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# QUANTUM ELECTRODYNAMICS IN AN EXTERNAL CONSTANT FIELD

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The generating functional of all quantum Green functions in an external field is found. The Green function of a scalar particle in a constant and uniform external field is calculated with the help of modified perturbation theory. Its infrared asymptotics is studied. It is shown that an enhancement of the pole takes place, as in the vacuum case.

## 1. Introduction

There has been great interest in electromagnetic processes in external fields during the last few years. This is accounted for, on the one hand, by the creation of the intensive electromagnetic fields in lasers and the appearance of electrons and photons of high energies, and, on the other hand, by possible astrophysical applications. In the above processes one can verify quantum electrodynamics in the domain of high energies and large fields. The study of this phenomenon is of fundamental importance because a correct description of the phenomenon in intensive external fields is connected with the necessity of going beyond the framework of the usual perturbation theory. For this reason the processes in the $e^2$ approximation in different external fields [1–14] have been studied in detail lately with the external field taken into account exactly (in this connection the constant uniform field and the plane wave field or their combination are of a special interest because in these external fields one can exactly solve the Dirac and Klein-Gordon equations and find the one-particle Green functions). However, the contribution of radiative corrections at high energies and intensive fields becomes essential. For this reason it becomes of special interest to work out such methods of calculation of radiative corrections to the processes in an external field which, in their final result, go essentially beyond the scope of the usual perturbation theory with the external field kept exactly.

This is due to the fact that the generalized (with the external field being taken into account exactly) Feynman diagram technique encounters essential difficulties caused by the necessity of correct calculation of the vertex function in a self-consistent approximation, since otherwise the results obtained are gauge noninvariant, and the corresponding generalized Ward identities in the presence of an external field [15–17] are not correct.

The most adequate method for solving this problem is the functional method developed in papers [17, 19]. With the help of this method a closed expression for the generating functional of the Green function is obtained in [16, 17, 20] and the modified perturbation theory is constructed.

The method of modified perturbation theory proposed in [16, 17, 20] leads to a result which goes far beyond the scope of the usual perturbation theory. This method appeared to be especially fruitful in quantum electrodynamics, because in the first order of the modified perturbation theory the contribution of the soft photons, both virtual and real, is summed completely. In particular, the first order of the modified perturbation theory gave the possibility of obtaining the correct asymptotic behaviour of the Green function and the cross section of the processes in QED [16, 20, 21] in the double logarithmic approximation. Not long ago [22] the method of the modified perturbation theory was also used in statistic quantum electrodynamics.

These general results (which are beyond the scope of the usual perturbation theory) will allow us to essentially advance in solving the problem of finding the quantum Green function in the presence of a real external field.

In the present paper we consider the calculation of the one-particle Green function of a scalar field in a constant uniform external field by means of the modified perturbation theory. The Green function is found in the representation of eigenfunctions of the Klein-Gordon equation in a constant field. Such functions were first obtained in papers [12, 13]. It was shown there that Dirac's eigenfunctions (in the spinor case) in a constant field diagonalize not only the Green function without radiative correction, but also the Green function in the $e^2$ approximation (see also [8]). The use of eigenfunctions appears fruitful in the modified perturbation theory because they contain considerable information about the dynamics of the system in a constant field.

## 2. Generating functional in QED with external field

The principal value of the functional method is the generating functional of all Green functions:

$$Z[\xi^*, \xi, J] = C_v^{-1}{}_{\text{out}}\langle 0|S[\xi^*, \xi, J]|0\rangle_{\text{in}}, \tag{2.1}$$

where

$$S[\xi^*, \xi, J] = T\exp\left\{ i\int \mathcal{L}_s(x)\,\mathrm{d}^4x \right\},$$

$$\mathcal{L}_s(x) = J_\mu(x)A^\mu(x) + \xi^*(x)\varphi(x) + \varphi^+(x)\xi(x).$$

$C_v = {}_{\text{out}}\langle 0|0\rangle_{\text{in}}$ is the probability amplitude for the vacuum to remain the vacuum, $|0\rangle_{\text{in}}$, $|0\rangle_{\text{out}}$ are initial and final vacuum states, $J_\mu(x)$ is the source of the electromagnetic field $A^\mu(x)$, $\xi(x)$ and $\xi^*(x)$ are sources of the scalar fields $\varphi^+(x)$ and $\varphi(x)$ respectively. Operators are taken in the usual Heisenberg representation (i.e. the interaction between the fields is taken into account in the absence of the external sources) and satisfy the equations

$$\begin{aligned}
&\left[\mathcal{P}_\mu(x)\mathcal{P}^\mu(x) - m^2\right]\varphi(x) = 0, \qquad \mathcal{P}_\mu(x) = i\partial_\mu - eA_\mu(x) - eA_\mu^{\text{ext}}(x),\\
&\left[\mathcal{P}_\mu^*(x)\mathcal{P}^{*\mu}(x) - m^2\right]\varphi^+(x) = 0, \qquad \mathcal{P}_\mu^*(x) = i\partial_\mu + eA_\mu(x) + eA_\mu^{\text{ext}}(x),\\
&\left[\Box g_{\mu\nu} + (d_l - 1)d_l^{-1}\partial_\mu\partial_\nu\right]A^\nu(x) = I_\mu(x), \qquad \Box = -\partial_\mu\partial^\mu,\\
&I_\mu(x) = e\varphi^+(x)\mathcal{P}_\mu(x)\varphi(x) - e\left(\mathcal{P}_\mu^*(x)\varphi^+(x)\right)\varphi(x).
\end{aligned} \tag{2.2}$$

$A_\mu^{\text{ext}}$ is an external electromagnetic field, $d_l$ is a gauge parameter. Using the method proposed in [16, 17, 20] one may show that the solution of the functional equations for the generating functional (2.1) resulting from (2.2) has the form

$$\begin{aligned}
Z[\xi^*, \xi, J] = C\exp\Big\{ &\Pi\Big(A^{\text{ext}} + \frac{\delta}{i\delta J}\Big)\\
&- i\int d^4x\, d^4y\, \xi^*(x) G\Big(x, y|A^{\text{ext}} + \frac{\delta}{i\delta J}\Big)\xi(y)\Big\}\\
&\times\exp\Big\{\tfrac{1}{2}i\int d^4x\, d^4y\, J_\mu(x) D^{\mu\nu}(x-y) J_\nu(y)\Big\},
\end{aligned} \tag{2.3}$$

where $G(x, y|a)$ is the causal Green function of the scalar particle in the "external" field $a_\mu(x) = A_\mu^{\text{ext}}(x) - i\delta/\delta J^\mu(x)$ satisfying the equation

$$\left[(i\partial_\mu - ea_\mu(x))(i\partial^\mu - ea^\mu(x)) - m^2\right]G(x, y|a) = \delta^{(4)}(x-y). \tag{2.4}$$

$\Pi(A^{\text{ext}} + \delta/i\delta J)$ is the polarization correction to the action of the classical electromagnetic field:

$$\Pi\Big(A^{\text{ext}} + \frac{\delta}{i\delta J}\Big) = \operatorname{Sp}\ln\Big[G\Big(\ |A^{\text{ext}} + \frac{\delta}{i\delta J}\Big)\Big/ G(\ |0)\Big]. \tag{2.5}$$

$C$ is a constant to be determined by the condition $Z[0,0,0] = 1$.

Further, it is convenient to find a functional representation for the Green function in an arbitrary field $a_\mu(x)$. To do this let us write $G(x, y|a)$ in the form of the inverse operator

$$\begin{aligned}
&G(x, y|a) = -i\int_0^\infty d\nu \exp\{i(\pi^2 - m^2 + i\varepsilon)\nu\}\delta^{(4)}(x-y),\\
&\pi_\mu = i\partial_\mu - ea_\mu.
\end{aligned} \tag{2.6}$$

Following paper [20] let us write the Green function (2.6) in the form of a functional integral:

$$G(x, y|a) = -i\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{-ip(x-y)} \int_0^\infty \mathrm{d}\nu\, \mathrm{e}^{-i(m^2 - i\varepsilon)\nu}$$

$$\times \int \mathrm{D}^4 t \exp\left\{-i\int_0^\nu \left[t_\mu(\nu')t^\mu(\nu') - 2p_\mu t^\mu(\nu')\right]\mathrm{d}\nu'\right\}\phi(x, p, \nu|t), \tag{2.7}$$

where the function $\phi(x, p, \nu|t)$ satisfies the equation

$$-i\frac{\partial\phi}{\partial\nu} = 2\pi_\mu t^\mu(\nu)\phi, \quad \phi(\nu = 0) = 1.$$

$\mathrm{D}^4 t$ is the normalized volume element in the functional space

$$\mathrm{D}^4 t = \prod_{\nu'=0}^{\nu} \mathrm{d}^4 t(\nu')\left[\int \prod_{\nu'=0}^{\nu} \mathrm{d}^4 t(\nu') \exp\left\{-i\int_0^\nu t_\mu(\nu')t^\mu(\nu')\,\mathrm{d}\nu'\right\}\right]^{-1}.$$

Let us consider the subsidiary function $\tilde{\phi}(x, p, \nu|A^{\mathrm{ext}}, u)$ satisfying the equation

$$-i\frac{\partial\tilde{\phi}}{\partial\nu} = \left(2\pi_\mu t^\mu(\nu) + u_\mu(\nu)t^\mu(\nu)\right)\tilde{\phi}, \qquad \tilde{\phi}(\nu = 0) = 1.$$

Note that $\tilde{\phi}(u = 0) = \phi$. It is not difficult to find the solution of this equation:

$$\tilde{\phi} = \exp\left\{-2i\int_0^\nu \left[ea_\mu\left(x - 2\int_{\nu'}^\nu t(\lambda)\,\mathrm{d}\lambda\right) + \tfrac{1}{2}u_\mu(\nu')\right]t^\mu(\nu')\,\mathrm{d}\nu'\right\}.$$

Separating the external field $A_\mu^{\mathrm{ext}}$ and the "radiative field" $\delta/i\delta J^\mu$ from $a_\mu$ and making some simple transformations we obtain [20, 23]

$$G\left(x, y|A^{\mathrm{ext}} + \frac{\delta}{i\delta J}\right) = -i\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{-ip(x-y)} \int_0^\infty \mathrm{d}\nu\, \mathrm{e}^{-i(m^2 - i\varepsilon)\nu}$$

$$\times \exp\left\{-2e\int_0^\nu P^\mu(\nu')\frac{\delta}{\delta J^\mu(x(\nu'))}\,\mathrm{d}\nu'\right\} Y(x, p, \nu|A^{\mathrm{ext}}, u)|_{u=0},$$

$$Y(x, p, \nu|A^{\mathrm{ext}}, u) = \int \mathrm{D}^4 t \exp\left\{-i\int_0^\nu \left[\mathfrak{L}(\nu'|t, A^{\mathrm{ext}}) - u_\mu(\nu')t^\mu(\nu')\right]\mathrm{d}\nu'\right\}, \tag{2.8}$$

where

$$\mathfrak{L}(\nu'|t, A^{\mathrm{ext}}) = \left[t_\mu(\nu') - 2p_\mu + 2eA_\mu^{\mathrm{ext}}\left(x - 2\int_{\nu'}^\nu t(\lambda)\,\mathrm{d}\lambda\right)\right]t^\mu(\nu')$$

is the classical action of the particle in the external field $A_\mu^{\text{ext}}$,

$$P^\mu(\nu') = -i\frac{\delta}{\delta u_\mu(\nu')}, \qquad x(\nu') = x - 2\int_{\nu'}^{\nu} P(\lambda)\,\mathrm{d}\lambda .$$

Substituting (2.8) into (2.3) we obtain for the generating functional $Z[\xi^*, \xi, J]$ the expression

$$Z[\xi^*, \xi, J] = C\exp\Bigg\{\int \mathrm{d}^4x \int_{s_0}^{\infty} \frac{\mathrm{d}s}{s}\,\mathrm{e}^{-i(m^2 - i\varepsilon)s}$$

$$\times\Bigg(-\langle 1\rangle_{x,x}^{A^{\text{ext}}=0} + \Bigg\langle \exp\Bigg(-2e\int_0^s P^\mu(s')\frac{\delta}{\delta J^\mu(x(s'))}\,\mathrm{d}s'\Bigg)\Bigg\rangle_{x,x}\Bigg)\Bigg\}$$

$$\times\exp\Bigg\{-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\xi^*(x)\int_0^{\infty}\mathrm{d}\nu\,\mathrm{e}^{-i(m^2 - i\varepsilon)\nu}$$

$$\times\Bigg\langle \exp\Bigg(-2e\int_0^{\nu} P^\mu(\nu')\frac{\delta}{\delta J^\mu(x(\nu'))}\,\mathrm{d}\nu'\Bigg)\Bigg\rangle_{xy}\xi(y)\Bigg\}$$

$$\times\exp\Big\{\tfrac{1}{2}i\int \mathrm{d}^4x\,\mathrm{d}^4y\,J^\mu(x)D_{\mu\nu}(x-y)J^\nu(y)\Big\}, \tag{2.9}$$

where

$$\Big\langle A\Big(\frac{\delta}{\delta u}\Big)\Big\rangle_{x,y} = A\Big(\frac{\delta}{\delta u}\Big)\tilde{Y}(x, y, \nu|A^{\text{ext}}, u)|_{u=0},$$

$$\tilde{Y}(x, y, \nu|A^{\text{ext}}, u) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-ip(x-y)}Y(x, p, \nu|A^{\text{ext}}, u),$$

$$D_{\mu\sigma}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} D_{\mu\sigma}(k)\mathrm{e}^{ikx},$$

$$D_{\mu\sigma}(k) = -\big(g_{\mu\sigma}k^2 + (d_l - 1)k_\mu k_\sigma\big)\int_{-it_0}^{\infty} t\,\mathrm{e}^{ik^2t}\mathrm{d}t. \tag{2.10}$$

Some remarks about expression (2.9) are in order. It is known that the expression for $Z[\xi^*, \xi, J]$ contains divergences before the programme of renormalization is carried out. It is convenient to carry out this programme after all the calculations have been done. To do so it is necessary to regularize $Z[\xi^*, \xi, J]$ (2.9), so that the divergences are absent during the calculation. As in papers [16, 20] this can be done by introducing the cut-off $s_0$ for the proper-time of the particle in the polarization term in (2.9) and the cut-off $t_0$ for the photon proper-time in (2.9). As long as $s_0$ and $t_0$ are not zero, the whole expression (2.9) is finite. After carrying out the programme of renormalization it is necessary to remove the regularization by making $s_0$ and $t_0$ zero. The resulting renormalized expressions should not depend on $s_0$ and $t_0$.

In (2.9) one may carry out the partial functional differentiation over the sources $J^\mu$ of the electromagnetic field $A_\mu$ by expanding (2.9) into a series in powers of the sources $\xi^*$ and $\xi$:

$$C^{-1}Z[\xi^*,\xi,J]=C^{-1}Z[0,0,J]$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\prod_{k=1}^{n}\int \mathrm{d}^4x_k\mathrm{d}^4y_k\,\xi^*(x_k)\xi(y_k)\int_0^\infty \mathrm{d}\nu_k\,\mathrm{e}^{-i(m^2-i\varepsilon)\nu_k}$$

$$\times\exp\left\{\sum_{n_1,n_2=1}^{n}A_{n_1,n_2}+\sum_{n_1=1}^{n}B_{n_1}(J)\right\}\exp\left\{+\int \mathrm{d}^4z\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s}\,\mathrm{e}^{-i(m^2-i\varepsilon)s}\right.$$

$$\times\left[\exp\left[\sum_{n_1=1}^{n}R_{n_1}-2e\int_0^s\mathcal{P}^\mu(s')\frac{\delta}{\delta J^\mu(z(s'))}\,\mathrm{d}s'\right]\right.$$

$$\left.\left.\times\tilde{Y}(z,z;s|A^{\mathrm{ext}},v)|_{v=0}-\tilde{Y}(z,z;s|0,0)\right]\right\}$$

$$\times\prod_{l=1}^{n}\tilde{Y}(x_l,y_l;\nu_l|A^{\mathrm{ext}},u_l)|_{u=0}$$

$$\times\exp\left\{\tfrac{1}{2}i\int \mathrm{d}^4x\,\mathrm{d}^4y\,J^\mu(x)D_{\mu\nu}(x-y)J^\nu(y)\right\},\qquad(2.11)$$

where

$$A_{n_1,n_2}=2ie^2\int_0^{\nu_{n_1}}\mathrm{d}\nu'\int_0^{\nu_{n_2}}\mathrm{d}\nu''\,P^\mu_{n_1}(\nu')D_{\mu\nu}\big(x_{n_1}(\nu')-x_{n_2}(\nu'')\big)P^\nu_{n_2}(\nu''),\qquad(2.12)$$

$$B_{n_1}(J)=-2ie\int_0^{\nu_{n_1}}\mathrm{d}\nu'\int \mathrm{d}^4x\,J^\mu(x)D_{\mu\nu}\big(x-x_{n_1}(\nu')\big)P^\nu_{n_1}(\nu'),\qquad(2.13)$$

$$R_{n_1}=4ie^2\int_0^s\mathrm{d}s'\int_0^{\nu_{n_1}}\mathrm{d}\nu'\,\mathcal{P}^\mu(s')D_{\mu\nu}\big(z(s')-x_{n_1}(\nu')\big)P^\nu_{n_1}(\nu'),\qquad(2.14)$$

$$P^\mu_{n_1}(\nu')=-i\frac{\delta}{\delta u_{n_1\mu}(\nu')},\qquad x_{n_1}(\nu')=x_{n_1}-2\int_{\nu'}^{\nu_{n_1}}P_{n_1}(\lambda)\,\mathrm{d}\lambda,$$

$$\mathcal{P}^\mu(s')=-i\frac{\delta}{\delta v_\mu(s')},\qquad z(s')=z-2\int_{s'}^{s}\mathcal{P}(\lambda)\,\mathrm{d}\lambda,\qquad(2.15)$$

and

$$Z[0,0,J] = C\exp\left(\Pi\left(A^{\mathrm{ext}} + \frac{\delta}{i\delta J}\right)\right)\exp\left\{\tfrac{1}{2}i\int \mathrm{d}^4x\,\mathrm{d}^4y\,J^\mu(x)\,D_{\mu\nu}(x-y)\,J^\nu(y)\right\}$$

is the generating functional for the photon Green functions. To carry out the complete functional differentiation over the sources $J_\mu$ one should expand $\Pi(A^{\mathrm{ext}} + \delta/i\delta J)$ into a series with respect to the functions $G(|A^{\mathrm{ext}} + \delta/i\delta J)$. In this way we obtain the following expression for $Z[\xi^*, \xi, J]$:

$$\begin{aligned}
C^{-1}Z[\xi^*,\xi,J] &= C^{-1}Z[0,0,J] + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\prod_{k=1}^{n}\int \mathrm{d}^4x_k\,\mathrm{d}^4y_k\,\xi^*(x_k)\xi(y_k)\\
&\quad\times\int_0^\infty \mathrm{d}\nu_k\,\mathrm{e}^{-i(m^2-i\varepsilon)\nu_k}\exp\left\{\sum_{n_1,n_2=1}^{n}A_{n_1,n_2} + \sum_{n_1=1}^{n}B_{n_1}(J)\right.\\
&\qquad\left. - \int \mathrm{d}^4z\int_{s_0}^\infty \frac{\mathrm{d}s}{s}\,\mathrm{e}^{-i(m^2-i\varepsilon)s}\tilde{Y}(z,z,s|0,0)\right\}\\
&\quad\times\left\{1 + \sum_{r=1}^{\infty}\frac{1}{r!}\prod_{p=1}^{r}\int \mathrm{d}^4z_p\int_{s_0}^{\infty}\frac{\mathrm{d}s_p}{s_p}\,\mathrm{e}^{-i(m^2-i\varepsilon)s_p}\right.\\
&\qquad\times\exp\left\{\sum_{n_1=1}^{n}\sum_{\alpha=1}^{r}C_{n_1,\alpha} + \sum_{\alpha=1}^{r}B_{1\alpha}(J) + \sum_{\alpha,\alpha_1=1}^{r}A_{1\alpha,\alpha_1}\right\}\\
&\qquad\left.\times\prod_{a=1}^{r}\tilde{Y}(z_a,z_a,s_a|A^{\mathrm{ext}},v_a)|_{v=0}\right\}\\
&\quad\times\prod_{l=1}^{n}\tilde{Y}(x_l,y_l,\nu_l|A^{\mathrm{ext}},u_l)|_{u=0}\\
&\quad\times\exp\left\{\tfrac{1}{2}i\int \mathrm{d}^4x\,\mathrm{d}^4y\,J^\mu(x)\,D_{\mu\nu}(x-y)\,J^\nu(y)\right\},
\end{aligned}\tag{2.16}$$

where

$$C_{n_1,\alpha} = 4ie^2\int_0^{\nu_{n_1}}\mathrm{d}\nu'\int_0^{s_\alpha}\mathrm{d}s'\,P^\mu_{n_1}(\nu')\,D_{\mu\nu}\big(x_{n_1}(\nu') - z_\alpha(s')\big)\,\mathcal{P}^\nu_\alpha(s'),\tag{2.17}$$

$$B_{1\alpha}(J) = -2ie\int_0^{s_\alpha}\mathrm{d}s'\int \mathrm{d}^4x\,J^\mu(x)\,D_{\mu\nu}\big(x - z_\alpha(s')\big)\,\mathcal{P}^\nu_\alpha(s'),$$

$$A_{1\alpha,\alpha_1} = 2ie^2\int_0^{s_\alpha}\mathrm{d}s'\int_0^{s_{\alpha_1}}\mathrm{d}s''\,\mathcal{P}^\mu_\alpha(s')\,D_{\mu\nu}\big(z_\alpha(s') - z_{\alpha_1}(s'')\big)\,\mathcal{P}^\nu_{\alpha_1}(s''),$$

$$\mathcal{P}^\mu_\alpha(s') = -i\frac{\delta}{\delta v_{\alpha\mu}(s')},\qquad z_\alpha(s') = z_\alpha - 2\int_{s'}^{s_\alpha}\mathcal{P}_\alpha(\lambda)\,\mathrm{d}\lambda.\tag{2.18}$$

The expression for $Z[\xi^*, \xi, J]$ will have the simplest form if we neglect the polarization effects. Then we obtain:

$$Z[\xi^*,\xi,J]=\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\prod_{k=1}^{n}\int d^4x_k d^4y_k \xi^*(x_k)\xi(y_k)\right.$$

$$\times\int_0^{\infty}d\nu_k\, e^{-i(m^2-i\varepsilon)\nu_k}\exp\left\{\sum_{n_1,n_2=1}^{n}A_{n_1,n_2}+\sum_{n_1=1}^{n}B_{n_1}(J)\right\}$$

$$\left.\times\prod_{l=1}^{n}\tilde{Y}(x_l,y_l,v_l|A^{\text{ext}},u_l)|_{u=0}\right]$$

$$\times\exp\left\{\tfrac{1}{2}i\int d^4x\, d^4y J^{\mu}(x)D_{\mu\nu}(x-y)J^{\nu}(y)\right\}. \tag{2.19}$$

If we are interested only in the Green functions of the scalar particle, the expression for the generating functional becomes still more simplified:

$$Z[\xi^*,\xi,\mathbf{0}]=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\prod_{k=1}^{n}\int d^4x_k d^4y_k \xi^*(x_k)\xi(y_k)$$

$$\times\exp\left\{\sum_{n_1,n_2=1}^{n}A_{n_1,n_2}\right\}\prod_{l=1}^{n}\tilde{Y}(x_l,y_l,\nu_l|A^{\text{ext}},u_l)|_{u=0}. \tag{2.20}$$

Using (2.16) one can write the one-particle scalar Green function in the form

$$G(x,y)=-i\int_0^{\infty}d\nu\exp\{-i(m^2-i\varepsilon)\nu+A_{1,1}\}$$

$$\times\left(1+\sum_{r=1}^{\infty}\frac{1}{r!}\prod_{p=1}^{r}\int d^4z_p\int_{s_0}^{\infty}\frac{ds_p}{s_p}e^{-i(m^2-i\varepsilon)s_p}\right.$$

$$\left.\times\exp\left\{\sum_{\alpha=1}^{r}C_{1\alpha}+\sum_{\alpha,\alpha_1=1}^{r}A_{1\alpha,\alpha_1}\right\}\prod_{a=1}^{r}\tilde{Y}(z_a,z_a,s_a|A^{\text{ext}},v_a)|_{v=0}\right)$$

$$\times\exp\left\{-\int d^4z\int_{s_0}^{\infty}\frac{ds}{s}e^{-i(m^2-i\varepsilon)s}\tilde{Y}(z,z,s|0,0)\right\}\tilde{Y}(x,y,\nu|A^{\text{ext}},u)|_{u=0}. \tag{2.21}$$

When neglecting polarization effects this expression takes an especially simple form:

$$G(x,y)=-i\int_0^{\infty}d\nu\exp\{-i(m^2-i\varepsilon)\nu\}\times\langle\exp A_{1,1}\rangle_{x,y}. \tag{2.22}$$

Consider now expression (2.20) in detail. Suppose we know a complete set of solutions of the eigenvalue problem.

$$\pi_\mu^{\text{ext}}(x)\pi^{\text{ext}\,\mu}(x)\psi_{\{k\}}(x)=p^2\psi_{\{k\}}(x), \qquad \pi_\mu^{\text{ext}}(x)=i\partial_\mu-eA_\mu^{\text{ext}}(x), \tag{2.23}$$

where $\{k\}$ is a set of quantum numbers ($\{k\}$ contains the quantum number $p^2$). Let us call such functions eigenfunctions. In the constant uniform field such functions were first obtained in [12, 13]. Suppose, further, that the eigenfunctions form a complete orthonormalized system of functions with respect to the scalar product

$$(\varphi,\psi)\equiv\int d^4x\,\varphi^*(x)\psi(x); \tag{2.24}$$

$$\left(\psi_{\{k\}},\psi_{\{k'\}}\right)=\delta_{\{k\},\{k'\}},$$

$$\sum_{\{k\}}\psi_{\{k\}}(x)\psi^*_{\{k\}}(y)=\delta^{(4)}(x-y). \tag{2.25}$$

(If the eigenvalues are discrete, then $\delta_{\{k\},\{k'\}}$ in (24) is Kronecker's symbol and if they are continuous, then $\delta_{\{k\},\{k'\}}$ is the $\delta$-function.) With the help of the eigenfunctions one can construct the causal Green function of the Bose particle in the external field $A_\mu^{\text{ext}}$ without radiative corrections:

$$G^{(0)}(x,y|A^{\text{ext}})=\sum_{\{k\}}\frac{\psi_{\{k\}}(x)\psi^*_{\{k\}}(y)}{p^2-m^2+i\varepsilon}, \tag{2.26}$$

which satisfies the equation

$$\left\{\pi_\mu^{\text{ext}}(x)\pi^{\text{ext}\,\mu}(x)-m^2\right\}G^{(0)}(x,y|A^{\text{ext}})=\delta^{(4)}(x-y). \tag{2.27}$$

If we represent $(p^2-m^2+i\varepsilon)^{-1}$ in an exponential form, the Green function (2.26) can be written as

$$G^{(0)}(x,y|A^{\text{ext}})=-i\int_0^\infty d\nu\,e^{-i(m^2-i\varepsilon)\nu}\sum_{\{k\}}e^{ip^2\nu}\psi_{\{k\}}(x)\psi^*_{\{k\}}(y). \tag{2.28}$$

On the other hand, the Green function $G^{(0)}(x,y|A^{\text{ext}})$ can be obtained using expression (2.8):

$$G^{(0)}(x,y|A^{\text{ext}})=-i\int_0^\infty d\nu\,e^{-i(m^2-i\varepsilon)\nu}\tilde{Y}(x,y,\nu|A^{\text{ext}},0). \tag{2.29}$$

Comparing (2.28) and (2.29) we have

$$\tilde{Y}(x,y,\nu|A^{\text{ext}},0)=\sum_{\{k\}}e^{ip^2\nu}\psi_{\{k\}}(x)\psi^*_{\{k\}}(y). \tag{2.30}$$

Using (2.24) and (2.30) we obtain an important property of eigenfunctions:

$$\int d^4x\, d^4y\, \psi^*_{\{k\}}(x) \tilde{Y}(x, y, \nu | A^{\text{ext}}, 0) \psi_{\{k'\}}(y) = e^{ip^2\nu} \delta_{\{k\},\{k'\}}. \tag{2.31}$$

Let us represent the sources $\xi^*(x)$ and $\xi(y)$ as a superposition of the eigenfunctions

$$\xi^*(x) = \sum_{\{k\}} \alpha^*_{\{k\}} \psi^*_{\{k\}}(x), \qquad \xi(x) = \sum_{\{k\}} \alpha_{\{k\}} \psi_{\{k\}}(x).$$

Then the expression for the generating functional $Z[\alpha^*, \alpha]$ (2.20) has the form

$$Z[\alpha^*, \alpha] = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{\substack{\{k_1\},\dots,\{k_n\} \\ \{k'_1\},\dots,\{k'_n\}}} \prod_{p=1}^{n} \alpha^*_{\{k_p\}} \alpha_{\{k'_p\}}$$

$$\times \int_0^{\infty} d\nu_p\, e^{-i(m^2 - i\varepsilon)\nu_p} \left\langle \left\langle \exp\left\{ \sum_{n_1, n_2 = 1}^{n} A_{n_1, n_2} \right\} \right\rangle \right\rangle_{k_1,\dots,k_n; k'_1,\dots,k'_n}, \tag{2.32}$$

where

$$\left\langle \left\langle A\left(\frac{\delta}{\delta u}\right) \right\rangle \right\rangle_{k_1,\dots,k_n; k'_1,\dots,k'_n} = \int d^4x_1 \dots d^4x_n\, d^4y_1 \dots d^4y_n$$

$$\times \prod_{l=1}^{n} \psi^*_{\{k_l\}}(x_l) A\left(\frac{\delta}{\delta u}\right) \prod_{p=1}^{n} \tilde{Y}(x_p, y_p, \nu_p | A^{\text{ext}}, u_p)|_{u=0} \psi_{\{k'_p\}}(y_p). \tag{2.33}$$

Expression (2.32) allows one to find the representation of Green functions in the basis of eigenfunctions.

## 3. Green functions in constant field

Consider the constant and uniform external field

$$A^{\text{ext}}_{\mu}(x) = -\tfrac{1}{2} F_{\mu\nu} x^{\nu}. \tag{3.1}$$

As can be seen from above the calculation of the generating functional and the Green functions reduces to the calculation of the functional integral (2.8), i.e. the function $\tilde{Y}(x, y, \nu|A^{\text{ext}}, u)$. In the constant field (1) the functional integral (2.8) reduces to the gaussian one. The function $Y(x, p, \nu|A^{\text{ext}}, u)$ was found in ref. [23]:

$$Y(x, p, \nu|A^{\text{ext}}, u) = [\det \operatorname{ch}(eF\nu)]^{-1/2}$$
$$\times \exp\left\{ i \iint_0^\nu d\nu' d\nu'' \pi^\mu(x, p, \nu') G_{\mu\sigma}(\nu', \nu''; \nu) \pi^\sigma(x, p, \nu'') \right\}, \tag{3.2}$$

where

$$\pi_\mu(x, p, \nu') = p_\mu + \tfrac{1}{2} eF_{\mu\nu} x^\nu + \tfrac{1}{2} u_\mu(\nu')$$
$$G(\nu', \nu''; \nu) = \delta(\nu' - \nu'') + eF \exp\{2eF(\nu' - \nu'')\}[\varepsilon(\nu' - \nu'') + \operatorname{th}(eF\nu)]$$
$$\varepsilon(\nu' - \nu'') = \begin{cases} 1, & \nu' > \nu'' \\ 0, & \nu' = \nu'' \\ -1, & \nu' < \nu'' \end{cases}. \tag{3.3}$$

Thus, the integral over $p_\mu$ in (2.10) also reduces to the gaussian form and we find

$$\tilde{Y}(x, y, \nu|A^{\text{ext}}, u) = \frac{i}{(4\pi)^2} \exp\left\{ \tfrac{1}{2} iexFy - i(x-y) L(\nu)(x-y) - \tfrac{1}{2} \operatorname{Sp} \ln \frac{\operatorname{sh}(eF\nu)}{eF} \right\} K(x-y, \nu|u),$$

$$L(\nu) = \tfrac{1}{4} eF \operatorname{cth}(eF\nu),$$

$$K(x-y, \nu|u) = \exp\left\{ \tfrac{1}{4} i \iint_0^\nu d\nu' d\nu'' u^\mu(\nu') \tilde{G}_{\mu\gamma}(\nu', \nu''; \nu) u^\gamma(\nu'') + \tfrac{1}{2} i(x-y) \frac{eF \exp(eF\nu)}{\operatorname{sh}(eF\nu)} \int_0^\nu \exp(-2eF\nu') u(\nu') d\nu' \right\},$$

$$\tilde{G}(\nu', \nu''; \nu) = G(\nu', \nu''; \nu) - \frac{2eF}{\operatorname{sh}(eF\nu)} \exp\{2eF(\nu' - \nu'')\}. \tag{3.4}$$

Substituting expression (3.4) into (2.21) we find the quantum Green function of the scalar particle:

$$G(x,y)=\frac{i}{(4\pi)^2}\int_0^\infty \mathrm{d}\nu$$

$$\times\exp\left\{-i(m^2-i\varepsilon)\nu+A_{1,1}-\int \mathrm{d}^4z\int_{s_0}^\infty\frac{\mathrm{d}s}{s}\mathrm{e}^{-i(m^2-i\varepsilon)s}\frac{i}{(4\pi s)^2}\right\}$$

$$\times\left(1+\sum_{r=1}^\infty\frac{1}{r!}\prod_{p=1}^r\int \mathrm{d}^4z_p\int_{s_0}^\infty\frac{\mathrm{d}s_p}{s_p}\mathrm{e}^{-i(m^2-i\varepsilon)s_p}\right.$$

$$\times\exp\left\{\sum_{\alpha=1}^r C_{1\alpha}+\sum_{\alpha,\alpha_1=1}^r A_{1\alpha,\alpha_1}\right\}\frac{i^r}{(4\pi)^{2r}}\prod_{a=1}^r$$

$$\left.\left.\times\exp\left\{-\tfrac{1}{2}\operatorname{Sp}\ln\frac{\operatorname{sh}(eFs_a)}{eF}+\tfrac{1}{4}i\iint_0^{s_a}\mathrm{d}s'\,\mathrm{d}s''\,v_a^\mu(s')\tilde{G}_{\mu\gamma}(s',s'';s_a)v_a^\gamma(s'')\right\}\right)\right|_{v=0}$$

$$\times\exp\left\{\tfrac{1}{2}iexFy-i(x-y)L(v)(x-y)-\tfrac{1}{2}\operatorname{Sp}\ln\frac{\operatorname{sh}(eF\nu)}{eF}\right.$$

$$+\tfrac{1}{4}i\iint_0^\nu \mathrm{d}\nu'\,\mathrm{d}\nu''u^\mu(\nu')\tilde{G}_{\mu\gamma}(\nu',\nu'';\nu)u^\gamma(\nu'')+\tfrac{1}{2}i(x-y)\frac{eF\mathrm{e}^{eF\nu}}{\operatorname{sh}(eF\nu)}$$

$$\left.\left.\times\int_0^\nu \mathrm{e}^{-2eF\nu'}u(\nu')\,\mathrm{d}\nu'\right\}\right|_{u=0}. \tag{3.5}$$

If we perform the substitution $z_p+x=z'_p$ in the integral over $z_p$ in (3.5) it is not difficult to see that the Green function $G(x,y)$ has the following structure:

$$G(x,y)=\exp\{\tfrac{1}{2}iexFy\}G_1(x-y), \tag{3.6}$$

where $G_1(x-y)$ is a scalar function of the difference $x-y$ and of the constant tensor $F$. Hence it follows that $G_1(x-y)$ may depend only upon arguments of the form $(x-y)K(F)(x-y)$, where $K(F)$ is an even function of the field.

The eigenfunctions of the problem (2.23) in the external field (3.1) can be found in a covariant (see appendix) form:

$$\psi_{\{k\}}(x)=N_{\{k\}}\exp\{ie\phi(x)-ip_1(\underset{\sim}{n}\cdot x)-ip_2(\underset{\sim}{m}\cdot x)-\tfrac{1}{2}\rho^2\}$$

$$\times H_n(\rho)D_\nu[\omega\,\mathrm{e}^{i\pi/4}\tau], \tag{3.7}$$

where $\{k\} = \{p_1, p_2, n, p^2, \omega\}$, $\omega = \pm 1$, $n = 0, 1, 2, \dots$; the numbers $p_1, p_2$ are eigenvalues of the operators $i\partial/\partial(n_-x)$ and $i\partial/\partial(m_-x)$ respectively, $\nu = -\frac{1}{2}(1 + i\lambda)$, $\lambda = [p^2 + |e|\mathcal{H}(2n+1)]/|e|\mathcal{E}$, $\mathcal{E}, \mathcal{H} = [(\mathcal{F}^2 + \mathcal{G}^2)^{1/2} \mp \mathcal{F}]^{1/2}$; $\mathcal{G}, \mathcal{F}$ are the invariants of the field (which we suppose are not zero), $\rho = (|e|\mathcal{H})^{1/2}((m_+x) + p_2/e\mathcal{H})$, $\tau = (2|e|\mathcal{E})^{1/2}(n_+x) + p_1/e\mathcal{E})$, $\phi(x) = -\frac{1}{2}\mathcal{E}(n_+x)(n_-x) - \frac{1}{2}\mathcal{H}(m_+x)(m_-x)$, $H_n(\rho)$ are Hermite polynomials, $D_\nu[\omega e^{i(\pi/4)}\tau]$ are the parabolic cylinder functions, and $n_\pm$, $m_\pm$ are vectors which are solutions of the equations

$$Fn_\pm = \mathcal{E}n_\mp, \qquad Fm_\pm = \mp\mathcal{H}m_\mp, \qquad n_\pm^2 = \pm 1,$$

$$m_\pm^2 = -1, \qquad (n_+n_-) = (n_+m_\pm) = (n_-m_\pm) = (m_-m_+) = 0,$$

$$N_{\{k\}} = \exp\{\tfrac{1}{4}i\pi\nu\}\Gamma(-\nu)(\mathcal{H}/2\mathcal{E})^{1/4}(2^n n!\sqrt{\pi})^{-1/2}.$$

Using the integral representation for the parabolic cylinder functions, one may show that the eigenfunctions (3.7) form a complete orthonormal system of functions in the sense of (2.24) and (2.25). Note that the following important property of eigenfunctions (3.7) is valid:

$$\int d^4x\, d^4y\, \psi^*_{\{k\}}(x) \exp\{\tfrac{1}{2}iexFy - i(x-y)T(F)(x-y)\}\psi_{\{k'\}}(y) \sim \delta_{\{k\},\{k'\}}. \tag{3.8}$$

Relation (3.8) permits us to prove that eigenfunctions (3.7) diagonalize not only the Green function $G^{(0)}(x, y|A^{\text{ext}})$, but also the quantum Green function $G(x, y)$, eq. (3.5). Indeed, suppose that the function $G_1(x-y)$ can be expanded as a Fourier integral:

$$G_1(x-y) = \int dk_1 \dots dk_l f(k_1, \dots k_l) \exp\left\{-i(x-y)\sum_{j=1}^{l} k_j K_j(F)(x-y)\right\}. \tag{3.9}$$

Here $l$ is the number of the independent structure matrices $K(F)$ on which the function $G_1(x-y)$ may depend. Then we have

$$\int d^4x\, d^4y\, \psi^*_{\{k\}}(x) G(x, y) \psi_{\{k'\}}(y) = \int dk_1 \dots dk_l f(k_1, \dots, k_l)$$

$$\times \int d^4x\, d^4y\, \psi^*_{\{k\}}(x) \exp\left\{\tfrac{1}{2}iexFy - i(x-y)\sum_{j=1}^{l} k_j K_j(F)(x-y)\right\}\psi_{\{k'\}}(y). \tag{3.10}$$

Choosing in (3.8) $T(F)=\Sigma_{j=1}^{l} k_j K_j(F)$ we obtain

$$\int \mathrm{d}^4x\, \mathrm{d}^4y\, \psi^*_{\{k\}}(x) G(x, y) \psi_{\{k'\}}(y) \sim \delta_{\{k\},\{k'\}}. \tag{3.11}$$

Proceeding from this we can write

$$G(x, y) = \sum_{\omega=\pm 1} \sum_{n=0}^{\infty} \int \frac{\mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}p^2}{(2\pi)^4} \psi_{\{k\}}(x) G_{\{k\}} \psi^*_{\{k\}}(y), \tag{3.12}$$

$$G_{\{k\}} = \int \mathrm{d}^4x\, \mathrm{d}^4y\, \psi^*_{\{k\}}(x) G(x, y) \psi_{\{k\}}(y). \tag{3.13}$$

Neglecting the polarization effects we obtain for $G(x, y)$

$$G_{\{k\}} = \frac{-i}{(4\pi)^2} \int_0^{\infty} \mathrm{d}\nu\, \mathrm{e}^{-i(m^2 - i\varepsilon)\nu} \langle\langle \exp A_{1,1} \rangle\rangle_{k,k}. \tag{3.14}$$

Let us rewrite the definition of the average $\langle\langle\ \rangle\rangle_{k,k}$ in a more convenient, for our subsequent work, form:

$$\left\langle\left\langle A\left(\frac{\delta}{\delta u}\right)\right\rangle\right\rangle_{k,k} = \int \mathrm{d}^4x\, \mathrm{d}^4y\, \psi^*_{\{k\}}(x) \left\langle A\left(\frac{\delta}{\delta u}\right)\right\rangle_{x,y} \psi_{\{k\}}(y). \tag{3.15}$$

Relation (3.14) permits one to develop the modified perturbation theory for $G_{\{k\}}$. To obtain the first order of the modified perturbation theory let us expand $G_{\{k\}}$ into a series with respect to the radiative interaction. To within $e^2$ we have

$$G_{\{k\}} = \frac{-i}{(4\pi)^2} \int_0^{\infty} \mathrm{d}\nu\, \mathrm{e}^{-i(m^2 - i\varepsilon)} J_{\{k\}}(\nu) \big(1 + \langle\langle A_{1,1} \rangle\rangle_{k,k} / J_{\{k\}}(\nu)\big),$$

$$J_{\{k\}}(\nu) = \langle\langle 1 \rangle\rangle_{k,k}. \tag{3.16}$$

According to [16, 17, 20] the first order of the modified perturbation theory consists in the replacement

$$1 + \langle\langle A_{1,1} \rangle\rangle_{k,k} / J_{\{k\}}(\nu) \to \exp\{\langle\langle A_{1,1} \rangle\rangle_{k,k} / J_{\{k\}}(\nu)\}.$$

Thus in the first order of the modified perturbation theory one has

$$\langle\langle \exp A_{1,1} \rangle\rangle_{k,k} = J_{\{k\}}(\nu) \exp\{\langle\langle A_{1,1} \rangle\rangle_{k,k} / J_{\{k\}}(\nu)\}, \tag{3.17}$$

and for the Green functions in this approximation we obtain

$$G^{(1)}_{\{k\}} = \frac{-i}{(4\pi)^2}\int_0^\infty \mathrm{d}\nu\, J_{\{k\}}(\nu)\exp\left\{-i(m^2-i\varepsilon)\nu + \frac{\langle\langle A_{1,1}\rangle\rangle_{k,k}}{J_{\{k\}}(\nu)}\right\}. \tag{3.18}$$

Substituting the solutions (3.7) into (3.16) and using expression (A.25) we find

$$J_{\{k\}}(\nu) = (4\pi)^2\exp(ip^2\nu)\,\delta_{\{K\},\{K\}} \tag{3.19}$$

Finally we have

$$G^{(1)}_{\{k\}} = -i\int_0^\infty \mathrm{d}\nu\exp\left\{i(p^2-m^2+i\varepsilon)\nu + \frac{\langle\langle A_{1,1}\rangle\rangle_{k,k}}{J_{\{k\}}(\nu)}\right\}. \tag{3.20}$$

## 4. Calculation of $\langle\langle A_{1,1}\rangle\rangle_{k,k}$

Using the definition of $\langle\langle A_{1,1}\rangle\rangle_{k,k}$ from (3.15) we can write

$$\langle\langle A_{1,1}\rangle\rangle_{k,k} = \int \mathrm{d}^4x\,\mathrm{d}^4y\,\psi^*_{\{k\}}(x)\langle A_{1,1}\rangle_{x,y}\psi_{\{k\}}(y), \tag{4.1}$$

$$\langle A_{1,1}\rangle_{x,y} = -\frac{2e^2}{(4\pi)^2}\exp\left\{\tfrac{1}{2}iexFy - i(x-y)L(\nu)(x-y) - \tfrac{1}{2}\,\mathrm{Sp}\ln\frac{\mathrm{sh}(eF\nu)}{eF}\right\}A_{1,1}(x-y,\nu), \tag{4.2}$$

$$A_{1,1}(x-y,\nu) = \iint_0^\nu \mathrm{d}\nu'\,\mathrm{d}\nu''\,a(x-y,\nu',\nu'',\nu), \tag{4.3}$$

$$a(x-y,\nu',\nu'',\nu) = P^\mu(\nu')D_{\mu\nu}(x(\nu')-x(\nu''))P^\nu(\nu'')K(x-y,\nu|u)_{u=0}, \tag{4.4}$$

where $K(x-y,\nu|u)$ is determined by (3.4). To find (4.4) let us use the integral representation of the photon Green function

$$D_{\mu\nu}(x(\nu')-x(\nu'')) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}D_{\mu\nu}(k)\exp\left\{2i\int_0^\nu t(\lambda;\nu',\nu'')k_\alpha P^\alpha(\lambda)\,\mathrm{d}\lambda\right\},$$

$$t(\lambda;\nu',\nu'') = \theta(\lambda-\nu'') - \theta(\lambda-\nu'). \tag{4.5}$$

As a result we obtain

$$a(x-y,\nu',\nu'',\nu)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}D_{\mu\gamma}(k)\exp\{ik\mathfrak{L}k+2iqk\}$$

$$\times\left\{-\tfrac{1}{2}i\tilde{G}^{\mu\gamma}(\nu',\nu'';\nu)+\left[K_\beta f_1^{\gamma\beta}(\nu',\nu'',\nu)+f_3^{\gamma}(\nu'',\nu)\right]\right.$$

$$\left.\times\left[k_\alpha f_2^{\mu\alpha}(\nu',\nu'',\nu)+f_3^{\mu}(\nu',\nu)\right]\right\}, \qquad (4.6)$$

$$f_1(\nu',\nu'',\nu)=\int_0^\nu\tilde{G}(\nu'',\lambda;\nu)t(\lambda;\nu',\nu'')\,\mathrm{d}\lambda,$$

$$f_2(\nu',\nu'',\nu)=\int_0^\nu\tilde{G}(\nu',\lambda;\nu)t(\lambda;\nu',\nu'')\,\mathrm{d}\lambda,$$

$$f_3(\nu',\nu)=\tfrac{1}{2}(x-y)\frac{eF}{\mathrm{sh}(eF\nu)}\exp\{eF(\nu-2\nu')\}, \qquad (4.7)$$

$$\mathfrak{L}=\iint_0^\nu\mathrm{d}\lambda\,\mathrm{d}\lambda'\,t(\lambda;\nu',\nu'')\tilde{G}(\lambda,\lambda';\nu)t(\lambda';\nu',\nu''),$$

$$q=\int_0^\nu f_3(\lambda,\nu)t(\lambda;\nu',\nu'')\,\mathrm{d}\lambda.$$

The integrals over $\lambda$ and $\lambda'$ in (4.7) are not difficult to calculate.

$$\mathfrak{L}=\frac{\mathrm{sh}(eFs)\mathrm{sh}[eF(\nu-s)]}{eF\,\mathrm{sh}(eF\nu)}, \qquad s=|\nu'-\nu''|,$$

$$q=\tfrac{1}{4}(x-y)\frac{\mathrm{e}^{eF\nu}}{\mathrm{sh}(eF\nu)}\left[\mathrm{e}^{-2eF\nu'}-\mathrm{e}^{-2eF\nu''}\right],$$

$$f_1(\nu',\nu'',\nu)=\tfrac{1}{2}\varepsilon(\nu'-\nu'')+\tfrac{1}{2}\left(\mathrm{e}^{-2eF(\nu'-\nu'')}-1\right)\left(\varepsilon(\nu'-\nu'')+\mathrm{cth}(eF\nu)\right),$$

$$f_2(\nu',\nu'',\nu)=\tfrac{1}{2}\varepsilon(\nu'-\nu'')+\tfrac{1}{2}\left(\mathrm{e}^{2eF(\nu'-\nu'')}-1\right)\left(\varepsilon(\nu'-\nu'')-\mathrm{cth}(eF\nu)\right). \quad (4.8)$$

To integrate expression (4.6) over $K_\mu$ it is convenient to use the representation

(2.10), having rewritten it first in a more convenient form:

$$D_{\mu\nu}(k) = D^{d}_{\mu\nu}(k) + D^{\chi}_{\mu\nu}(k),$$

$$D^{d}_{\mu\nu}(k) = g_{\mu\nu}(k^2 + i\varepsilon)^{-1} = -ig_{\mu\nu}\int_{-it_0}^{\infty} e^{ik^2t}dt,$$

$$D^{\chi}_{\mu\nu}(k) = -k_\mu k_\nu (d_l - 1)\int_{-it_0}^{\infty} t\,e^{ik^2t}dt. \tag{4.9}$$

In accordance with this procedure it is also convenient to represent expression (6) in the form of two terms:

$$a(x-y, \nu', \nu'', \nu) = a^{d}(x-y, \nu', \nu'', \nu) + a^{\chi}(x-y, \nu', \nu'', \nu). \tag{4.10}$$

Then for $a^{d}(x-y, \nu', \nu'', \nu)$ and $a^{\chi}(x-y, \nu', \nu'', \nu)$ we obtain, respectively,

$$\begin{aligned} a^{d}(x-y, \nu', \nu'', \nu) = &-i\left[-\tfrac{1}{2}i\tilde{G}^{\mu}_{\mu}(\nu', \nu''; \nu) + f^{\mu}_{3}(\nu', \nu) f_{3\mu}(\nu'', \nu)\right] \\ &\times \int_{-it_0}^{\infty} I\,dt - i f^{\mu\beta}_{1}(\nu', \nu'', \nu) f^{\alpha}_{2\mu}(\nu', \nu'', \nu) \\ &\times \int_{-it_0}^{\infty} I_{\alpha\beta}\,dt - i\left(f^{\mu\beta}_{1}(\nu', \nu'', \nu) f_{3\mu}(\nu', \nu)\right. \\ &\left. + f^{\mu\beta}_{2}(\nu', \nu'', \nu) f_{3\mu}(\nu'', \nu)\right)\int_{-it_0}^{\infty} I_{\beta}\,dt, \end{aligned} \tag{4.11}$$

$$\begin{aligned} a^{\chi}(x-y, \nu', \nu'', \nu) = &-(d_l - 1)\Big\{\left[-\tfrac{1}{2}i\tilde{G}^{\mu\sigma}(\nu', \nu'', \nu) + f^{\mu}_{3}(\nu', \nu) f^{\sigma}_{3}(\nu'', \nu)\right] \\ &\times \int_{-it_0}^{\infty} I_{\mu\sigma}\,dt + f^{\sigma\beta}_{1}(\nu', \nu'', \nu) f^{\mu\alpha}_{2}(\nu', \nu'', \nu) \\ &\times \int_{-it_0}^{\infty} I_{\mu\sigma\alpha\beta}\,dt + \left(f^{\sigma\beta}_{1}(\nu', \nu'', \nu) f^{\mu}_{3}(\nu', \nu)\right. \\ &\left. + f^{\mu\beta}_{2}(\nu', \nu'', \nu) f^{\sigma}_{3}(\nu'', \nu)\right)\int_{-it_0}^{\infty} I_{\mu\sigma\beta}\,dt\Big\}, \end{aligned} \tag{4.12}$$

$$I_{\alpha_1\ldots\alpha_n} = \int \frac{d^4k}{(2\pi)^4} k_{\alpha_1}\ldots k_{\alpha_n} \exp\{ik\Lambda k + 2iqk\},$$

$$\Lambda = \mathfrak{L} + t. \tag{4.13}$$

All the integrals $I_{\alpha_1\ldots\alpha_n}$ involved in (4.11) and (4.12) are expressed by means of the integral $I$:

$$I=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\exp(ik\Lambda k+2iqk)=\frac{1}{(4\pi)^2}(\det\Lambda)^{-1/2}\exp(-iq\Lambda^{-1}q),\tag{4.14}$$

$$I_\alpha=-(\Lambda^{-1}q)_\alpha I,\qquad I_{\alpha\beta}=\left[\tfrac{1}{2}i\Lambda^{-1}_{\alpha\beta}+(\Lambda^{-1}q)_\alpha(\Lambda^{-1}q)_\beta\right]I,$$

$$I_{\mu\alpha\beta}=-\left[\tfrac{1}{2}i\left(\Lambda^{-1}_{\mu\alpha}(\Lambda^{-1}q)_\beta+\Lambda^{-1}_{\mu\beta}(\Lambda^{-1}q)_\alpha+\Lambda^{-1}_{\alpha\beta}(\Lambda^{-1}q)_\mu\right)\right.$$
$$\left.+(\Lambda^{-1}q)_\mu(\Lambda^{-1}q)_\alpha(\Lambda^{-1}q)_\beta\right]I,$$

$$I_{\mu\nu\alpha\beta}=\left\{-\tfrac{1}{4}\left(\Lambda^{-1}_{\mu\nu}\Lambda^{-1}_{\alpha\beta}+\Lambda^{-1}_{\mu\alpha}\Lambda^{-1}_{\nu\beta}+\Lambda^{-1}_{\mu\beta}\Lambda^{-1}_{\nu\alpha}\right)\right.$$
$$+\tfrac{1}{2}i\left(\Lambda^{-1}_{\mu\nu}(\Lambda^{-1}q)_\alpha(\Lambda^{-1}q)_\beta+\Lambda^{-1}_{\mu\alpha}(\Lambda^{-1}q)_\nu(\Lambda^{-1}q)_\beta\right.$$
$$+\Lambda^{-1}_{\mu\beta}(\Lambda^{-1}q)_\nu(\Lambda^{-1}q)_\alpha+\Lambda^{-1}_{\nu\alpha}(\Lambda^{-1}q)_\mu(\Lambda^{-1}q)_\beta+\Lambda^{-1}_{\nu\beta}(\Lambda^{-1}q)_\mu(\Lambda^{-1}q)_\alpha$$
$$\left.\left.+\Lambda^{-1}_{\alpha\beta}(\Lambda^{-1}q)_\mu(\Lambda^{-1}q)_\nu\right)+(\Lambda^{-1}q)_\mu(\Lambda^{-1}q)_\nu(\Lambda^{-1}q)_\alpha(\Lambda^{-1}q)_\beta\right\}I.\tag{4.15}$$

Substituting (4.15) into (4.11) and (4.12) we obtain after cumbersome calculations

$$a^d(x-y,\nu',\nu'',\nu)=(4\pi)^{-2}\int_{-it_0}^{\infty}(\det\Lambda)^{-1/2}\exp\left\{-i(x-y)\frac{\tau_1\Lambda^{-1}}{8\,\mathrm{sh}^2(eF\nu)}(x-y)\right\}$$
$$\times\left\{-2i\delta(s)+2^{-5}(x-y)\frac{\eta\Lambda^{-2}}{\mathrm{sh}^2(eF\nu)}(x-y)\right.$$
$$\left.+\tfrac{1}{8}i\,\mathrm{Sp}\Lambda^{-1}\left(1+4eFt\frac{\mathrm{ch}[eF(\nu-2s)]}{\mathrm{sh}(eF\nu)}\right)\right\}\mathrm{d}t,\tag{4.16}$$

$$a^\times(x-y,\nu',\nu'',\nu)=-i(4\pi)^{-2}(d_l-1)\int_{-it_0}^{\infty}t(\det\Lambda)^{-1/2}$$
$$\times\exp\left\{-i(x-y)\frac{\tau_1\Lambda^{-1}}{8\,\mathrm{sh}^2(eF\nu)}(x-y)\right\}$$

$$\times\Bigg\{(4t)^{-1}\delta(s)+i2^{-4}(x-y)\Bigg[\Lambda^{-2}\frac{eF\tau_1\operatorname{ch}[eF(\nu-2s)]}{\operatorname{sh}^3(eF\nu)}$$

$$+\Lambda^{-1}\frac{2e^2F^2\operatorname{ch}(2eFs)}{\operatorname{sh}^2(eF\nu)}-\Lambda^{-1}\frac{\tau_4\operatorname{sh}[eF(\nu-2s)]}{\operatorname{sh}^3(eF\nu)}\Bigg](x-y)$$

$$+2^{-4}\Bigg[\tfrac{1}{4}(x-y)\Lambda^{-2}\frac{\tau_4}{\operatorname{sh}^2(eF\nu)}(x-y)-i\operatorname{Sp}\Lambda^{-1}\frac{\operatorname{sh}[eF(\nu-2s)]}{\operatorname{sh}(eF\nu)}\Bigg]^2$$

$$+\tfrac{1}{8}\operatorname{Sp}\Bigg[\Lambda^{-1}\frac{2eF\operatorname{ch}[eF(\nu-2s)]}{\operatorname{sh}(eF\nu)}-\Lambda^{-2}\frac{\operatorname{sh}^2[eF(\nu-2s)]}{\operatorname{sh}^2(eF\nu)}\Bigg]\Bigg\}\,\mathrm{d}t, \tag{4.17}$$

$$\tau_1=\operatorname{ch}(2eFs)-1,\qquad \tau_2=2eFt+\operatorname{sh}(2eFs),$$

$$\tau_3=\operatorname{sh}(2eFs)+2eFt\operatorname{ch}(2eFs),\quad \tau_4=\tau_1+2eFt\operatorname{sh}(2eFs),$$

$$\eta=\tau_1+4eFt\tau_3,\qquad \Lambda=(2eF)^{-1}(\tau_2-\tau_1\operatorname{cth}(eF\nu)). \tag{4.18}$$

Note that $a^d$ and $a^\chi$ depend on $\nu'$ and $\nu''$ only in the combination $s=|\nu'-\nu''|$. Therefore, when integrating over $\nu'$ and $\nu''$ in (4.3) one has

$$\iint_0^\nu \mathrm{d}\nu'\,\mathrm{d}\nu'\,\phi(s)=2\int_0^\nu(\nu-s)\phi(s)\,\mathrm{d}s, \tag{4.19}$$

where $\phi(s)$ is an arbitrary function. Thus

$$A_{1,1}(x-y,\nu)=2\int_0^\nu(\nu-s)a(x-y,\nu,s)\,\mathrm{d}s. \tag{4.20}$$

If in accordance with the breaking up of the photon function $D_{\mu\nu}(k)$ into the parts (9) one represents analogously $A_{1,1}(x-y,\nu)$ in the form of the sum of two terms, then $A^\chi_{1,1}(x-y,\nu)$ can be considerably simplified. For doing this note the identity

$$2i\frac{\partial I}{\partial s}=\Bigg\{\tfrac{1}{4}(x-y)\Lambda^{-2}\frac{\tau_4}{\operatorname{sh}^2(eF\nu)}(x-y)-i\operatorname{Sp}\Lambda^{-1}\frac{\operatorname{sh}[eF(\nu-2s)]}{\operatorname{sh}(eF\nu)}\Bigg\}I. \tag{4.21}$$

Using this relation and integrating over $s$ by parts, the expression for $A^\chi_{1,1}(x-y,\nu)$

may be represented in the form

$$A^{\chi}_{1,1}(x-y,\nu)=\tfrac{1}{2}i(4\pi)^{-2}(d_l-1)\int_{-it_0}^{\infty}\frac{\mathrm{d}t}{t}\left[\exp\left\{-i\frac{(x-y)^2}{4t}\right\}-1\right]. \tag{4.22}$$

Finally we obtain

$$\begin{aligned}A_{1,1}(x-y,\nu)={}&i\nu(8t_0)^{-1}+\tfrac{1}{2}i(4\pi)^{-2}\int_0^{\nu}(\nu-s)\,\mathrm{d}s\int_{-it_0}^{\infty}(\det\Lambda)^{-1/2}\\&\times\exp\left\{-i(x-y)\frac{\tau_1\Lambda^{-1}}{8\,\mathrm{sh}^2(eF\nu)}(x-y)\right\}\\&\times\left\{(x-y)\Lambda^{-2}\frac{\eta}{\mathrm{sh}^2(eF\nu)}(x-y)\right.\\&\left.+4i\,\mathrm{Sp}\Lambda^{-1}\left(1+4eFt\frac{\mathrm{ch}[eF(\nu-2s)]}{\mathrm{sh}(eF\nu)}\right)\right\}\\&+\tfrac{1}{2}i(4\pi)^{-2}(d_l-1)\int_{-it_0}^{\infty}\frac{\mathrm{d}t}{t}\left(\mathrm{e}^{-i(x-y)^2/4t}-1\right).\end{aligned} \tag{4.23}$$

Substitute now (4.23) into (4.1). The result of this substitution is written in the form

$$\begin{aligned}\langle\langle A_{1,1}\rangle\rangle_{k,k}={}&\left\{-i\frac{\alpha}{\pi}\nu t_0^{-1}J_{k,k}(L)+\frac{i\alpha}{2\pi}\int_0^{\nu}(\nu-s)\,\mathrm{d}s\int_{-it_0}^{\infty}(\det\Lambda)^{-1/2}\,dt\right.\\&\times\left[K^{\alpha\beta}J_{k,k;\alpha\beta}\left(L+\frac{\tau_1\Lambda^{-1}}{8\,\mathrm{sh}^2(eF\nu)}\right)\right.\\&+4i\,\mathrm{Sp}\Lambda^{-1}\left(1+4eFt\frac{\mathrm{ch}[eF(\nu-2s)]}{\mathrm{sh}(eF\nu)}\right)\\&\left.\times J_{k,k}\left(L+\frac{\tau_1\Lambda^{-1}}{8\,\mathrm{sh}^2(eF\nu)}\right)\right]-\frac{\alpha}{4\pi}(d_l-1)\\&\left.\times\int_{-it_0}^{\infty}\frac{\mathrm{d}t}{t}\left[J_{k,k}\left(L+\frac{1}{4t}\right)-J_{k,k}(L)\right]\right\}\left(\det\frac{\mathrm{sh}(eF\nu)}{eF}\right)^{-1/2},\\\alpha={}&\frac{e^2}{4\pi},\qquad K=\Lambda^{-2}\frac{\eta}{\mathrm{sh}^2 eF\nu}\end{aligned} \tag{4.24}$$

where

$$J_{k,k'}(\Omega)=\int \mathrm{d}^4x\,\mathrm{d}^4y\,\psi^*_{\{k\}}(x)\exp\{\tfrac{1}{2}iexFy-i(x-y)\Omega(F)(x-y)\}\,\psi_{\{k'\}}(y), \tag{4.25}$$

$$J_{k,k';\alpha\beta}(\Omega)=\int \mathrm{d}^4x\,\mathrm{d}^4y\,(x-y)_\alpha(x-y)_\beta\psi^*_{\{k\}}(x)$$
$$\times\exp\{\tfrac{1}{2}iexFy-i(x-y)\Omega(F)(x-y)\}\,\psi_{\{k'\}}(y). \tag{4.26}$$

Differentiating (4.25) over $\Omega^{\alpha\beta}$ one can find the connection between these two integrals:

$$J_{k,k';\alpha\beta}(\Omega)=i\frac{\partial J_{k,k'}(\Omega)}{\partial\Omega^{\alpha\beta}}. \tag{4.27}$$

Making use of the integral representations for the parabolic cylinder functions and Hermite polynomials [24], the integrals (4.25) can be calculated to give

$$J_{k,k'}(\Omega)=(4\pi)^2\left[\det(16\Omega^2-e^2F^2)\right]^{-1/4}\exp\{-ip\chi(F)p\}\,\delta_{\{k\},\{k'\}},$$
$$\chi(F)=(2eF)^{-1}\ln\frac{4\Omega-eF}{4\Omega+eF}, \tag{4.28}$$

and $p_\mu$ is a four-vector:

$$p_\mu p^\mu=p^2,\qquad p_\mu p^\mu=p_\parallel^2+p_\perp^2,$$
$$p_\parallel^2=pPp=p^2+|e|\mathcal{H}(2n+1),\qquad p_\perp^2=pLp=-|e|\mathcal{H}(2n+1),$$
$$P_{\mu\nu}=(\mathcal{H}^2+\mathcal{E}^2)^{-1}\left(F_{\mu\nu}^2+g_{\mu\nu}\mathcal{H}^2\right),\quad L_{\mu\nu}=(\mathcal{H}^2+\mathcal{E}^2)^{-1}\left(g_{\mu\nu}\mathcal{E}^2-F_{\mu\nu}^2\right).$$

The integral $J_{k,k';\alpha\beta}(\Omega)$ enters into (4.24) together with the matrix $K=\Lambda^{-2}\eta/\mathrm{sh}^2(eF\nu)$. Differentiating (4.28) over $\Omega^{\alpha\beta}$ we obtain

$$K^{\alpha\beta}J_{k,k';\alpha\beta}(\Omega)=4J_{k,k'}(\Omega)\left\{p\frac{K}{16\Omega^2-e^2F^2}p-i\,\mathrm{Sp}\,\frac{2K\Omega}{16\Omega^2-e^2F^2}\right\}. \tag{4.29}$$

From (4.27) we have

$$J_{k,k'}(L) = (4\pi)^2 e^{ip^2\nu}\delta_{\{k\},\{k'\}}\left(\det\frac{\operatorname{sh} eF\nu}{eF}\right)^{1/2}$$

$$J_{k,k'}\left(L + \frac{\tau_1\Lambda^{-1}}{8\operatorname{sh}^2(eF\nu)}\right) = J_{k,k'}(L)(\det\Lambda)^{1/2}\left(\det\frac{\tau_2^2-\tau_1^2}{4e^2F^2}\right)^{-1/4}$$

$$\times\exp\{-ip\psi(s,t)p\},$$

$$\psi(s,t) = (2eF)^{-1}\ln\frac{\tau_2+\tau_1}{\tau_2-\tau_1},$$

$$J_{k,k'}\left(L+\frac{1}{4t}\right) = J_{k,k}(L)\left(\det\frac{\tau_2^2-\tau_1^2}{4e^2F^2}\right)^{-1/4}_{s=\nu}\exp\{-ip\psi(\nu,t)p\}.$$

$$\left(\det\frac{\operatorname{sh} eF\nu}{eF}\right)^{1/2} \tag{4.30}$$

Note that $J_{k,k}(L) = J_{\{k\}}(\nu)$. Substituting (4.29), (4.30) into (4.24) we get

$$\begin{aligned}\frac{\langle\langle A_{1,1}\rangle\rangle_{k,k}}{J_{\{k\}}(\nu)} &= -i\frac{\alpha}{\pi}\nu t_0^{-1} - i\frac{\alpha}{2\pi}\int_0^\nu(\nu-s)\,ds\int_{-it_0}^\infty M(s,t;p,F)\,dt\\ &\quad+\frac{\alpha}{4\pi}(d_l-1)\int_{-it_0}^\infty t^{-1}\left[t^2g(\nu,t;p,F)-1\right]dt\\ &\equiv b_1(\nu;p,F),\end{aligned} \tag{4.31}$$

$$M(s,t;p,F) = g(s,t;p,F)\left\{p\frac{\eta}{\tau_2^2-\tau_1^2}p + \tfrac{1}{2}i\operatorname{Sp}\frac{R}{\tau_2^2-\tau_1^2}\right\},$$

$$R = eF\left[2eFt(3-2\operatorname{ch}(2eFs)) + (1-8e^2F^2t^2)\operatorname{sh}(2eFs)\right],$$

$$g(s,t;p,F) = \left[\det\left(\frac{\tau_2^2-\tau_1^2}{4e^2F^2}\right)\right]^{-1/4}\exp\{-ip\psi(s,t)p\}. \tag{4.32}$$

This expression diverges when $t_0\to 0$. Before proceeding to the renormalization consider the case $F=0$. In this limit

$$M(s,t;p,0) = g(s,t;p,0)\left\{p^2\frac{(s+2t)^2}{2(s+t)^2} + \frac{i}{2(s+t)}\right\},$$

$$g(s,t;p,0) = \frac{\exp\{-ip^2s^2/(s+t)\}}{(s+t)^2}.$$

Thus

$$\frac{\langle\langle A_{1,1}\rangle\rangle_{k,k}}{J_{\{k\}}(\nu)} = -i\frac{3\alpha}{4\pi}\nu t_0^{-1} - i\frac{\alpha}{2\pi}\int_0^\nu (\nu - s)\,\mathrm{d}s \int_{-it_0}^\infty \mathrm{e}^{-ip^2s^2/(s+t)}$$

$$\times p^2\frac{s+2t}{(t+s)^3}\,\mathrm{d}t - \frac{\alpha}{4\pi}\int_0^\nu \mathrm{d}s \int_{-it_0}^\infty \frac{\mathrm{e}^{-ip^2s^2/(s+t)}}{(t+s)^2}\,\mathrm{d}t$$

$$+\frac{\alpha}{4\pi}(d_l - 1)\int_{-it_0}^\infty t^{-1}\left(\frac{t^2}{(t+\nu)^2}\,\mathrm{e}^{-ip^2\nu^2/(t+\nu)} - 1\right)\mathrm{d}t$$

$$\equiv b_1(\nu; p), \tag{4.33}$$

which coincides with the results of [20]. Consider the Green function

$$G^{(1)}_{\{k\}}\big|_{F=0} = -i\int_0^\infty \mathrm{d}\nu \exp\{i(p^2 - m^2 + i\varepsilon)\nu + b_1(\nu; p)\}. \tag{4.34}$$

Let us separate the terms linear in $\nu$ from $b_1(\nu; p)$ and rewrite (4.33) in the following form:

$$b_1(\nu, p) = -i\nu M_0(p) + N_0(\nu, p), \tag{4.35}$$

where

$$M_0(p) = \frac{3\alpha}{4\pi}t_0^{-1} + \frac{\alpha}{2\pi}\int_0^\infty \mathrm{d}s \int_{-it_0}^\infty M(s, t; p, 0)\,\mathrm{d}t,$$

$$N_0(\nu, p) = i\frac{\alpha}{2\pi}\int_0^\nu s\,\mathrm{d}s \int_{-it_0}^\infty M(s, t; p, 0)\,\mathrm{d}t + i\frac{\alpha}{2\pi}\int_\nu^\infty \mathrm{d}s \int_0^\infty M(s, t; p, 0)\,\mathrm{d}t$$

$$-\frac{\alpha}{4\pi}\int_0^\nu \mathrm{d}s \int_{-it_0}^\infty g(s, t; p, 0)\,\mathrm{d}t$$

$$+\frac{\alpha}{4\pi}(d_l - 1)\int_{-it_0}^\infty \frac{\mathrm{d}t}{t}\left[t^2 g(\nu, t; p, 0) - 1\right]. \tag{4.37}$$

Note that if we calculate the mass operator $\overline{M}_0(p^2, m^2)$ of a scalar particle in the $e^2$ approximation and make the substitution $m^2 \to p^2$ then we get just $M_0(p)$:

$$M_0(p) = \overline{M}_0(p^2, m^2 \to p^2). \tag{4.38}$$

Therefore, the renormalization of $M_0(p)$ may be performed in a standard way by choosing the normalization point as $p^2 = m_e^2$ where $m_e^2$ is the experimental mass squared [20]. To do this let us represent $M_0(p)$ in the form

$$M_0(p) = M_0(p^2 = m_e^2) + (p^2 - m_e^2) \left.\frac{\partial M_0(p)}{\partial p^2}\right|_{p^2 = m_e^2} + M_{0R}(p), \quad (4.39)$$

where $M_{0R}(p)$ is the renormalized mass operator. Let us make in (4.34) the substitution

$$\nu = z_2 \nu', \quad (4.40)$$

where $z_2$ is the renormalization constant of the wave function, and $\delta m^2$ is the mass renormalization

$$\delta m^2 = m_e^2 - m^2. \quad (4.41)$$

Then

$$G_{\{k\}}^{(1)}\Big|_{F=0} = -iz_2 \int_0^\infty d\nu \exp\Bigg\{ i(p^2 - m_e^2)(z_2 - 1)\nu + i\delta m^2 \nu$$

$$- i\nu M_0(p^2 = m_e^2) - i\nu \left.\frac{\partial M_0(p)}{\partial p^2}\right|_{p^2 = m_e^2} (p^2 - m_e^2)$$

$$+ i(p^2 - m_e^2)\nu - i\nu M_{0R}(p) + N_0(\nu; p)\Bigg\}. \quad (4.42)$$

The constants $z_2$ and $\delta m^2$ are chosen from the conditions

$$\delta m^2 = M_0(p^2 = m_e^2), \quad (4.43)$$

$$z_2 - 1 = \left.\frac{\partial M_0(p)}{\partial p^2}\right|_{p^2 = m_e^2}. \quad (4.44)$$

The integrals entering into (4.43) and (4.44) are easily calculated:

$$\delta m^2 = \frac{3\alpha}{4\pi} m_e^2 \left[ (m_e^2 t_0)^{-1} + \tfrac{3}{2} + \ln(\gamma m_e^2 t_0)^{-1} \right], \quad (4.45)$$

$$z_2 - 1 = \frac{3\alpha}{4\pi} \left[ \tfrac{1}{2} + \ln(\gamma m_e^2 t_0)^{-1} \right], \quad (4.46)$$

where $\gamma$ is Euler's constant. The function $N_0(\nu, p)$, eq. (4.37), is equal to

$$N_0(\nu, p) = -\frac{\alpha}{4\pi} d_l \ln(\gamma m_e^2 t_0)^{-1} + \frac{\alpha}{2\pi}(3 - d_l) \ln(m_e^2 \nu) + a(\nu, p)$$

$$a(\nu, p) \to 0, \qquad \nu \to \infty. \quad (4.47)$$

The diverging term $-(\alpha/4\pi)d_l\ln(\gamma m_e^2 t_0)^{-1}$ should also be related to the $z_2$ factor:

$$\tilde{z}_2 = z_2 \exp\left(-\frac{\alpha}{4\pi} d_l \ln\left(\gamma m_e^2 t_0\right)^{-1}\right).$$

In the $e^2$ approximation we have

$$\tilde{z}_2 - 1 = \frac{\alpha}{4\pi}\left[(3-d_l)\ln\left(\gamma m_e^2 t_0\right)^{-1} + \tfrac{1}{2}\right]. \tag{4.48}$$

Let us return now to expression (4.31) and rewrite $b_1(\nu, p, F)$ in the following form:

$$b_1(\nu, p, F) = -i\frac{\alpha}{\pi}\nu t_0^{-1} - i\frac{\alpha}{2\pi}\nu\int_0^\infty \mathrm{d}s \int_{-it_0}^\infty M(s, t; p, F)\,\mathrm{d}t$$
$$+ N_0(\nu, p) + N(\nu, p, F) + i\frac{\alpha}{2\pi}\nu\int_\nu^\infty \mathrm{d}s\int_0^\infty M(s, t; p, F)\,\mathrm{d}t, \tag{4.49}$$

$$N(\nu, p, F) = i\frac{\alpha}{2\pi}\int_0^\nu s\,\mathrm{d}s\int_0^\infty M_{1\mathrm{R}}(s, t; p, F)\,\mathrm{d}t$$
$$+ \frac{\alpha}{4\pi}(d_l - 1)\int_0^\infty t\left[g(\nu, t; p, F) - g(\nu, t; p, 0)\right]\mathrm{d}t,$$
$$M_{1\mathrm{R}}(s, t; p, F) = M(s, t; p, F) - M(s, t; p, 0), \tag{4.50}$$

where $N_0(\nu, p)$ is determined by (4.37). In the last term in (4.49) and (4.50) we have put the lower proper-time-integration limit equal to zero everywhere; on removing the regularization they are finite. One can verify this fact by expanding the integrals into power series with respect to the field and calculating the corresponding integrals. Consider the first two terms in (4.49) linear in $\nu$. Let us write $M(s, t; p, F)$ in the form

$$M(s, t; p, F) = M\left(s, t; p^2 = m_e^2, 0\right) + \left(p^2 - m_e^2\right)\left.\frac{\partial M(s, t; p, 0)}{\partial p^2}\right|_{p^2 = m_e^2}$$
$$+ M_\mathrm{R}(s, t; p, F),$$
$$M_\mathrm{R}(s, t; p, F) = M(s, t; p, F) - \exp\left(-im_e^2\frac{s^2}{t+s}\right)$$
$$\times (t+s)^{-3}\left\{m_e^2\frac{(s+2t)^2}{2(s+t)} + i\right\} - \exp\left\{-im_e^2\frac{s^2}{t+s}\right\}$$
$$\times\left[\frac{(s+t)^2 + t^2}{t+s} - im_e^2 s^2 (s+2t)^2 (s+t)^{-2}\right]. \tag{4.51}$$

Then the integrals over $s$ and $t$ in $M_\mathrm{R}(s, t; p, F)$ are finite when $t_0 \to 0$ and we obtain for $b_1(\nu; p, F)$

$$b_1(\nu; p, F) = -i\nu\delta m^2 - i(p^2 - m_e^2)(z_2 - 1)\nu - i\nu M_R(p, F) + i\frac{\alpha}{2\pi}\nu\int_\nu^\infty ds \int_0^\infty M(s, t; p, F)\,dt + i\frac{\alpha}{2\pi}\int_0^\nu s\,ds\int_0^\infty M_{1R}(s, t; p, F)\,dt + N_0(\nu, p) + \frac{\alpha}{4\pi}(d_l - 1)\int_0^\infty t g_R(\nu, t; p, F)\,dt, \tag{4.52}$$

$$M_R(p, F) = \frac{\alpha}{2\pi}\int_0^\infty ds\int_0^\infty M_R(s, t; p, F)\,dt,$$

$$g_R(s, t; p, F) = g(s, t; p, F) - g(s, t; p, 0), \tag{4.53}$$

where $\delta m^2$ and $z_2 - 1$ are determined by (4.43) and (4.44). Thus we see that the procedure of renormalization in the presence of a constant field is the same as in the vacuum ($F = 0$) and the constants of renormalization do not depend upon the external field.

If we write the Green function $G^{(1)}_{\{k\}}$ in the form

$$G^{(1)}_{\{k\}} = -i\int_0^\infty d\nu \exp\{i(p^2 - m^2 + i\varepsilon)\nu + b_1(\nu, p, F)\}, \tag{4.54}$$

and make the change of variables $\nu = z_2\nu'$ and the mass renormalization $m^2 = m_e^2 - \delta m^2$, we obtain

$$G^{(1)}_{\{k\}} = \tilde{z}_2 G^{(1)}_{\{k\}R}, \tag{4.55}$$

where $G^{(1)}_{\{k\}R}$ is the renormalized Green function

$$G^{(1)}_{\{k\}R} = -i\int_0^\infty d\nu \exp\{i(p^2 - m_e^2 + i\varepsilon)\nu + b_{1R}(\nu, p, F)\}, \tag{4.56}$$

$$b_{1R}(\nu, p, F) = -i\nu M_R(p, F) + N_R(\nu, p, F) + i\frac{\alpha}{2\pi}\nu\int_\nu^\infty ds\int_0^\infty M(s, t; p, F)\,dt,$$

$$N_R(\nu, p, F) = N(\nu, p, F) + N_{0R}(\nu, p),$$

$$N_{0R}(\nu, p) = N_0(\nu, p) + \frac{\alpha}{4\pi}d_l \ln(\gamma m_e^2 t_0)^{-1}. \tag{4.57}$$

Note that if we calculate the mass operator in a constant field in the $e^2$ approximation and write it in the form

$$M(p, F) = \int_0^\infty ds\int_{-it_0}^\infty \overline{M}(s, t; p, F, m^2)\,dt, \tag{4.58}$$

then the function $M(s, t; p, F)$, eq. (4.32), is connected with $\overline{M}(s, t; p, F, m^2)$ in the following way:

$$M(s,t;p,F)=\overline{M}(s,t;p,F,m^2\to p^2). \tag{4.59}$$

Then our result for $M(s,t;p,F)$ coincides with that of ref. [8], i.e. $M(s,t;p,F)$ is the kernel of the mass operator in the $e^2$ approximation.

To find the infrared asymptotics of the Green function one should calculate the asymptotic behaviour of $b_{1R}(\nu;p,F)$ when $\nu\to\infty$. The terms linear in $\nu$ contribute to the position of the pole of the Green function. From (4.57) and (4.54) we see that the pole of the Green function is at the point

$$p^2-m_e^2-M_R(p,F)=0. \tag{4.60}$$

In the vacuum there occurs, besides, the enhancement of the pole near the point $p^2=m_e^2$ [16, 17, 20] owing to the fact that the function $N_{0R}(\nu;p)$ behaves logarithmically when $\nu\to\infty$, i.e.

$$\lim_{\nu\to\infty}N_{0R}(\nu,p)\sim\frac{\alpha}{2\pi}(3-d_l)\ln m_e^2\nu.$$

Let us find the asymptotic behaviour of $N_R(\nu;p,F)$ when $\nu\to\infty$. To do so we shall act in the following way: when $\nu\to\infty$ we write

$$N_R(\nu;p,F)=\frac{N_R(\nu;p,F)}{\ln(m_e^2\nu)}\ln m_e^2\nu$$

$$\underset{\nu\to\infty}{=}f(p,F)\ln(m_e^2\nu),$$

$$N_R(\nu,p,F)=N(\nu,p,F)+N_{0R}(\nu,p), \tag{4.61}$$

where

$$f(p,F)=\lim_{\nu\to\infty}\nu\frac{\partial N_R(\nu;p,F)}{\partial\nu}. \tag{4.62}$$

Substituting expressions (4.57) and (4.50) into (4.62) we obtain

$$f(p,F)=\lim_{\nu\to\infty}\left\{i\frac{\alpha}{2\pi}\nu^2\int_0^\infty M(\nu,t;p,F)\,\mathrm{d}t\right.$$

$$\left.+\frac{\alpha}{4\pi}(d_l-1)\nu\int_0^\infty t\,\partial g(\nu,t;p,F)/\partial\nu\,\mathrm{d}t\right\}. \tag{4.63}$$

Thus, the Green function $G^{(1)}_{\{k\}}$ near the point $p^2-m_e^2-M_R(p,F)=0$ has the form

$$G^{(1)}_{\{k\}}\sim\left[p^2-m_e^2-M_R(p,F)\right]^{-(1+f(p,F))}. \tag{4.64}$$

If $F=0$ then

$$f(p,0)=\frac{\alpha}{2\pi}(3-d_l), \tag{4.65}$$

and we obtain the well-known results [16,20]. If $F\neq 0$, the function $f(p,F)$ depends upon the parameters $p^2$, $\chi=-(eFp)^2m^{-6}$ and the invariants $\mathcal{F}$ and $\mathcal{G}$. As the function $f(p,F)$ is linear in the constant $\alpha$ it is necessary to substitute $p^2\to m_e^2$ near the pole (4.60), so as to remain within the accuracy.

Consider the crossed field ($\mathcal{F}=\mathcal{G}=0$). In this case $f(p,F)$ depends upon one parameter $\chi$ and we have

$$f(p,F)=\begin{cases}\dfrac{\alpha}{2\pi}(3-d_l), & \chi\ll 1\\[2mm] \dfrac{\alpha}{12\pi}(29+d_l), & \chi\gg 1\end{cases}. \tag{4.66}$$

For constant magnetic field ($\mathcal{G}=0,\mathcal{F}<0$) if $-\mathcal{F}/m^6e^2\ll 1$, $\chi\gg-\mathcal{F}/m^4e^2$, the results of the calculations for the function coincide with those for the crossed field.

## Appendix

Here we find the eigenfunctions of Klein-Gordon's equation in a constant and uniform external field $A_\mu^{\text{ext}}(x)=-\frac{1}{2}F_{\mu\nu}x^\nu$. These functions satisfy the equations

$$\pi_\mu^{\text{ext}}(x)\pi^{\text{ext}\,\mu}(x)\psi_{\{k\}}(x)=p^2\psi_{\{k\}}(x). \tag{A.1}$$

Consider isotropic orthogonal eigenvectors of the tensor $F_{\mu\nu}$:

$$F_{\mu\nu}n^\nu=\mathcal{E}n_\mu,\qquad F_{\mu\nu}\bar{n}^\nu=-\mathcal{E}\bar{n}_\mu,\qquad F_{\mu\nu}m^\nu=i\mathcal{H}m_\mu,$$

$$F_{\mu\nu}\bar{m}^\nu=-i\mathcal{H}\bar{m}_\mu,\qquad \mathcal{E},\mathcal{H}=\left[(\mathcal{F}^2+\mathcal{G}^2)^{1/2}\mp\mathcal{F}\right]^{1/2}. \tag{A.2}$$

The vectors (A.2) are normalized by the conditions $(n\bar{n})=-(m\bar{m})=2$. If we introduce the vectors $n_\pm=\frac{1}{2}(n\pm\bar{n})$, $m_+=\frac{1}{2}(m+\bar{m})$, $m_-=\frac{1}{2}i(\bar{m}-m)$ satisfying the equations

$$Fn_\pm=\mathcal{E}n_\mp,\qquad Fm_\pm=\mp\mathcal{H}m_\mp,\qquad n_\pm^2=\pm 1,$$

$$m_\pm^2=-1,\qquad (n_+n_-)=(n_\pm m_+)=(n_\pm m_-)=(m_-m_+)=0,$$

the tensor $F_{\mu\nu}$ and the field $A_\mu^{\text{ext}}(x)$ can be represented in the form

$$F_{\mu\nu}=\mathcal{E}(n_{-\mu}n_{+\nu}-n_{+\mu}n_{-\nu})+\mathcal{H}(m_{-\mu}m_{+\nu}-m_{+\mu}m_{-\nu}), \tag{A.3}$$

$$A_\mu^{\text{ext}}(x)=-\tfrac{1}{2}\mathcal{E}\left[n_{-\mu}(n_+x)-n_{+\mu}(n_-x)\right]-\tfrac{1}{2}\mathcal{H}\left[m_{-\mu}(m_+x)-m_{+\mu}(m_-x)\right]. \tag{A.4}$$

Let us perform the gauge transformation of the potential

$$A'^{\text{ext}}_\mu(x) = A^{\text{ext}}_\mu(x) + \partial_\mu \phi(x),$$
$$\phi(x) = -\tfrac{1}{2}\mathcal{E}(n_+x)(n_-x) - \tfrac{1}{2}\mathcal{H}(m_+x)(m_-x). \tag{A.5}$$

This transformation is due to the fact that in terms of the potential $A'^{\text{ext}}_\mu(x)$ it is possible to separate the variables in eq. (A.1). If we transform the functions in accord with the gauge transformation (4.5)

$$\psi_{\{k\}}(x) = \exp(ie\phi(x))\,\psi'_{\{k\}}(x), \tag{A.6}$$

then the functions $\psi'_{\{k\}}(x)$ satisfy the equations

$$\pi'^{\text{ext}}_\mu(x)\,\pi'^{\text{ext}\,\mu}(x)\,\psi'_{\{k\}}(x) = p^2\psi'_{\{k\}}(x),$$
$$\pi'^{\text{ext}}_\mu(x) = i\partial_\mu - eA'^{\text{ext}}_\mu(x). \tag{A.7}$$

Substitute (A.5) into (A.7). Then we have

$$\left[-\frac{\partial^2}{\partial(n_+x)^2} - \left(i\frac{\partial}{\partial(n_-x)} + e\mathcal{E}(n_+x)\right)^2 + \frac{\partial^2}{\partial(m_+x)^2} - \left(i\frac{\partial}{\partial(m_-x)} + e\mathcal{H}(m_+x)\right)^2\right]\psi'_{\{k\}}(x) = p^2\psi'_{\{k\}}(x). \tag{A.8}$$

Carrying out the separation of the variables in (A.8) we obtain

$$\psi'_{\{k\}}(x) = N_{\{k\}}\exp\{-ip_1(n_-x) - ip_2(m_-x) - \tfrac{1}{2}\rho^2\}H_n(\rho)D_\nu[\omega\,e^{i(\pi/4)}\tau],$$
$$\rho = (|e|\mathcal{H})^{1/2}[(m_+x) + p_2/e\mathcal{H}], \qquad \tau = (2|e|\mathcal{E})^{1/2}[(n_+x) + p_1/e\mathcal{E}],$$
$$\nu = -\tfrac{1}{2}(1 + i\lambda), \qquad \lambda = \frac{[p^2 + |e|\mathcal{H}(2n+1)]}{|e|\mathcal{E}}, \qquad n = 0,1,2,\dots, \qquad \omega = \pm 1. \tag{A.9}$$

$H_n(\rho)$ are the Hermite polynomials, $D_\nu[\omega\,e^{i(\pi/4)}\tau]$ are the parabolic cylinder functions, $p_1$ and $p_2$ are eigenvalues of the operator integrals of motion $i\partial/\partial(n_-x)$ and $i\partial/\partial(m_-x)$, respectively, $N_{\{k\}}$ is a normalizing factor. Let us verify that for the functions (A.9) the orthogonality condition (2.24) does take place and find in this way the normalizing factor. Using the orthogonality condition of the Hermite functions we have

$$(\psi_{\{k\}},\psi_{\{k'\}}) = N^*_{\{k\}}N_{\{k'\}}(2\pi)^2(2e^2\mathcal{E}\mathcal{H})^{-1/2}$$

$$\times 2^n n!\sqrt{\pi}\,\delta_{n,n'}\delta(p_1-p_1')\delta(p_2-p_2')T, \tag{A.10}$$

$$T = \int_{-\infty}^{+\infty} \mathrm{d}\tau\, D_{\nu^*}[\omega \mathrm{e}^{-i(\pi/4)}\tau]\, D_{\nu'}[\omega' \mathrm{e}^{i(\pi/4)}\tau]. \tag{A.11}$$

To calculate the integral $T$ let us make use of the integral representation for the parabolic cylinder functions [24]:

$$D_\nu[\omega\,\mathrm{e}^{i(\pi/4)}\tau] = \Gamma^{-1}(-\nu)\exp\{-\tfrac{1}{4}i\pi\nu - \tfrac{1}{4}i\tau^2\}\int_0^\infty t^{-\nu-1}\exp\{-\tfrac{1}{2}it^2 - it\tau\}\,\mathrm{d}t. \tag{A.12}$$

As a result we obtain

$$T = 2|e|\mathcal{E}(2\pi)^2[\Gamma(-\nu)\Gamma(-\nu^*)]^{-1}\exp\{\tfrac{1}{4}i\pi(\nu^*-\nu)\}\,\delta_{\omega,\omega'}\delta(p^2-p^{2\prime}). \tag{A.13}$$

If the normalizing factor $N_{\{k\}}$ is equal to

$$N_{\{k\}} = (2\pi)^{-2}\left(\frac{\mathcal{H}}{2\mathcal{E}}\right)^{1/4}(2^n n!\sqrt{\pi})^{-1/2}\Gamma(-\nu)\mathrm{e}^{i(\pi\nu/4)}, \tag{A.14}$$

the functions (A.9) are normalized by condition (2.24). Let us show that the functions (A.9) form a complete set in the sense of (2.25):

$$\sum_{n=0}^{\infty}\sum_{\omega=\pm1}\int\frac{\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p^2}{(2\pi)^4}\psi_{\{k\}}(x)\psi^*_{\{k\}}(y)$$

$$= \left(\frac{\mathcal{H}}{2\mathcal{E}}\right)^{1/2}\int \mathrm{d}p_1\,\mathrm{d}p_2\exp\{-ip_1(n_-z)-ip_2(m_-z)\}$$

$$\times\sum_{n=0}^{\infty}v_n(\rho)v_n(\rho')\int_{-\infty}^{+\infty}J_{p^2}\,\mathrm{d}p^2, \tag{A.15}$$

$$z = -x+y,$$

$$J_{p^2} = \mathrm{e}^{i(\pi/4)(\nu-\nu^*)}\Gamma(-\nu)\Gamma(-\nu^*)\sum_{\omega=\pm1}D_\nu[\omega\,\mathrm{e}^{i(\pi/4)}\tau]\,D_{\nu^*}[\omega\,\mathrm{e}^{-i(\pi/4)}\tau'],$$

$$\nu^* = -\nu-1,\qquad \tau' = (2|e|\mathcal{E})^{1/2}[(n_+y)+p_1/e\mathcal{E}], \tag{A.16}$$

$$\rho' = [|e|\mathcal{H}]^{1/2}[(m_+y)+p_2/e\mathcal{H}],$$

where $v_n(\rho) = (2^n n!\sqrt{\pi})^{-1/2}\exp(-\frac{1}{2}\rho^2)H_n(\rho)$ are the Hermite functions. Let us find $\int_{-\infty}^{+\infty} J_{p^2}\,\mathrm{d}p^2$. To do this let us use the identity $\Gamma(-\nu)\Gamma(-\nu^*) = -\pi/\sin(\pi\nu)$ and introduce the new variable $\nu = -\frac{1}{2} - \frac{1}{2}i[p^2 + |e|\mathcal{H}(2n+1)]/|e|\mathcal{E}$. Then

$$\int_{-\infty}^{+\infty} J_{p^2}\,\mathrm{d}p^2 = +i2|e|\mathcal{E}\int_{-1/2-i\infty}^{-1/2+i\infty} J_\nu\,\mathrm{d}\nu, \tag{A.17}$$

$$J_\nu = \pi\frac{\exp\{\frac{1}{2}i\pi(\nu+\frac{1}{2})\}}{\sin(\pi\nu)}\sum_{\omega=\pm 1} D_\nu[\omega\,\mathrm{e}^{i(\pi/4)}\tau]D_{-\nu-1}[\omega\,\mathrm{e}^{-i(\pi/4)}\tau']. \tag{A.18}$$

The integral over $\nu$ is calculated with the help of the Cherry formula [24]

$$\int_{c-i\infty}^{c+i\infty}\frac{\mathrm{e}^{i(\pi/2)(\nu+1/2)}}{\sin(\pi\nu)}\sum_{\omega=\pm 1} D_\nu[\omega\,\mathrm{e}^{i(\pi/4)}\tau]D_{-\nu-1}[\omega\,\mathrm{e}^{-i(\pi/4)}\tau']\mathrm{d}\nu$$
$$= -4\pi i\delta(\tau-\tau'), \qquad -1 < \operatorname{Re} c < 0. \tag{A.19}$$

Thus, choosing $c = -\frac{1}{2}$ in (17) we obtain

$$\int_{-\infty}^{+\infty} J_{p^2}\,\mathrm{d}p^2 = -(2|e|\mathcal{E})^{1/2}(2\pi)^2\delta(n_+z). \tag{A.20}$$

We see that this expression does not depend on the quantum numbers. Therefore one may integrate over $p_2$ and sum over $n$ in (A.15) using the condition of completeness of the Hermite functions:

$$\sum_{n=0}^{\infty} v_n(\rho)v_n(\rho') = \delta(\rho-\rho') = (|e|\mathcal{H})^{-1/2}\delta(m_+z). \tag{A.21}$$

After integrating over $p_1$ we obtain the condition of completeness (2.25):

$$\sum_{\{k\}}\psi_{\{k\}}(x)\psi^*_{\{k\}}(y) = \delta^{(4)}(x-y). \tag{A.22}$$

To calculate the integral

$$J_{k,k'}(\Omega) = \int \mathrm{d}^4x\,\mathrm{d}^4y\,\psi^*_{\{k\}}(x)\exp\{\tfrac{1}{2}iexFy - iz\Omega(F)z\}\psi_{\{k'\}}(y), \tag{A.23}$$

which occurs when calculating $\langle\langle A_{1,1}\rangle\rangle_{k,k'}$ let us use the method of refs. [12,13]. Using this method we obtain

$$J_{k,k'}(\Omega) = (4\pi)^2[\det(16\Omega^2 - e^2F^2)]^{-1/4}\exp\{-ip\chi(F)p\}\delta_{\{k\},\{k'\}},$$
$$\chi(F) = (2eF)^{-1}\ln\frac{4\Omega - eF}{4\Omega + eF}. \tag{A.24}$$

Proceeding from this expression we shall calculate the integral occurring in (2.24):

$$J_{k,k';\alpha\beta}(\Omega)=i\frac{\partial J_{k,k'}(\Omega)}{\partial\Omega^{\alpha\beta}}=iJ_{k,k'}(\Omega)$$

$$\times\frac{\partial}{\partial\Omega^{\alpha\beta}}\left\{-ip\chi(F)p-\tfrac{1}{4}\operatorname{Sp}\ln(16\Omega^2-e^2F^2)\right\}. \tag{A.25}$$

Let $\mathcal{F}(\Omega)$ be an arbitrary function of $\Omega$ which can be expanded as a series in $\Omega$:

$$\mathcal{F}_{\mu\nu}(\Omega)=\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{F}^{(n)}(0)\Omega^n_{\mu\nu}.$$

Let us find the derivative of $\mathcal{F}_{\mu\nu}(\Omega)$ with respect to $\Omega^{\alpha\beta}$:

$$\frac{\partial\mathcal{F}_{\mu\nu}(\Omega)}{\partial\Omega^{\alpha\beta}}=\sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{F}^{(n)}(0)\sum_{k=0}^{n-1}\Omega^k_{\mu\alpha}\Omega^{n-k-1}_{\beta\nu}. \tag{A.26}$$

Here we make use of the relation

$$\frac{\partial\Omega^n_{\mu\nu}}{\partial\Omega^{\alpha\beta}}=\sum_{k=0}^{n-1}\Omega^k_{\mu\alpha}\Omega^{n-k-1}_{\beta\nu}. \tag{A.27}$$

Expression (A.25) contains derivatives of two types: $\partial p\mathcal{F}(\Omega)p/\partial\Omega^{\alpha\beta}$ and $\partial\operatorname{Sp}\mathcal{F}(\Omega)/\partial\Omega^{\alpha\beta}$. Using (A.26) we have

$$\frac{\partial}{\partial\Omega^{\alpha\beta}}\operatorname{Sp}\mathcal{F}(\Omega)=\mathcal{F}'_{\beta\alpha}(\Omega),$$

$$\frac{\partial}{\partial\Omega^{\alpha\beta}}p\mathcal{F}(\Omega)p=\sum_{n=1}^{\infty}\frac{\mathcal{F}^{(n)}(0)}{n!}\sum_{k=0}^{n-1}(p\Omega^k)_\alpha(\Omega^{n-k-1}p)_\beta. \tag{A.28}$$

But the expressions (A.28) are included in (3.24) together with the factor $K=\Lambda^{-2}\eta/\operatorname{sh}^2(eF\nu)$. Then we obtain finally from (A.28)

$$K^{\alpha\beta}\frac{\partial}{\partial\Omega^{\alpha\beta}}\operatorname{Sp}\mathcal{F}(\Omega)=\operatorname{Sp}\left[K\mathcal{F}'(\Omega)\right],$$

$$K^{\alpha\beta}\frac{\partial}{\partial\Omega^{\alpha\beta}}p\mathcal{F}(\Omega)p=p\mathcal{F}'(\Omega)Kp. \tag{A.29}$$

Using relations (A.29) we have

$$K^{\alpha\beta}J_{k,k';\alpha\beta}(\Omega)=4J_{k,k'}(\Omega)\left[p\frac{K}{16\Omega^2-e^2F^2}p-i\operatorname{Sp}(2K\Omega)/(16\Omega^2-e^2F^2)\right]. \tag{A.30}$$

Substituting the expressions

$$\Omega(F) = L(\nu), \qquad \Omega(F) = L(\nu) + \frac{\tau_1 \Lambda^{-1}}{8\,\mathrm{sh}^2(eF\nu)}, \qquad \Omega(F) = L(\nu) + \frac{1}{4t},$$

instead of $\Omega(F)$ into (A.24) we obtain the relations (3.30).

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