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ANOTHER VERSION FOR OPERATORIAL QUANTIZATION OF DYNAMICAL SYSTEMS WITH IRREDUCIBLE CONSTRAINTS

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An alternative version is proposed for operatorial canonical quantization of dynamical systems subject to irreducible first- and second-class constraints, and which exploits a modified way of defining the extra degrees of freedom needed for conversion of the original second-class constraints to (effective) first-class constraints. The alternative version considered is shown to be canonically equivalent to the previously suggested formulation. It is also shown that both formulations belong to an infinite class of canonically-equivalent solutions of the basic generating equations which correspond to the most general effective constraints.

1. Introduction

In the works of the two first-named authors [1,2] a general formulation was suggested for a method of operatorial quantization of dynamical systems with irreducible first- and second-class constraints. The operatorial quantization is based in the idea of conversion of the original second-class constraints into effective first-class constraints by considering some additional degrees of freedom needed to this end. After the effective first-class constraints appear, they are subjected (along with the initial first-class constraints, if any) to the standard procedure within the general method of operatiorial canonical quantization of dynamical systems with first-class constraints [3,4].

In the recent paper [5] the authors considered an alternative version of the formulation of canonical quantization of dynamical systems with irreducible second-class constraints. This formulation also included the conversion of the constraints of second kind into those of first kind as its basic method. The difference is that now the way of introducing additional degrees of freedom was modified. The modification reduced to the strict requirement that the effective first-class constraints be abelian. The effective constraints and the hamiltonian were sought for in the form of a series in powers of additional variables, with the zeroth terms coinciding with the original second-class constraints and the original hamiltonian, respectively. The quantization was performed not at the level of operators, but within the formal path integral, the basic equations being formulated in terms of the classical Poisson brackets. The general case, when the original first-class constraint might also be present in the theory, was not treated.

In the present paper we propose an operatorial extension of the scheme suggested in ref. [5]. When doing so we also admit that the original irreducible constraints in the theory are of first and second kinds. In sect. 2 the operatorial version is given to the modified method of the introduction of additional variables and of the construction of effective first-class constraints. In sect. 3 the operator-valued generating equations of the gauge algebra of the original first-class constraints are formulated in the presence of second-class constraints. In sect. 4 the *full* unitarizing Hamiltonian is constructed. In sect. 5 the case of pure second-class constraints is studied in more detail. In sect. 6 explicit expressions for the generating operators of the gauge algebra, as well as an explicit form of extended involution relations, are given in the presence of second-class constraints, referring to the gauge theories of rank-1 as an example. In sect. 7 the general structure of solution of the generating equations of arbitrary-rank gauge algebra is considered. In sect. 8 it is shown that the operatorial formulation attained is related through a canonical transformation with the formalism proposed earlier in refs. [1,2]. In the concluding sect. 9, some mutually complementary peculiar properties of the above two formulations are discussed and it is shown that they belong to an infinite class of canonically equivalent solutions of the basic generating equations, that corresponds to the most general effective constraints.

Notations. Similar to the previous papers, we are exploiting the following basic notations. The grassmanian parity and the ghost number of a quantity A are denoted, respectively, as $\epsilon(A)$ and $\mathrm{gh}(A)$. The supercommutator of operators A and B is denoted as

$$[A, B] \equiv AB - BA(-1)^{\epsilon(A)\epsilon(B)}. \tag{1.1}$$

All canonical pairs of operators are written in the standard way

$$(Q^A, P_A), \qquad \epsilon(Q^A) = \epsilon(P_A), \qquad \mathrm{gh}(Q^A) = -\mathrm{gh}(P_A). \tag{1.2}$$

The only nonzero equal-time supercommutators for them are

$$[Q^A, P_B] = i\hbar\, \delta^A_B. \tag{1.3}$$

If the ghost number of some canonical pairs, when first defined, is not indicated, it should be understood as to be equal to zero.

2. Conversion of second-class constraints into effective abelian first-class constraints

Let

$$(q^i, p_i), \qquad \epsilon(q^i) = \epsilon(p_i), \qquad i = 1, \dots, n, \tag{2.1}$$

be canonical operator pairs of dynamical variables of an original phase space. Further, let, in a dynamical system with the hamiltonian $H_0(q, p)$, the first-class constraints

$$T'_a(q, p), \qquad \epsilon(T'_a) \equiv \epsilon'_a, \qquad a = 1, \dots, m', \tag{2.2}$$

and the second-class constraints

$$T''_\alpha(q, p), \qquad \epsilon(T''_\alpha) \equiv \epsilon''_\alpha, \qquad \alpha = 1, \dots, 2m'', \tag{2.3}$$

be given in terms of the original operators (1.1). All the constraints are assumed to be linear independent.

Following refs. [1, 2], let us consider the new operators

$$\Phi^\alpha, \qquad \epsilon(\Phi^\alpha) = \epsilon''_\alpha, \qquad \alpha = 1, \dots, 2m'', \tag{2.4}$$

such that

$$[\Phi^\alpha, \Phi^\beta] = i\hbar\omega^{\alpha\beta}, \tag{2.5}$$

where $\omega^{\alpha\beta}$ is a constant simplectic matrix

$$\epsilon(\omega^{\alpha\beta}) = \epsilon''_\alpha + \epsilon''_\beta, \qquad \omega^{\beta\alpha} = -\omega^{\alpha\beta}(-1)^{\epsilon''_\alpha \epsilon''_\beta}. \tag{2.6}$$

Let us define, further, the operators

$$T_\alpha(q, p, \Phi), \qquad \epsilon(T_\alpha) = \epsilon''_\alpha, \qquad \alpha = 1, \dots, 2m'', \tag{2.7}$$

by the equation

$$[T_\alpha, T_\beta] = 0, \tag{2.8}$$

and the condition

$$T_\alpha(q, p, 0) = T''_\alpha(q, p), \tag{2.9}$$

where the second-class constraints (2.3) are involved in the r.h.s.

Eqs. (2.8) and (2.9) convert the original second-class constraints (2.3) in the original phase space (2.1) to effective abelian first-class constraints (2.7) in the direct sum of the phase spaces (2.1) (2.4). At the classical level, i.e. in terms of the Poisson brackets, eq. (2.8) was suggested in ref. [5].

3. Generating the gauge algebra of first-class constraints

Let us now define the canonical operator pairs of ghosts

$$\left(C'^{a}, \bar{\mathscr{P}}'_{a}\right), \qquad a=1,\ldots,m', \tag{3.1}$$

$$\epsilon(C'^{a})=\epsilon\left(\bar{\mathscr{P}}'_{a}\right)=\epsilon'_{a}+1, \qquad \mathrm{gh}(C'^{a})=-\mathrm{gh}\left(\bar{\mathscr{P}}'_{a}\right)=1, \tag{3.2}$$

and the generating operators

$$\Omega'(q,p,\Phi,C',\bar{\mathscr{P}}'), \qquad H'(q,p,\Phi,C',\bar{\mathscr{P}}'),$$

using the equations

$$[\Omega',\Omega']=0, \qquad [\Omega',T_{\beta}]=0, \tag{3.3}$$

$$\epsilon(\Omega')=1, \qquad \mathrm{gh}(\Omega')=1, \tag{3.4}$$

$$[H',\Omega']=0, \qquad [H',T_{\beta}]=0, \tag{3.5}$$

$$\epsilon(H')=0, \qquad \mathrm{gh}(H')=0, \tag{3.6}$$

and the conditions

$$\Omega'(q,p,0,C',0)=T'_{a}(q,p)C'^{a}, \tag{3.7}$$

$$H'(q,p,0,0,0)=H_{0}(q,p). \tag{3.8}$$

Effective abelian constraints (2.7) are involved in the second equations of eqs. (3.3) and (3.5), while the r.h.s. of eqs. (3.7) and (3.8) contain the original first-class constraints (2.2) and the original hamiltonian, respectively. Eqs. (3.3)–(3.6) are generating equations for the operator gauge algebra of the first-class constraints in the presence of second-class constraints.

4. The unitarizing hamiltonian

Let us define now another set of canonical pairs of ghost operators

$$\left(C''^{\alpha}, \bar{\mathscr{P}}''_{\alpha}\right), \qquad \alpha=1,\ldots,2m'', \tag{4.1}$$

$$\epsilon(C''^{\alpha})=\epsilon\left(\bar{\mathscr{P}}''_{\alpha}\right)=\epsilon''_{\alpha}+1, \qquad \mathrm{gh}(C''^{\alpha})=-\mathrm{gh}\left(\bar{\mathscr{P}}''_{\alpha}\right)=1, \tag{4.2}$$

and also the canonical pairs of antighost operators relative to eqs. (3.1) and (4.1)

$$\left(\mathscr{P}'^{a}, \bar{C}'_{a}\right), \qquad a = 1, \ldots, m', \tag{4.3}$$

$$\epsilon\left(\mathscr{P}'^{a}\right) = \epsilon\left(\bar{C}'_{a}\right) = \epsilon'_{a} + 1, \qquad \mathrm{gh}\left(\mathscr{P}'^{a}\right) = -\mathrm{gh}\left(\bar{C}'_{a}\right) = 1, \tag{4.4}$$

$$\left(\mathscr{P}''^{\alpha}, \bar{C}''_{\alpha}\right), \qquad \alpha = 1, \ldots, 2m'', \tag{4.5}$$

$$\epsilon\left(\mathscr{P}''^{\alpha}\right) = \epsilon\left(\bar{C}''_{\alpha}\right) = \epsilon''_{\alpha} + 1, \qquad \mathrm{gh}\left(\mathscr{P}''^{\alpha}\right) = -\mathrm{gh}\left(\bar{C}''_{\alpha}\right) = 1, \tag{4.6}$$

and, finally, the canonical operator pairs of dynamically active Lagrange multipliers

$$\left(\lambda'^{a}, \pi'_{a}\right), \qquad a = 1, \ldots, m', \qquad \epsilon\left(\lambda'^{a}\right) = \epsilon\left(\pi'_{a}\right) = \epsilon'_{a}, \tag{4.7}$$

$$\left(\lambda''^{\alpha}, \pi''_{\alpha}\right), \qquad \alpha = 1, \ldots, 2m'', \qquad \epsilon\left(\lambda''^{\alpha}\right) = \epsilon\left(\pi''_{\alpha}\right) = \epsilon''_{\alpha}. \tag{4.8}$$

The full generating operator is given as

$$\Omega = \Omega' + \pi'_{a}\mathscr{P}'^{a} + \Omega'' + \pi''_{\alpha}\mathscr{P}''^{\alpha}, \tag{4.9}$$

where Ω' is defined by eqs. (3.3), (3.4) and (3.7), and the designation is used as

$$\Omega'' \equiv T_{\alpha}C''^{\alpha}, \tag{4.10}$$

effective abelian constraints (2.7) being involved in the r.h.s. of eq. (4.10). Now we are in a position to define the full unitarizing hamiltonian by the standard formula

$$H = H' + (i\hbar)^{-1}[\Psi, \Omega], \tag{4.11}$$

where H' is fixed for eqs. (3.5), (3.6) and (3.8), Ω is defined as in eq. (4.9), while the operator Ψ,

$$\epsilon(\Psi) = 1, \qquad \mathrm{gh}(\Psi) = -1, \tag{4.12}$$

fixes the gauge as

$$\Psi = \bar{\mathscr{P}}'_{a}\lambda'^{a} + \bar{C}'_{a}\chi'^{a} + \bar{\mathscr{P}}''_{\alpha}\lambda''^{\alpha} + \bar{C}''_{\alpha}\chi''^{\alpha}. \tag{4.13}$$

The full hamiltonian (4.11) determines the time evolution of the dynamical system. The genuine physical states are BRST singlets

$$\Omega|\Phi\rangle = 0, \qquad |\Phi\rangle \neq \Omega|\text{anything}\rangle \tag{4.14}$$

with zero ghost number.

5. The case of pure second-class constraints

Let us study in more detail the case when there is no original first-class constraints (2.2) in the system, so that we deal, in fact, with the second-class constraints (2.3) alone. Then one has in eq. (4.11)

$$\Omega = T_\alpha C''^\alpha + \pi''_\alpha \mathscr{P}''^\alpha, \tag{5.1}$$

$$\Psi = \bar{\mathscr{P}}''_\alpha \lambda''^\alpha + \bar{C}''_\alpha \chi''^\alpha, \tag{5.2}$$

and also

$$H' = H'(q, p, \Phi), \qquad [H', T_\beta] = 0, \tag{5.3}$$

$$H'(q, p, 0) = H_0(q, p). \tag{5.4}$$

Let us seek for a solution to equations (2.8) that determines the effective abelian constraints (2.7), by expanding them in a Weyl-symmetric series in powers of the operators (2.4)

$$T_\alpha(q, p, \Phi) = \exp\left(\Phi^\alpha \frac{\partial}{\partial \varphi^\alpha}\right) \times \sum_{n=0}^{\infty} X_{\alpha\alpha_n \ldots \alpha_1}(q, p) \varphi^{\alpha_1} \ldots \varphi^{\alpha_n}|_{\varphi=0}. \tag{5.5}$$

Here, the variables

$$\varphi^\alpha, \qquad \epsilon(\varphi^\alpha) = \epsilon''_\alpha, \qquad \alpha = 1, \ldots, 2m'' \tag{5.6}$$

are the classical counterparts of the operators (2.4).

The coefficients in eq. (5.5) have the antisymmetric property

$$X_{\alpha\alpha_n \ldots \alpha_1} = X_{\alpha(\alpha_n \ldots \alpha_1)}, \tag{5.7}$$

with the symmetrization operation

$$\alpha_n \ldots \alpha_1 \to (\alpha_n \ldots \alpha_1), \tag{5.8}$$

defined for any quantity $K_{\alpha_n \ldots \alpha_1}$ as

$$K_{(\alpha_n \ldots \alpha_1)} \equiv K_{\beta_n \ldots \beta_1} S^{\beta_1 \ldots \beta_n}_{\alpha_n \ldots \alpha_1}, \tag{5.9}$$

$$n! S^{\beta_1 \ldots \beta_n}_{\alpha_n \ldots \alpha_1} \equiv \left(\varphi^{\beta_1} \ldots \varphi^{\beta_n} \frac{\overleftarrow{\partial}}{\partial \varphi^{\alpha_n}} \cdots \frac{\overleftarrow{\partial}}{\partial \varphi^{\alpha_1}} \right). \tag{5.10}$$

It follows from eq. (5.9) that

$$X_\alpha(q,p) = T''_\alpha(q,p). \tag{5.11}$$

The substitution of expansion (5.5) into eq. (2.8) gives rise to the following relation for the coefficients

$$\mathfrak{S}_{\alpha\beta(\gamma_n\dots\gamma_1)} = 0, \qquad n = 0,1,\dots, \tag{5.12}$$

where

$$\mathfrak{S}_{\alpha\beta\gamma_n\dots\gamma_1} \equiv \sum_{m=0}^{n}\sum_{l=0}^{\infty} C^n_{ml}\left(\tfrac{1}{2}i\hbar\right)^l \times\Big\{X_{\alpha\gamma_n\dots\gamma_{m+1}\delta_l\dots\delta_1}X_\beta{}^{\delta_1\dots\delta_l}{}_{\gamma_m\dots\gamma_1}(-1)^{\epsilon''_\beta\epsilon^n_m} - X_{\beta\gamma_n\dots\gamma_{m+1}\delta_l\dots\delta_1}X_\alpha{}^{\delta_1\dots\delta_l}{}_{\gamma_m\dots\gamma_1}(-1)^{\epsilon''_\alpha(\epsilon''_\beta+\epsilon^n_m)}\Big\}, \tag{5.13}$$

$$C^n_{ml} \equiv \frac{(n-m+l)!\,(m+l)!}{(n-m)!\,l!\,m!}, \tag{5.14}$$

$$X_\alpha{}^{\gamma_1\dots\gamma_l}{}_{\delta_m\dots\delta_1} \equiv \omega^{\gamma_1\beta_1}\dots\omega^{\gamma_l\beta_l}X_{\alpha\beta_l\dots\beta_1\delta_m\dots\delta_1}(-1)^{\sum_{k=1}^{l}\epsilon''_{\beta_k}(\epsilon''_\alpha+1+\epsilon^l_k)}, \tag{5.15}$$

$$\epsilon^p_q \equiv \sum_{j=q+1}^{p}\epsilon''_{\gamma_j}. \tag{5.16}$$

Assume that all the coefficients in the expansion (5.5) are known. Put, then, an operator $\tilde{A}(q,p,\Phi)$ into correspondence with the operator $A(q,p)$

$$A(q,p) \mapsto \tilde{A}(q,p,\Phi), \tag{5.17}$$

using the equation

$$\left[\tilde{A}, T_\beta\right] = 0, \qquad \tilde{A}(q,p,0) = A(q,p). \tag{5.18}$$

To find a solution of this equation, and Ansatz analogous to the Weyl-symmetric expansion (5.5) is also used

$$\tilde{A}(q,p,\Phi) = \exp\left\{\Phi^\alpha\frac{\partial}{\partial\varphi^\alpha}\right\}\sum_{n=0}^{\infty}Y_{\alpha_n\dots\alpha_1}(q,p)\varphi^{\alpha_1}\dots\varphi^{\alpha_n}\big|_{\varphi=0}, \tag{5.19}$$

where

$$Y_{\alpha_n\dots\alpha_1} = Y_{(\alpha_n\dots\alpha_1)}, \tag{5.20}$$

$$Y(q,p) = A(q,p). \tag{5.21}$$

The substitution of expansions (5.5) and (5.19) into eq. (5.18) produces the following the following relations for the coefficients

$$\mathfrak{H}_{\beta(\gamma_n \dots \gamma_1)} = 0, \qquad n = 0, 1, \dots, \tag{5.22}$$

where

$$\mathfrak{H}_{\beta\gamma_n \dots \gamma_1} \equiv \sum_{m=0}^{n} \sum_{l=0}^{\infty} C^n_{ml} \left(\tfrac{1}{2} i\hbar\right)^l \times \Big\{ Y_{\gamma_n \dots \gamma_{m+1}\delta_l \dots \delta_1} X_\beta{}^{\delta_1 \dots \delta_l}{}_{\gamma_m \dots \gamma_1} (-1)^{\epsilon''_\beta \epsilon^n_m} - X_{\beta\gamma_n \dots \gamma_{m+1}\delta_l \dots \delta_1} Y^{\delta_1 \dots \delta_l}{}_{\gamma_m \dots \gamma_1} (-1)^{\epsilon(A)(\epsilon''_\beta + \epsilon^n_m)} \Big\}, \tag{5.23}$$

$$Y^{\gamma_1 \dots \gamma_l}{}_{\delta_m \dots \delta_1} \equiv \omega^{\gamma_1 \beta_1} \dots \omega^{\gamma_l \beta_l} Y_{\beta_l \dots \beta_1 \delta_m \dots \delta_1} (-1)^{\sum_{k=1}^{l} \epsilon''_{\beta_k}(\epsilon(A) + 1 + \epsilon'_k)}. \tag{5.24}$$

Given all the coefficients in expansion (5.19), solution to eqs. (5.3) and (5.4) can be represented, in the sense of the correspondence (4.17)–(4.19), as

$$H'(q, p, \Phi) = \tilde{H}_0(q, p, \Phi). \tag{5.25}$$

6. The case when original first-class constraints are also present and generate the rank-1 gauge algebra

This case is of most practical interest for the majority of known gauge systems. The corresponding solution of the generating equations (3.3)–(3.8) is written in the Weyl basis relative to the ghosts as

$$\Omega' = \tilde{T}'_a C'^a + \tfrac{1}{6}\left(\bar{\mathcal{P}}'_a \tilde{U}^a_{bc} C'^c C'^b + C'^b \bar{\mathcal{P}}'_a \tilde{U}^a_{bc} C'^c + C'^c C'^b \bar{\mathcal{P}}'_a \tilde{U}^a_{bc} \right)(-1)^{\epsilon'_b}, \tag{6.1}$$

$$H' = \tilde{H}_0 + \tfrac{1}{2}\left(\bar{\mathcal{P}}'_a \tilde{V}^a_b C'^b - C'^b \bar{\mathcal{P}}'_a \tilde{V}^a_b (-1)^{\epsilon'_b} \right). \tag{6.2}$$

Here the extended structure operators

$$\tilde{T}'_a = \tilde{T}'_a(q, p, \Phi), \qquad \tilde{U}^a_{bc} = \tilde{U}^a_{bc}(q, p, \Phi), \tag{6.3}$$

$$\tilde{H}_0 = \tilde{H}_0(q, p, \Phi), \qquad \tilde{V}^a_b = \tilde{V}^a_b(q, p, \Phi), \tag{6.4}$$

correspond, in the sense of eqs. (5.17)–(5.19), to the original structure operators

$$T'_a(q, p), \qquad U^a_{bc}(q, p), \tag{6.5}$$

$$H_0(q, p), \qquad V^a_b(q, p), \tag{6.6}$$

respectively.

The substitution of eqs. (6.1) and (6.2) into equations (3.3), and (3.5) results in relations for the operators (6.3) and (6.4). We only explicitly give here the extended involution relations, that appear in the lowest order with respect to the ghosts

$$[\tilde{T}'_a, \tilde{T}'_b] = \tfrac{1}{2}i\hbar\left(\tilde{T}'_c\tilde{U}^c_{ab} + \tilde{U}^c_{ab}\tilde{T}'_c(-1)^{(\epsilon'_a+\epsilon'_b+1)\epsilon'_c}\right)$$

$$+\left(\tfrac{1}{2}i\hbar\right)^2[\tilde{U}^c_{ad}, \tilde{U}^d_{cb}](-1)^{(\epsilon'_a+1)\epsilon'_c}, \tag{6.7}$$

$$[\tilde{H}_0, \tilde{T}'_b] = \tfrac{1}{2}i\hbar\left(\tilde{T}'_c\tilde{V}^c_b + \tilde{V}^c_b\tilde{T}'_c(-1)^{(\epsilon'_b+1)\epsilon'_c}\right)$$

$$+\left(\tfrac{1}{2}i\hbar\right)^2[\tilde{V}^c_d, \tilde{U}^d_{cb}](-1)^{\epsilon'_c}. \tag{6.8}$$

7. Structure of solution of the generating equations in the general case of an irreducible gauge algebra

In the (most general) case of an irreducible arbitrary-rank gauge algebra the solution of the generating equations (3.3)–(3.8), in the Weyl basis relative to the ghosts, is sought for in the form

$$\Omega' = D\bar{\Omega}'|_0, \qquad H' = D\bar{H}'|_0, \tag{7.1}$$

where

$$D \equiv \exp\left\{C'^a\frac{\partial}{\partial C^a} + \bar{\mathcal{P}}'_a\frac{\partial}{\partial\bar{\mathcal{P}}_a}\right\}, \tag{7.2}$$

and the variables

$$(C^a, \bar{\mathcal{P}}_a), \qquad a = 1, \dots, m', \tag{7.3}$$

$$\epsilon(C^a) = \epsilon(\bar{\mathcal{P}}_a) = \epsilon'_a + 1 \equiv \epsilon_a, \qquad \mathrm{gh}(C^a) = -\mathrm{gh}(\bar{\mathcal{P}}_a) = 1, \tag{7.4}$$

are classical counterparts to the ghost operators (3.1), and the substitution symbol $|_0$ implies setting the classical variables (7.3) equal to zero in (7.1).

Owing to eqs. (3.3) and (3.5), we have

$$\bar{\Omega}'\Delta\bar{\Omega}' = 0, \tag{7.5}$$

$$\bar{H}'\Delta\bar{\Omega}' - \bar{\Omega}'\Delta\bar{H}' = 0, \tag{7.6}$$

where

$$\Delta \equiv \exp\left\{\tfrac{1}{2}i\hbar\left(\frac{\overleftarrow{\partial}}{\partial C^a}\frac{\overrightarrow{\partial}}{\partial\bar{\mathcal{P}}_a} - \frac{\overrightarrow{\partial}}{\partial C^a}\frac{\overleftarrow{\partial}}{\partial\bar{\mathcal{P}}_a}\right)\right\}. \tag{7.7}$$

In order to solve these equations again the series in powers of the classical variables (7.3) are used for Ansätze

$$\bar{\Omega}' = \sum_{m,n\geqslant 0} \bar{\mathscr{P}}_{a_m} \dots \bar{\mathscr{P}}_{a_1} \tilde{X}^{a_1 \dots a_m}_{b_n \dots b_1}(q, p, \Phi) C^{b_1} \dots C^{b_n}, \tag{7.8}$$

$$\bar{H}' = \sum_{m,n\geqslant 0} \bar{\mathscr{P}}_{a_m} \dots \bar{\mathscr{P}}_{a_1} \tilde{Y}^{a_1 \dots a_m}_{b_n \dots b_1}(q, p, \Phi) C^{b_1} \dots C^{b_n}. \tag{7.9}$$

The tilde over an operator means the correspondence (5.17)–(5.19)* is used throughout.

The coefficients in expansions (7.8) and (7.9) have the symmetry properties

$$\tilde{X}^{a_1 \dots a_m}_{b_n \dots b_1} = \tilde{X}^{[a_1 \dots a_m]}_{[b_n \dots b_1]}, \tag{7.10}$$

$$\tilde{Y}^{a_1 \dots a_m}_{b_n \dots b_1} = \tilde{Y}^{[a_1 \dots a_m]}_{[b_n \dots b_1]}, \tag{7.11}$$

where the symmetrization

$$a_1 \dots a_m \to [a_1 \dots a_m], \qquad b_n \dots b_1 \to [b_n \dots b_1] \tag{7.12}$$

is defined for any quantity $K^{a_1 \dots a_m}_{b_n \dots b_1}$ as

$$K^{[a_1 \dots a_m]}_{[b_n \dots b_1]} \equiv \bar{S}^{a_1 \dots a_m}_{c_m \dots c_1} K^{c_1 \dots c_m}_{d_n \dots d_1} S^{d_1 \dots d_n}_{b_n \dots b_1}, \tag{7.13}$$

$$m!\, \bar{S}^{a_1 \dots a_m}_{b_m \dots b_1} \equiv \left(\frac{\vec{\partial}}{\partial \bar{\mathscr{P}}_{a_1}} \cdots \frac{\vec{\partial}}{\partial \bar{\mathscr{P}}_{a_m}} \bar{\mathscr{P}}_{b_m} \dots \bar{\mathscr{P}}_{b_1} \right), \tag{7.14}$$

$$n!\, S^{b_1 \dots b_n}_{a_n \dots a_1} \equiv \left(C^{b_1} \dots C^{b_n} \frac{\overleftarrow{\partial}}{\partial C^{a_n}} \cdots \frac{\overleftarrow{\partial}}{\partial C^{a_1}} \right). \tag{7.15}$$

The substitution of (7.8) and (7.9) into (7.5) and (7.6) creates the relations for the coefficients

$$\mathscr{L}^{[a_1 \dots a_m]}_{[b_n \dots b_1]} = 0, \qquad m = 0, 1, \dots, \qquad n = 0, 1, \dots, \tag{7.16}$$

$$\mathscr{M}^{[a_1 \dots a_m]}_{[b_n \dots b_1]} = 0, \qquad m = 0, 1, \dots, \qquad n = 0, 1, \dots, \tag{7.17}$$

* Note an awkwardness in notations. The coefficients in eqs. (7.8) and (7.9) correspond, in the sense of eqs. (5.17)–(5.19) to the operators $X^{a_1 \dots a_m}_{b_n \dots b_1}(q, p)$, $Y^{a_1 \dots a_m}_{b_n \dots b_1}(q, p)$, and are not to be identified with any of the operators (5.7), (5.15), (5.20) and (5.24).

where

$$\mathscr{L}^{a_1\dots a_m}_{b_n\dots b_1} \equiv \sum_{p=0}^{m}\sum_{q=0}^{n}\left\{\tfrac{1}{2}\left[\tilde{X}^{a_1\dots a_p}_{b_n\dots b_{q+1}}, \tilde{X}^{a_{p+1}\dots a_m}_{b_q\dots b_1}\right](-1)^{\epsilon_q^{p0}}\right.$$

$$\left. + \sum_{r,s}{}' \tilde{X}^{a_1\dots a_p c_1\dots c_r}_{b_1\dots b_{q+1} d_s\dots d_1}\tilde{X}^{d_1\dots d_s a_{p+1}\dots a_m}_{c_r\dots c_1 b_q\dots b_1}(-1)^{\epsilon_q^{pr}} C^{pr}_{qs}\right\}, \quad (7.18)$$

$$\mathscr{M}^{a_1\dots a_m}_{b_n\dots b_1} \equiv \sum_{p=0}^{m}\sum_{q=0}^{n}\left\{\left[\tilde{Y}^{a_1\dots a_p}_{b_n\dots b_{q+1}}, \tilde{X}^{a_{p+1}\dots a_m}_{b_q\dots b_1}\right](-1)^{\epsilon_q'^{p0}}\right.$$

$$+ \sum_{r,s}{}'\left(\tilde{Y}^{a_1\dots a_p c_1\dots c_r}_{b_n\dots b_{q+1} d_s\dots d_1}\tilde{X}^{d_1\dots d_s a_{p+1}\dots a_m}_{c_r\dots c_1 b_q\dots b_1}(-1)^{\epsilon_q'^{pr}}\right.$$

$$\left.\left. - \tilde{X}^{a_1\dots a_p c_1\dots c_r}_{b_n\dots b_{q+1} d_s\dots d_1}\tilde{Y}^{d_1\dots d_s a_{p+1}\dots a_m}_{c_r\dots c_1 b_q\dots b_1}(-1)^{\epsilon_q''^{pr}}\right) C^{pr}_{qs}\right\}, \quad (7.19)$$

$$\sum_{r,s}{}' \equiv \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} - \delta_r^0\delta_s^0, \quad (7.20)$$

$$\epsilon_q^{pr} \equiv \left(\sum_{i=p+1}^{m}\epsilon_{a_i}+1\right)\left(\sum_{j=q+1}^{n}\epsilon_{b_j}+1\right)+1$$

$$+\left(\sum_{i=p+1}^{m}\epsilon_{a_i}+\sum_{i=q+1}^{n}\epsilon_{b_i}+1\right)\sum_{j=1}^{r}\epsilon_{c_j}, \quad (7.21)$$

$$C^{pr}_{qs} \equiv \frac{(p+r)!\,(q+r)!\,(m-p+s)!\,(n-q+s)!}{p!\,r!\,q!\,(m-p)!\,s!\,(n-q)!}\left(-\tfrac{1}{2}i\hbar\right)^r\left(\tfrac{1}{2}i\hbar\right)^s, \quad (7.22)$$

$$\epsilon_q'^{pr} \equiv \epsilon_q^{pr} + \sum_{i=p+1}^{m}\epsilon_{a_i} + \sum_{i=1}^{r}\epsilon_{c_i}, \quad (7.23)$$

$$\epsilon_q''^{pr} \equiv \epsilon_q^{pr} + \sum_{i=q+1}^{n}\epsilon_{b_i} + \sum_{i=1}^{r}\epsilon_{c_i}, \quad (7.24)$$

The expansion coefficients in eqs. (7.8) and (7.9) are related to the structure operators of the extended gauge algebra as

$$\tilde{X}^{a_1\dots a_m}_{b_n\dots b_1} = \delta_n^{m+1}\frac{(-1)^{E^{a_1\dots a_m}_{b_{m+1}\dots b_1}}}{m!\,(m+1)!}\tilde{U}^{a_1\dots a_m}_{b_{m+1}\dots b_1}, \quad (7.25)$$

$$\tilde{Y}^{a_1\dots a_m}_{b_n\dots b_1} = \delta_n^{m}\frac{(-1)^{E^{a_1\dots a_m}_{b_m\dots b_1}}}{(m!)^2}\tilde{V}^{a_1\dots a_m}_{b_m\dots b_1}, \quad (7.26)$$

where the following designation is used

$$E^{a_1\cdots a_m}_{b_n\cdots b_1} \equiv \sum_{i=2}^{m}\sum_{j=i}^{m}\epsilon'_{a_j} + \sum_{i=2}^{n}\sum_{j=i}^{n}\epsilon'_{b_j}. \tag{7.27}$$

The power (7.27) guarantees, as a consequence of eqs. (7.10) and (7.11), that the structure operators $\tilde{U}^{\cdots\cdots}, \tilde{V}^{\cdots\cdots}$ in the r.h.s. of eqs. (7.25) and (7.26) obey the natural antisymmetric property: the permutation of any two adjacent superscripts $(a_i \leftrightarrow a_{i+1})$ or subscripts $(b_{i+1} \leftrightarrow b_i)$ produces an extra sign factor

$$-(-1)^{\epsilon'_{a_i}\epsilon'_{a_{i+1}}} \quad \text{or} \quad -(-1)^{\epsilon'_{b_{i+1}}\epsilon'_{b_i}}, \tag{7.28}$$

respectively.

For $m = 0$ in the r.h.s. of eqs. (7.25) and (7.26) by virtue of eqs. (3.7) and (3.8) one has

$$\tilde{U}_b = \tilde{T}'_b(q, p, \Phi), \qquad \tilde{V} = \tilde{H}_0(q, p, \Phi). \tag{7.29}$$

In case of rank-one theories, when the expansions (7.8) and (7.9) stop short with the terms linear in $\bar{\mathscr{P}}$, one has exactly expressions (6.1) and (6.2) in eq. (7.1). In this case, eq. (7.16) for $m = 0$ and $n = 2$ and eq. (7.17) for $m = 0$, and $n = 1$ give the extended involution relations (6.7) and (6.8), respectively.

8. Correspondence with the formalism of refs. [1,2]

Here we shall discuss the correspondence between the modified formulation in the present paper and the one developed earlier in refs. [1,2]. The crucial point here is certainly the way of defining the additional variables and constructing the effective constraints. To get an adequate insight into the matter, a pure classical description is quite sufficient. Moreover, to avoid considering the now inessential sign factors, we confine ourselves here to the classical consideration of the pure boson case.

Let us first remember how the effective constraints are built in refs. [1,2]. The principle fact is that arbitrary second-class constraints t''_α can be presented as

$$t''_\alpha = \tau''_\beta(q, p)\, v^\beta_\alpha(q, p), \tag{8.1}$$

where the functions $\tau''_\beta(q, p)$ Poisson commute to produce the constant simplectic matrix $\omega_{\alpha\beta}$

$$\{\tau''_\alpha, \tau''_\beta\} = \omega_{\alpha\beta}, \tag{8.2}$$

and $v^\beta_\alpha(q, p)$ is an inversible matrix that depends, generally, on the canonical variables (q, p).

We define now the extra variables φ^{α} to be included into the Poisson bracket definition, so that

$$\{\varphi^{\alpha}, \varphi^{\beta}\} = \omega^{\alpha\beta}, \qquad \omega_{\alpha\beta}\omega^{\beta\gamma} = \delta_{\alpha}{}^{\gamma}. \tag{8.3}$$

Then, the functions

$$\tau'_{\alpha} \equiv \tau''_{\alpha} + \omega_{\alpha\beta}\varphi^{\beta}, \tag{8.4}$$

obey

$$\{\tau'_{\alpha}, \tau'_{\beta}\} = 0, \tag{8.5}$$

i.e. the function (8.4) is an abelian first-class constraint. However, at $\varphi = 0$, these constraints convert not to the original constraints (7.1), but to the functions

$$\tau''_{\alpha} = t''_{\beta}(v^{-1})^{\beta}_{\alpha}. \tag{8.6}$$

Let us, therefore, define new first-class constraints

$$t'_{\alpha} \equiv \tau'_{\beta}v^{\beta}_{\alpha} = t''_{\alpha} + \omega_{\gamma\beta}\varphi^{\beta}v^{\gamma}_{\alpha}. \tag{8.7}$$

For $\varphi = 0$ they coincide with eq. (8.1), but prove to be already nonabelian

$$\{t'_{\alpha}, t'_{\beta}\} = t'_{\gamma}u^{\gamma}_{\alpha\beta}. \tag{8.8}$$

Here, the structure coefficients

$$u^{\gamma}_{\alpha\beta} \equiv (v^{-1})^{\gamma}_{\mu}\left(\{v^{\mu}_{\alpha}, t'_{\beta}\} - \{v^{\mu}_{\beta}, t'_{\alpha}\}\right)$$
$$-\tfrac{1}{2}t'_{\delta}(v^{-1})^{\gamma}_{\mu}(v^{-1})^{\delta}_{\nu}\left(\{v^{\mu}_{\alpha}, v^{\nu}_{\beta}\} - \{v^{\mu}_{\beta}, v^{\nu}_{\alpha}\}\right) + t'_{\delta}u^{\gamma\delta}_{\alpha\beta}, \tag{8.9}$$

appear on the r.h.s. The functions

$$u^{\gamma\delta}_{\alpha\beta} = -u^{\gamma\delta}_{\beta\alpha} = -u^{\delta\gamma}_{\alpha\beta}, \tag{8.10}$$

are arbitrary and regular.

It is the operator-valued analogs of the functions (8.7) that are in fact used in refs. [1,2] for effective first-class constraints (see, e.g., eqs. (2.36)–(2.38) in ref. [1]).

Now we are going to see how the effective constraints are constructed in ref. [5] and to put these two constructions into correspondence with one another. Let t''_{α} be again the original classical second-class constraints. Then, the prescription of ref. [5]

is to directly look for *abelian* effective constraints t_α obeying the equation

$$t_\alpha = t_\alpha(q,p,\varphi), \qquad \{t_\alpha, t_\beta\} = 0, \qquad t_\alpha(q,p,0) = t''_\alpha(q,p) \tag{8.11}$$

in the form of a series

$$t_\alpha = t''_\alpha(q,p) + \sum_{n=1}^{\infty} x_{\alpha\alpha_n\dots\alpha_1}(q,p)\varphi^{\alpha_1}\dots\varphi^{\alpha_n}. \tag{8.12}$$

Within the approximation linear in φ,

$$t_\alpha = t''_\alpha + x_{\alpha\beta}\varphi^\beta + \mathrm{O}(\varphi^2), \tag{8.13}$$

one gets

$$\{t''_\alpha, t''_\beta\} = -x_{\alpha\mu}\omega^{\mu\nu}x_{\beta\nu}, \tag{8.14}$$

which corresponds to the classical approximation in our eqs. (5.12) and (5.13) for $n=0$. Equations for higher coefficients in expansion (8.12) also follow, quite naturally, from our eqs. (5.12) and (5.13) in classical approximation at $n = 1, 2, \dots$.

Let us ask ourselves the question: what is the relation between the two sets of effective first-class constraints? The constraints (8.12) are abelian by construction, whereas the constraints (8.7) are nonabelian according to (8.8) and (8.9). However, at $\varphi = 0$, both sets of constraints coincide with t''_α. The question is, is it possible to reduce the nonabelian constraints (8.7) to abelian ones by using the abelization procedure? It is known that any first-class constraints can be (locally) made abelian by forming a linear combination with regular coefficients. At first glance, the making of the constraints (8.7) abelian is trivial, since they are already linear combinations of abelian constraints (8.4). Although being abelian, the constraints (8.4) do not coincide with t''_α at $\varphi = 0$. Hence, we should try to meet simultaneously the two requirements: first, that the linear combinations of constraints (8.7) with regular invertible coefficients,

$$t'_\beta(q,p,\varphi)w^\beta_\alpha(q,p,\varphi), \tag{8.15}$$

be abelian, is

$$\{t'_\mu w^\mu_\alpha, t'_\nu w^\nu_\beta\} = 0, \tag{8.16}$$

and, second, that the condition

$$w^\mu_\alpha(q,p,0) = \delta^\mu{}_\alpha, \tag{8.17}$$

be obeyed, which is to guarantee the coincidence of (8.15) with $t''_\alpha(q,p)$ at $\varphi = 0$.

To solve equations (8.16) and (8.17), the power-series representation can be used

$$w_\alpha^\beta = \delta_\alpha^\beta + \sum_{n=1}^{\infty} y^\beta_{\alpha\alpha_n \dots \alpha_1}(q, p)\varphi^{\alpha_1} \dots \varphi^{\alpha_n}, \tag{8.18}$$

so that, within the accuracy linear in φ one has

$$w_\alpha^\beta = \delta_\alpha^\beta + y^\beta_{\alpha\gamma}\varphi^\gamma + O(\varphi^2). \tag{8.19}$$

The abelian constraints (8.12) are, naturally, identified with the linear combinations (8.15)

$$t_\alpha = t'_\beta w_\alpha^\beta, \tag{8.20}$$

whence it follows in the linear approximation (8.13) and (8.19) that

$$x_{\alpha\beta} = v_\alpha^\gamma w_{\gamma\beta} + t''_\gamma y^\gamma_{\alpha\beta}. \tag{8.21}$$

The first term in the r.h.s. of (8.21) gives exact solution to equations (8.1) and (8.14) on the hypersurface of constraints $t'' = 0$. The presence of the second term in eq. (8.21) then corresponds to the following general circumstance: any two regular functions that coincide on the hypersurface of constraints differ from one another, generally, by a linear combination of constraints with regular coefficients. Thereby, the eq. (8.21) determines the coefficients $y^\gamma_{\alpha\beta}$ up to a transformation

$$y^\gamma_{\alpha\beta} \to y^\gamma_{\alpha\beta} + t''_\delta z^{\gamma\delta}_{\alpha\beta}, \tag{8.22}$$

where

$$z^{\gamma\delta}_{\alpha\beta} = -z^{\delta\gamma}_{\alpha\beta}, \tag{8.23}$$

are arbitrary regular functions. In an analogous way one can find (within a natural arbitrariness) all the coefficients in expansion (8.18) in terms of those in expansion (8.12).

Thus, we see that to find abelian effective constraints (8.12) following the recipe of ref. [5] is to find the matrix (8.18) of the special abelization (8.16) and (8.17) for the effective nonabelian constraints (8.7), defined in refs. [1, 2].

On the other hand, we know [6] that making the first-class constraints abelian is associated with a ghost-depending canonical transformation of the fermion generating operator of the gauge algebra. The term $\bar{\mathscr{P}}_\alpha w_\beta^\alpha C^\beta$ in the expansion of the operator of this canonical transformation just explicitly includes the matrix of the linear combination. This is already enough to make it clear that the generating operators of the abelian gauge algebra of the constraints (8.12) and those of the nonabelian algebra of constraints (8.7) are related by just the abelizing canonical transformation.

Now we are in a position to formulate the final result of this section: the two formulations of operatorial quantization – that developed earlier in refs. [1, 2] and that considered in the present paper – are canonically equivalent. To be more precise, the generating operators of the gauge algebra and the unitarizing hamiltonians are related in the two formalisms by a canonical transformation depending on the ghosts that make the gauge algebra of the effective constraints abelian.

9. Conclusion

Here we shall compare some features of the two versions of operatorial quantization suggested, respectively, in refs. [1, 2] and in the present paper *in what concerns their results*. In a way, these features are complementary. This becomes especially obvious from the structure of the fermion generating operators, providing there is no first-class constraints originally in the theory. The generating fermion operator given by eq. (2.32) of ref. [1] is linear in the extra degree-of-freedom operators Φ, but is, generally, an endless series in powers of the ghost operators $\bar{\mathscr{P}}, C$. The corresponding generating fermion operator (4.10) of the present paper is, on the contrary, linear in C'', but, generally, *contains all powers of* Φ. The unitarizing hamiltonian is a series in powers of Φ in the both formalisms, but in the formalism of refs. [1, 2] it also is a series in powers of ghosts, while in the present formalism it is quadratic in the ghosts $(C'', \bar{\mathscr{P}}'')$, $(\mathscr{P}'', \bar{C}'')$.

It may seem that the formalism of refs. [1, 2] is much more complicated technically than that of the present paper. Still, we think that each formalism has its advantages and disadvantages. For instance, in the present formalism the effective constraints and hamiltonian operators are series in powers of the additional variables Φ that carry the statistics of the original second-class constraints. It is hardly probable that in any nontrivial cases these series are saturated by finite numbers of terms, although for every known example the series in powers of ghosts are known to include only finite numbers of terms in the theories with first-class constraints.

In view of the above mentioned, we adhere to the opinion that each of the two alternative versions of operatorial quantization has its special features that may make it advantageous in studying special models. Namely, the formalism of refs. [1, 2] is likely to be more appropriate when treating the theories where the second-class constraints include the "former" first-class constraints distorted by a strong violation of an initial gauge symmetry, say, by mass terms of no-Higgs origin. On the other hand, the present formalism may prove to be more proper for the theories in which the second-class constraints are not of a natural gauge origin. The case when the second-class constraints are polynomials and one can hope that the series in powers of the extra variables Φ contain only finite numbers of terms is likely to belong here too.

Thus, in refs. [1, 2] and in the present paper we have studied in detail two realizations of the general method of operatorial quantization in the presence of second-class constraints. One may ask whether the above two realizations saturate all the available possibilities. The answer is certainly no. There exists an infinite class of canonically equivalent formulations covered by our basic equations

$$[\Omega, \Omega] = 0, \qquad \epsilon(\Omega) = 1, \qquad \mathrm{gh}(\Omega) = 1, \tag{9.1}$$

$$[\mathscr{H}, \Omega] = 0, \qquad \epsilon(\mathscr{H}) = 0, \qquad \mathrm{gh}(\mathscr{H}) = 0, \tag{9.2}$$

$$H = \mathscr{H} + (i\hbar)^{-1}[\Psi, \Omega], \tag{9.3}$$

$$\epsilon(\Psi) = 1, \qquad \mathrm{gh}(\Psi) = -1, \tag{9.4}$$

$$\Omega = \Omega^{\dagger}, \qquad \mathscr{H} = \mathscr{H}^{\dagger}, \qquad \Psi = -\Psi^{\dagger}. \tag{9.5}$$

As applied to the systems with irreducible first- and second-class constraints (eqs. (2.2) and (2.3) respectively), the general solution has the form

$$\Omega = \Omega_{\min}(q, p; \Phi; C', \bar{\mathscr{P}}'; C'', \bar{\mathscr{P}}'') + \pi'_a \mathscr{P}'^a + \pi''_\alpha \mathscr{P}''^\alpha, \tag{9.6}$$

$$\mathscr{H} = \mathscr{H}(q, p; \Phi; C', \bar{\mathscr{P}}'; C'', \bar{\mathscr{P}}''), \tag{9.7}$$

under the conditions

$$\Omega_{\min}(q, p; 0; C', 0; C'', 0) = T'_a C'^a + T''_\alpha C''^\alpha, \tag{9.8}$$

$$\mathscr{H}(q, p; 0; 0, 0; 0, 0) = H_0. \tag{9.9}$$

The realization of refs. [1,2], as well as the present one, are two simplest particular solutions, gained in the framework of (9.1)–(9.9), specialized by their mutually complementary properties, as explained above.

In the general solution (9.6)–(9.9), the boundary conditions (9.8) contain the original first- and second-class constraints in a fully symmetrical way. Special for the second-class constraints here is only the dependence on the extra degrees of freedom Φ present in eqs. (9.6) and (9.7).

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