \tag{4.14}$$

$$|\psi_{\text{phys}}\rangle = \psi(z, \bar{z}; \hat{a}^\dagger)|0\rangle\ , \tag{4.15}$$

where ψ is a function of z, $\bar{z}$ and an analytic function of the creation operator $\hat{a}^\dagger$. Due to the SU(1,1) symmetry, these equations can be solved exactly and the general solution for physical states can be represented in the form

$$|\psi_{\text{phys}}\rangle = \hat{\mathcal{U}}|\psi_{\text{phys},0}\rangle\ , \tag{4.16}$$

where the unitary operator $\hat{\mathcal{U}}$ is a SU(1,1) group element and $|\psi_{\text{phys},0}\rangle$ is an initial condition independent of the point z on the Lobachevski plane

$$|\psi_{\text{phys},0}\rangle = \psi_{\text{phys},0}(\hat{a}^\dagger)|0\rangle\ , \tag{4.17}$$

where $\psi_{\text{phys},0}$ is an arbitrary analytic function of $\hat{a}^\dagger$.

We will look for the operator $\hat{\mathcal{U}}$ in the form

$$\hat{\mathcal{U}} = \mathrm{e}^{-\bar{\alpha}\hat{L}_+} e^{\beta\hat{L}_0} \mathrm{e}^{\alpha\hat{L}_-}\ , \tag{4.18}$$

$$\hat{\mathcal{U}}^\dagger = \hat{\mathcal{U}}^{-1}\ , \tag{4.19}$$

where $\alpha = \alpha(z, \bar{z})$ and $\beta = \beta(z, \bar{z})$, $\bar{\beta} = \beta$, are some functions of z and $\bar{z}$ to be found from Eq. (4.14). Using the su(1,1) commutation relations, one finds

$$\begin{aligned}\frac{\partial\hat{\mathcal{U}}}{\partial\bar{z}} &= -\frac{\partial\alpha}{\partial\bar{z}}\hat{L}_+\hat{\mathcal{U}} + \frac{\partial\beta}{\partial\bar{z}}\mathrm{e}^{-\alpha\hat{L}_+}\hat{L}_0\mathrm{e}^{\beta\hat{L}_0}\mathrm{e}^{\bar{\alpha}\hat{L}_-} \\ &= \left(\frac{\partial\beta}{\partial\bar{z}}\hat{L}_0 + \left(\alpha\frac{\partial\beta}{\partial\bar{z}} - \frac{\partial\alpha}{\partial\bar{z}}\right)\hat{L}_+\right)\hat{\mathcal{U}},\end{aligned} \tag{4.20}$$

and the equations for α and β are (for definiteness we set $\varepsilon = -1$ in Eq. (4.12))

$$\frac{\partial\beta}{\partial\bar{z}} = -\frac{z}{1 - z\bar{z}}\ , \qquad \alpha\frac{\partial\beta}{\partial\bar{z}} - \frac{\partial\alpha}{\partial\bar{z}} = \frac{1}{1 - z\bar{z}}\ . \tag{4.21}$$

This yields a solution for the unitary operator $\hat{\mathcal{U}}$

$$\hat{\mathcal{U}} = \mathrm{e}^{-\bar{z}\hat{L}_+}\mathrm{e}^{\ell n(1 - z\bar{z})\hat{L}_0}\mathrm{e}^{z\hat{L}_-}\ . \tag{4.22}$$

The second equation $\mathcal{D}|\psi_{\rm phys}\rangle = 0$ is satisfied automatically due to the unitarity of $\hat{\mathcal{U}}$. Note that $\hat{\mathcal{U}}$ can also be represented in the "anti-Wick" form

$$\hat{\mathcal{U}} = \mathrm{e}^{zL_-}\mathrm{e}^{-\ell n(1-z\bar{z})\hat{L}_0}\mathrm{e}^{-\bar{z}\hat{L}_+} \ . \tag{4.23}$$

Remarkably, Eqs. (4.16), (4.17) and (4.22) coincide with the definition of the generalized coherent states for the group SU(1,1) [15].

Comparing to the general solution on an arbitrary symplectic manifold $\mathcal{M}$, the parallel transport operator which is generally represented by the P-exponent is evaluated exactly since the structure group of the Fock bundle is reduced to SU(1,1).

To summarize, we have found that the physical quantum states, i.e. solutions of the constraints equations of the BFF-converted dynamical system with first-class constraints (4.14), coincide with the generalized coherent states of SU(1,1). Or, in geometric terms, generalized coherent states are covariantly constant sections of the Fock bundle with the flat connection (4.12) defined by the first-class constraints. Note that this definition of coherent states allows for a generalization to the case when no group symmetry is present. Indeed, one can define generalized coherent states for any symplectic manifold $\mathcal{M}$ as covariantly constant sections of the Fock bundle over $\mathcal{M}$ with flat connection defined by the BFV-BRST operator of Ref. [1]. Of course, no exact solution can be obtained in the general case when no symmetry is present, as the master equations can be solved only perturbatively.

Let us consider a state $\hat{\mathcal{U}}|0\rangle$ that is gauge equivalent to the vacuum vector $|0\rangle$ (vacuum state in the covariant gauge). It coincides with the standard SU(1,1) coherent state [15]

$$|\zeta\rangle = (1 - z\bar{z})^{1/2\hbar}\mathrm{e}^{\zeta\hat{L}_+}|0\rangle\,, \quad \zeta = -\bar{z}, \tag{4.24}$$

or, expanding the exponential,

$$|\zeta\rangle = (1 - \zeta\bar{\zeta})^{1/2\hbar}\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+\frac{1}{\hbar})}{n!\Gamma(\frac{1}{\hbar})}\right)^{1/2}(\zeta)^n|n\rangle\ , \tag{4.25}$$

where $|n\rangle$ form an orthonormal basis in the Fock space (see Eqs. (4.2))

$$|n\rangle = \left(\frac{\Gamma(\frac{1}{\hbar})}{n!\Gamma(n+\frac{1}{\hbar})}\right)^{1/2} L_+^n|0\rangle = (\hbar^n n!)^{-1/2}(\hat{a}^\dagger)^n|0\rangle\ , \quad \langle n|m\rangle = \delta_{n,m}\ . \tag{4.26}$$

The inner product of two coherent states in distinct points z and w of the Lobachevski plane is given by

$$\langle w|z\rangle = (1 - z\bar{z})^{1/2\hbar}(1 - w\bar{w})^{1/2\hbar}(1 - z\bar{w})^{-1/\hbar} \tag{4.27}$$

(for $z = w$ one obviously has $\langle z|z\rangle = 1$).

4.4. Quantum observables of the gauge-invariant system and covariant symbols of Berezin

Quantum observables of our gauge-invariant dynamical system must commute with the first-class constraints

$$[\mathcal{D},\hat{\mathcal{A}}]=0, \qquad [\bar{\mathcal{D}},\hat{\mathcal{A}}]=0\,. \tag{4.28}$$

A general solution to these equations is obviously given by

$$\hat{\mathcal{A}}=\hat{\mathcal{U}}\hat{\mathcal{A}}_0\,\hat{\mathcal{U}}^\dagger, \tag{4.29}$$

where $\hat{\mathcal{A}}_0=\mathcal{A}_0(\hat{a}^\dagger,\hat{a})$ is an initial condition operator which is independent of z and $\bar{z}$, and $\hat{\mathcal{U}}$ is the SU(1,1) gauge transformation (4.18).

Let us consider a vacuum expectation value $\mathcal{A}(z,\bar{z})$ of the quantum observable $\mathcal{A}$

$$\begin{aligned}\mathcal{A}(z,\bar{z})&=\langle 0|\hat{\mathcal{A}}|0\rangle=\langle 0|\hat{\mathcal{U}}\hat{\mathcal{A}}_0\,\hat{\mathcal{U}}^\dagger|0\rangle\,,\\&=\langle \bar{z}|\hat{\mathcal{A}}_0|\bar{z}\rangle\,.\end{aligned} \tag{4.30}$$

One sees that it coincides with the matrix element of the initial condition operator $\hat{\mathcal{A}}_0$ sandwiched between the two SU(1,1) coherent states.

The resolution of unity on the Lobachevski plane reads as follows:

$$\int d\mu_\hbar(z,\bar{z})|z\rangle\langle z|=\hat{\mathbf{1}}\,, \tag{4.31}$$

with the measure

$$d\mu_\hbar(z,\bar{z})=\left(\frac{1}{\hbar}-1\right)\frac{dz\wedge d\bar{z}}{2\pi i(1-z\bar{z})^2}\,. \tag{4.32}$$

The normalization factor $(\frac{1}{\hbar}-1)$ is found from the vacuum expectation value of Eq. (4.31)

$$\int d\mu_\hbar(z,\bar{z})|\langle 0|z\rangle|^2=1. \tag{4.33}$$

Let us define an analytic continuation of the vacuum expectation value $\mathcal{A}(z,\bar{z})$ as a matrix element

$$\mathcal{A}(z,\bar{w})=\frac{\langle 0|\hat{\mathcal{U}}(z,\bar{z})\mathcal{A}_0\,\hat{\mathcal{U}}^\dagger(w,\bar{w})|0\rangle}{\langle 0|\hat{\mathcal{U}}(z,\bar{z})\,\hat{\mathcal{U}}^\dagger(w,\bar{w})|0\rangle}=\frac{\langle\bar{z}|\hat{\mathcal{A}}_0|\bar{w}\rangle}{\langle\bar{z}|\bar{w}\rangle}\,. \tag{4.34}$$

It coincides with $\mathcal{A}(z,\bar{z})$ at $z=w$, and the matrix element of the identity operator $\hat{\mathbf{1}}$ is equal to unity.

Now suppose we are given two quantum observables $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$. Then the vacuum expectation value $\mathcal{A}(z,\bar{z})$ of the product $\hat{\mathcal{A}}_1\hat{\mathcal{A}}_2$ can be calculated as follows:

$$\mathcal{A}(z,\bar{z})=\langle 0|\hat{\mathcal{A}}_1\hat{\mathcal{A}}_2|0\rangle=\langle 0|\hat{\mathcal{U}}\hat{\mathcal{A}}_{1,0}\hat{\mathcal{A}}_{2,0}\,\hat{\mathcal{U}}^\dagger|0\rangle$$

$$= \int d\mu_\hbar(w,\bar{w}) \langle \bar{z} | \hat{\mathcal{A}}_{1,0} | \bar{w} \rangle \langle \bar{w} | \hat{\mathcal{A}}_{2,0} | \bar{z} \rangle$$

$$= \left(\frac{1}{\hbar} - 1 \right) \int \frac{dw \wedge d\bar{w}}{2\pi i (1 - w\bar{w})^2} \mathcal{A}_1(z,\bar{w}) \mathcal{A}_2(w,\bar{z}) \left[\frac{(1 - z\bar{z})(1 - w\bar{w})}{(1 - z\bar{w})(1 - w\bar{z})} \right]^{1/\hbar} , \tag{4.35}$$

where we have used Eq. (4.30), the resolution of unity (4.32) and (4.34).

Now, the composition (4.35) is nothing but the Berezin multiplication formula for covariant symbols on the Lobachevski plane [16] (see also Ref. [15]), and covariant symbols introduced by Berezin can be interpreted as vacuum expectation values of quantum observables of the dynamical system with first-class constraints (4.12).

Thus, the theory of covariant symbols of Berezin can be deduced by quantizing first-class constraints.

To conclude this section, let us construct quantum observables corresponding to the classic expressions for the SU(1,1) generators (3.4). First, the initial conditions are given by Eqs. (4.2) (initial conditions are operators acting on the fiber at the point $z = 0$). Then the su(1,1) quantum observables are given by

$$\hat{\mathcal{L}}_\pm = \hat{\mathcal{U}} \hat{L}_\pm \hat{\mathcal{U}}^\dagger , \qquad \hat{\mathcal{L}}_0 = \hat{\mathcal{U}} \hat{L}_0 \hat{\mathcal{U}}^\dagger . \tag{4.36}$$

The corresponding covariant symbols defined as vacuum expectation values of (4.36) can be easily computed

$$L_0(z,\bar{z}) = \langle 0 | \hat{\mathcal{L}}_0 | 0 \rangle = \langle \bar{z} | \hat{L}_0 | \bar{z} \rangle = \frac{1}{2\hbar} \left(\frac{1 + z\bar{z}}{1 - z\bar{z}} \right) , \tag{4.37a}$$

$$L_+(z,\bar{z}) = \langle 0 | \hat{\mathcal{L}}_+ | 0 \rangle = \langle \bar{z} | \hat{L}_+ | \bar{z} \rangle = \frac{1}{\hbar} \left(\frac{z}{1 - z\bar{z}} \right) , \tag{4.37b}$$

$$L_-(z,\bar{z}) = \langle 0 | \hat{\mathcal{L}}_- | 0 \rangle = \langle \bar{z} | \hat{L}_- | \bar{z} \rangle = \frac{1}{\hbar} \left(\frac{\bar{z}}{1 - z\bar{z}} \right) , \tag{4.37c}$$

They coincide with the classical observables (3.3) up to the normalization factor $1/i\hbar$.

5. Gauge-invariant quantization on the sphere

Consider a stereographic projection of two-dimensional sphere with the Poisson brackets

$$[z,\bar{z}]^{\omega}_{\mathrm{PB}} = i(1 + z\bar{z})^2 . \tag{5.1}$$

The generators of SU(2) symmetry are hamiltonian vector fields with the hamiltonian functions

$$\mathcal{J}_0 = \frac{i}{2} \left(\frac{1 - z\bar{z}}{1 + z\bar{z}} \right) , \tag{5.2a}$$

$$\mathcal{J}_+ = \frac{i\bar{z}}{1+z\bar{z}} , \tag{5.2b}$$

$$\mathcal{J}_- = \frac{iz}{1+z\bar{z}} , \tag{5.2c}$$

which satisfy the SU(2) commutation relations

$$[\mathcal{J}_0, \mathcal{J}_\pm]^{\omega}_{\rm PB} = \pm \mathcal{J}_\pm , \tag{5.3}$$

$$[\mathcal{J}_+, \mathcal{J}_-]^{\omega}_{\rm PB} = 2\mathcal{J}_0 . \tag{5.4}$$

We will consider quantization in the covariant gauge. Let us first introduce the momenta p and $\bar{p}$ with the non-zero Poisson brackets

$$[z, p]_{\rm PB} = 1 , \qquad [\bar{z}, \bar{p}]_{\rm PB} = 1 . \tag{5.5}$$

Then the second-class constraints

$$\theta = p + \frac{i\bar{z}}{2(1+z\bar{z})} \approx 0, \qquad \bar{\theta} = p - \frac{iz}{2(1+z\bar{z})} \approx 0 \tag{5.6}$$

satisfy the relations

$$[\theta, \bar{\theta}]_{\rm PB} = i(1+z\bar{z})^{-2} . \tag{5.7}$$

By introducing the gauge degrees of freedom ϕ and $\bar{\phi}$,

$$[\phi, \bar{\phi}]_{\rm PB} = -i , \tag{5.8}$$

the second-class constraints are converted into the first-class ones

$$\mathcal{T} = p - \frac{1}{1+z\bar{z}}\{\bar{z}\mathcal{J}_0 + \varepsilon\mathcal{J}_-\} \approx 0 , \tag{5.9a}$$

$$\bar{\mathcal{T}} = \bar{p} + \frac{1}{1+z\bar{z}}\{z\mathcal{J}_0 + \varepsilon\mathcal{J}_+\} \approx 0, \quad \varepsilon = \pm 1 , \tag{5.9b}$$

where $\mathcal{J}$ are classical Holstein–Primakoff generators satisfying the Poisson bracket relations (5.3), (5.4)

$$\mathcal{J}_0 = \frac{i}{2}(2\phi\bar{\phi} - 1) , \tag{5.10a}$$

$$\mathcal{J}_+ = i(1 - \phi\bar{\phi})^{1/2}\bar{\phi} , \tag{5.10b}$$

$$\mathcal{J}_- = i(1 - \phi\bar{\phi})^{1/2}\phi . \tag{5.10c}$$

Now we can quantize our system with the first-class constraints (5.8) as usual

$$\hat{z} = z, \quad \hat{p} = -i\hbar\frac{\partial}{\partial z}, \quad \hat{\bar{z}} = \bar{z}, \quad \hat{\bar{p}} = -i\hbar\frac{\partial}{\partial \bar{z}} , \tag{5.11a}$$

$$\hat{a}^\dagger := \hat{\phi}, \qquad \hat{a} := \hat{\bar{\phi}}, \qquad [\hat{a}, \hat{a}^+] = \hbar , \tag{5.11b}$$

and the quantum first-class constraints define a SU(2) connection

$$\mathcal{D} := \frac{i}{\hbar}\hat{\mathcal{T}} = \frac{\partial}{\partial z} + \frac{1}{1+z\bar{z}}\{\bar{z}\hat{\mathcal{J}}_0 + \varepsilon\hat{\mathcal{J}}_-\} , \tag{5.12a}$$

$$\bar{\mathcal{D}} := \frac{i}{\hbar}\hat{\bar{\mathcal{T}}} = \frac{\partial}{\partial \bar{z}} - \frac{1}{1+z\bar{z}}\{z\hat{\mathcal{J}}_0 + \varepsilon\hat{\mathcal{J}}_+\} , \tag{5.12b}$$

where $\hat{\mathcal{J}}$ are quantum Holstein–Primakoff generators

$$\hat{\mathcal{J}}_0 = \frac{1}{2\hbar}(2\hat{a}^\dagger\hat{a} - 1) , \tag{5.13a}$$

$$\hat{\mathcal{J}}_+ = \frac{1}{\hbar}\hat{a}^\dagger(1 - \hat{a}^\dagger\hat{a})^{1/2} , \tag{5.13b}$$

$$\hat{\mathcal{J}}_- = \frac{1}{\hbar}(1 - \hat{a}^\dagger\hat{a})^{1/2}\hat{a} . \tag{5.13c}$$

Now let us consider Eqs. (4.14) for the physical quantum states. The SU(2) operator $\hat{\mathcal{U}}$ is easily found from (4.14) (we set $\varepsilon = -1$ in Eq. (5.11))

$$\hat{\mathcal{U}} = \mathrm{e}^{-\bar{z}\hat{\mathcal{J}}_+}\mathrm{e}^{\ell n(1+z\bar{z})\hat{\mathcal{J}}_0}\mathrm{e}^{z\hat{\mathcal{J}}_-} . \tag{5.14}$$

Consider a state $\hat{\mathcal{U}}|0\rangle$,

$$|\zeta\rangle = (1+\zeta\bar{\zeta})^{-1/2\hbar}\mathrm{e}^{\zeta\hat{\mathcal{J}}_+}|0\rangle , \quad \zeta = -\bar{z}. \tag{5.15}$$

Using Eq. (5.13b) one finds

$$|n) := \frac{1}{n!}\hat{\mathcal{J}}_+^n|0\rangle = \left[\frac{1}{n!}\prod_{k=0}^{n-1}\left(\frac{1}{\hbar} - k\right)\right]^{1/2} |n\rangle, \tag{5.16a}$$

$$|n\rangle = \frac{1}{\sqrt{\hbar^n n!}}(\hat{a}^\dagger)^n|0\rangle . \tag{5.16b}$$

The inner product is

$$(m|n) = \frac{1}{n!}\prod_{k=0}^{n-1}\left(\frac{1}{\hbar} - k\right)\delta_{m,n}. \tag{5.16c}$$

One sees that, due to the presence of the square root in Eqs. (5.13), (5.16a), for the states $|n)$ to be well defined and the inner product (5.16c) to be positive, the Planck constant must be an inverse of an integer, i.e. $\hbar = 1/(2j)$, $j = 1/2, 1, 3/2, 2\ldots$ (More precisely, it is the dimensionless product of the Planck constant and the square of the sphere radius r that must be integer. For the sake of simplicity we have set $r = 1$.) When $1/\hbar$ is non-integer, the product in (5.16c) contains both positive and negative multipliers. Moreover, when $\hbar = 1/(2j)$, $j = 1/2, 1, 3/2, 2\ldots$, due to the occurrence of zero at $k = 2j$ in the product $\prod(2j - k)$ in Eq. (5.16a), all states $|n)$ with $n \geqslant 2j + 1$ vanish identically, and for the state $\hat{\mathcal{U}}|0\rangle$ we obtain the following decomposition

$$|\zeta\rangle = (1+\zeta\bar{\zeta})^{-j}\sum_{m=-j}^{j}\left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2}(\zeta)^{j+m}|m\rangle . \tag{5.17}$$

Thus, it coincides with the SU(2) spin coherent state [15,28] and the states $(n!)^{-1}\hat{\mathcal{J}}_+^n|0\rangle$, $n = 0, 1, \ldots, 2j$, form a basis of the $(2j+1)$-dimensional representation of SU(2). The inner product of two coherent states in distinct points z and w is given by

$$\langle w|z\rangle = (1+w\bar{w})^{-j}(1+z\bar{z})^{-j}(1+\bar{w}z)^{2j} \,. \tag{5.18}$$

Thus, as expected, in the case of sphere the Hilbert space of physical quantum states, that are annihilated by the quantum first-class constraints (5.12) and have positive norm, is finite-dimensional and its dimension is equal to $1 + 1/\hbar = 2j + 1$.

The solution for quantum observables is given by Eq. (4.29), where $\hat{\mathcal{U}}$ is defined in (5.14). Repeating the consideration of Section 4.4, one arrives at the Berezin multiplication formula for covariant symbols on S^2 [16]

$$\mathcal{A}(z,\bar{z}) = (2j+1)\int \frac{\mathrm{d}w\wedge \mathrm{d}\bar{w}}{2\pi i(1+w\bar{w})^2}\mathcal{A}_1(z,\bar{w})\mathcal{A}_2(w,\bar{z}) \times \left[\frac{(1+z\bar{w})(1+w\bar{z})}{(1+z\bar{z})(1+w\bar{w})}\right]^{2j} \,. \tag{5.19}$$

6. Gauge-invariant quantization on Hermitian symmetric spaces

Let us now consider a general case where the original phase space $\mathcal{M}$ is an arbitrary hermitian symmetric space (HSS), $\mathcal{M} \simeq G/H$. There are two dual classes of HSS, compact and non-compact. A non-compact HSS can be realized as a bounded symmetric domain D in $\mathbb{C}^N$, and its dual compact HSS can be described by its stereographic projection [15,24].

Let G/H be an HSS, and g, h are the Lie algebras of G and H, respectively. Then g has a 3-graded structure

$$[M_a^b, M_c^d] = \Sigma_{ac}^{be} M_e^d - \Sigma_{ae}^{bd} M_c^e \,, \tag{6.1a}$$

$$[M_a^b, \bar{P}_c] = \Sigma_{ac}^{be}\bar{P}_e \,, \tag{6.1b}$$

$$[M_a^b, P^c] = -\Sigma_{ae}^{bc}P^e \,, \tag{6.1c}$$

$$[\bar{P}_a, P^b] = -2M_a^b, \quad a, b\ldots = 1, 2, \ldots, N, \tag{6.1d}$$

where M_a^b are generators of $H \subset \mathrm{U}(N)$, Σ_{ab}^{cd} are the structure constants of the corresponding Freighdental triple system, and $\bar{P}_a$ and P^a are raising and lowering generators similar to L_+ and L_- (see Ref. [25]).

Suppose a non-compact HSS G/H is realized as a bounded symmetric domain D in $\mathbb{C}^N$ with the coordinates z_α and $\bar{z}^\beta$, $\alpha, \beta = 1, 2, \ldots N$. Then the Kähler form

$$\omega = \frac{i}{2}\frac{\partial^2 \ln K(z,\bar{z})}{\partial z_\alpha \partial \bar{z}^\beta}\mathrm{d}z_\alpha \wedge \mathrm{d}\bar{z}^\beta \,, \tag{6.2}$$

where $K(z,\bar{z})$ is the Bergman kernel of D, defines non-linear Poisson brackets on D.

To quantize dynamical systems on (D,ω), the momenta p^α and $\bar{p}_\beta$ are introduced and subjected to the second-class constraints, which are then converted into the first-class ones by introducing extra gauge degrees of freedom. Next, the system with the first-class constraints is quantized.

The extended Hilbert space is a space of sections of the Fock bundle $\mathcal{R}D$ over D, $|\psi\rangle = \psi(z,\bar{z},\hat{a}^\dagger)|0\rangle$, where $\hat{a}_b^\dagger$ are creation operators for the gauge degrees of freedom,

$$[\hat{a}^a,\hat{a}_b^\dagger] = \hbar\delta_b^a \ . \tag{6.3}$$

The quantum first-class constraints define a flat G-connection in $\mathcal{R}D$

$$\mathcal{D}^\alpha = \frac{\partial}{\partial z_\alpha} + \Delta_b^{\alpha,a}\hat{M}_a^b + h_a^\alpha \hat{P}^a \ , \tag{6.4a}$$

$$\bar{\mathcal{D}}_\beta = \frac{\partial}{\partial \bar{z}^\beta} - \bar{\Delta}_{\beta,a}^b \hat{M}_b^a - \bar{h}_\beta^a \hat{\bar{P}}_a \ , \tag{6.4b}$$

where $\hat{M}_b^a$, $\hat{P}^a$ and $\hat{\bar{P}}_a$ are generators of a unitary representation of G in the Fock space of gauge degrees of freedom satisfying commutation relations (6.1), $(\Delta,\bar{\Delta})$ is an H-connection, $H \subset \mathrm{U}(N)$, and $(h,\bar{h})$ is a unitary frame. The commutativity

$$[\mathcal{D}^\alpha,\bar{\mathcal{D}}_\beta] = 0 \tag{6.5}$$

follows from the fact that G/H is an HSS [26].

Physical quantum states which are covariantly constant sections of the Fock bundle,

$$\mathcal{D}^\alpha|\psi_{\rm phys}\rangle = 0, \ \ \bar{\mathcal{D}}_\beta|\psi_{\rm phys}\rangle = 0 \ , \tag{6.6}$$

can be written in the form

$$|\psi_{\rm phys}\rangle = \hat{\mathcal{U}}|\psi_{\rm phys,0}\rangle \ , \tag{6.7}$$

where $\hat{\mathcal{U}}$ is a group element of G that can be represented in the form

$$\hat{\mathcal{U}} = \mathrm{e}^{-\bar{f}^a \hat{\bar{P}}_a}\mathrm{e}^{F_a^b \hat{M}_b^a}\mathrm{e}^{f_a \hat{P}^a} \tag{6.8}$$

with the functions $f^a, F_a{}^b$ to be found from Eq. (6.6). Thus, the physical quantum states of the gauge-invariant system coincide with the generalized coherent states for the HSS G/H [15].

Similarly, the quantum observables commuting with the first-class constraints have the form

$$\hat{\mathcal{A}} = \hat{\mathcal{U}}\hat{\mathcal{A}}_0\,\hat{\mathcal{U}}^\dagger \ , \tag{6.9}$$

and the theory of covariant symbols of Berezin [16] may be deduced similarly to the case of the Lobachevski plane and sphere.

7. Conclusion

The gauge-invariant approach to geometric quantization developed in Refs. [2–5,1] yields a complete quantum description of dynamical systems with non-trivial geometry and topology of the phase space. It synthesizes geometric, deformation and Berezin covariant symbols quantization approaches into a unified theory.

In this paper quantization on finite-dimensional symmetric spaces was considered. In this case all the master equations are exactly solvable due to the presence of symmetry. Infinite-dimensional dynamical systems with symmetry constitute the next physically important class. Applications to the models of conformal field theory may be especially interesting, as they could lead to some further insight into the geometric structure and symmetries of exactly solvable models. The next class of applications is to the quantization of dynamical systems without any global symmetry. In some cases the topological structure of the phase space may still lead to the exact non-perturbative solutions. Applications to the topological field theory [27] could be especially interesting. Gauge-invariant geometric quantization also provides a natural framework for a generalization of the notion of coherent states [15,28] to arbitrary symplectic manifolds.

Acknowledgements

The authors would like to take this opportunity to express their gratitude to the members of the Board of Trustees of ICAST for their generous support. Especially to Mr. Ira Jaffe, Chairman of the Board of Trustees of ICAST, and President, Jaffe, Raitt, Heuer and Weiss, P.C., and to Mary and George Keros for their interest in our work and kind hospitality.

The authors would also like to thank Dr. Edward David for his invaluable support and encouragement.

One of the authors (E.S.F.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

This work was supported in part by the NATO Linkage Grant No. 931717.

References

[1] E.S. Fradkin and V.Ya. Linetsky, Nucl. Phys. B 431 (1994) 569.
[2] I.A. Batalin and E.S. Fradkin, Phys. Lett. 180 (1986) 157; Nucl. Phys. B 279 (1987) 514.
[3] I.A. Batalin, E.S. Fradkin and T.E. Fradkina, Nucl. Phys. B 314 (1989) 158; B 323 (1989) 734; T.E. Fradkina, Sov. J. Nucl. Phys. 49 (1988) 598.
[4] I.A. Batalin and E.S. Fradkin, Nucl. Phys. B 326 (1989) 701.
[5] I.A. Batalin, E.S. Fradkin and T.E. Fradkina, Nucl. Phys. B 332 (1990) 723.
[6] E.S. Fradkin, in Proc. Xth Winter School in Karpacz, Acta Univ. Wratislav. 207 (1973) 93.
[7] E.S. Fradkin and G.A. Vilkovisky, Phys. Rev. D 8 (1973) 4241; Phys Lett. B 55 (1975) 224; Lett. Nuovo Cimento 13 (1975) 187; CERN Report TH 2332 (1977).
[8] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B 69 (1977) 309.

[9] E.S. Fradkin and T.E. Fradkina, Phys. Lett. B 72 (1978) 343.
[10] I.A. Batalin and E.S. Fradkin, Phys. Lett. B 122 (1983) 157; Phys. Lett. B 128 (1983) 303; J. Math. Phys. 25 (1984) 2426; Lett. Nuovo Cimento 13 (1975) 187.
[11] I.A. Batalin and E.S. Fradkin, Riv. Nuovo Cimento 10 (1986) 1; Ann. Inst. Henri Poincare 49 (1988) 145.
[12] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. 111 (1978) 61, 111.
[13] J. Vey, Comments Math. Helvet. 50 (1975) 421.
[14] A. Lichnerowicz, in Deformation Theory of Algebras and Structures, NATO ASI Ser. C 247 (1988) 855;
M. DeWilde and P. Lecomte, in Deformation Theory of Algebras and Structures, NATO ASI Ser. C 247 (1988) p. 897; Lett. Math. Phys. 7 (1983) 487.
[15] A. Perelomov, Generalized Coherent States and Applications (Springer-Verlag, 1986).
[16] F.A. Berezin, Commun. Math. Phys. 40 (1975) 153; Izv. Akad. Nauk SSSR, Ser. Mat. 36 (1972) 1134; 39 (1975) 363.
[17] B. Kostant, Lect. Notes Math. 170 (1970) 87;
J.-M. Souriau, Structures des systemes dynamiques (Dunod, Paris, 1970);
A.A. Kirilov, Elements of the theory of representations (Springer, New York, 1976).
[18] B. Kostant, Symp. Math. 14 (1974) 139.
[19] V. Guillemin and S. Sternberg, Geometric asymptotics, AMS Math. Surv. 14 (1977);
R.J. Blattner, Lect. Notes Math. 570 (1977) 11;
D.J. Simms and N.M.J. Woodhouse, Lect. Notes Phys. 53 (1976).
[20] J. Czyz, Rep. Math. Phys. 15 (1979) 57;
H. Hess, Lect. Notes Phys. 139 (1981) 1;
M. Forger and H. Hess, Commun. Math. Phys. 64 (1979) 269;
P.L. Robinson and R.J. Rownsley, AMS Mem. 81 (1989) 410.
[21] N.M.J. Woodhouse, Geometric quantization (Clarendon, Oxford, 1992).
[22] T. Holstein and H. Primakoff, Phys. Rev. 58 (1940) 1098.
[23] H.D. Doebner, B. Gruber and M. Lorente, J. Math. Phys. 30 (1989) 594.
[24] S. Helgason, Differential geometry and symmetric spaces (Academic Press, New York, London, 1962).
[25] M. Gunaydin, Mod. Phys. Lett. A 6 (1991) 1733.
[26] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. I and II (Wiley/Interscience, New York, 1963, 1969).
[27] E. Witten, Commun. Math. Phys. 117 (1988) 353; 121 (1989) 351;
S. Axelrod, S. Dela Pietra and E. Witten, J. Diff. Geom. 33 (1991) 787;
N.J. Hitchin, Commun. Math. Phys. 131 (1990) 347.
[28] J.R. Klauder and B.-S. Skagerstam, Eds., Coherent states (World Scientific, Singapore, 1985).