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FORMAL PATH INTEGRAL FOR THEORIES WITH NONCANONICAL COMMUTATION RELATIONS

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For theories with noncanonical commutation relations, the generating functional (S-matrix) is constructed at a formal level, as a Hamiltonian path integral.

Introduction

When quantizing dynamical systems within the Hamiltonian formalism, it is usually assumed that the phase variables satisfy, at equal times, the canonical commutation relations. Bearing in mind that the parametrization of the phase space is rather arbitrary, we must note that this method depends on a choice of the special class of canonical coordinates. A natural step towards achieving a completely coordinate-reparametrization invariant version of the Hamiltonian formalism would be to consider the most general commutation relations, subjected to the only restriction that they should be selfconsistent. Simultaneously, an interesting possibility would appear to extend the main results, to be gained in the process, onto the phase space with nonzero inner curvature.

In the present paper, we are going to obtain a formal expression for the generating functional, or the S-matrix, as a reparametrization-invariant path integral in the phase space for non-degenerate theories with noncanonical commutation relations of the most general type.

Unfortunately, our way of derivation of the final result is in this paper of a rather intuitive character and is not, to this extent, completely satisfactorily viewed upon from the point of the genuine operator approach. We will start from the classical description with the Poisson bracket defined using an arbitrary simplectic metric, and construct the classical action producing correct Hamiltonian equations of motion. Next we will construct an invariant analog of the canonical Liouville measure to be used, finally, for defining the path integral.

It is true that our results require, in principle, a farther justification to be given in the next publication within the framework of consistent operator quantization. Nevertheless, the expression obtained in this paper for the generating functional (the S-matrix) can well be used in the field theory in its present form if interbred with the usual rules of invariant regularization. In this sense, the status of our result is the same as that of Refs. 1–5 as well as of that of every paper exploiting the Hamiltonian path integral as a primary means to solving the quantization problem.

1. Classical Action

In what follows, only the pure Bose case will be considered, for simplicity. At the final step, we shall point out the necessary modification of the result necessary to account for the fermion degree of freedom, if present.

Let there be a dynamical system with N degrees of freedom, described by the original Hamiltonian

$$H = H(\Gamma), \tag{1.1}$$

given as a function of $2N$ Bose variables

$$\Gamma^A, \qquad A = 1, \ldots, 2N. \tag{1.2}$$

Define, for any two functions $X(\Gamma)$ and $Y(\Gamma)$ of the phase variables (1.2), the Poisson bracket

$$\{X, Y\} \equiv (\partial_A X)\omega^{AB}(\partial_B Y) \tag{1.3}$$

with the antisymmetric tensor field

$$\omega^{AB}(\Gamma) = -\omega^{BA}(\Gamma) \tag{1.4}$$

defining the simplectic metric, so that

$$\{\Gamma^A, \Gamma^B\} = \omega^{AB}(\Gamma). \tag{1.5}$$

The requirement that the Jacobi relation for the bracket (1.3) should be obeyed irrespective of the functions involved imposes the following basic condition on the metric

$$\omega^{AD}\partial_D\omega^{BC} + \text{cycle}\,(A, B, C) = 0. \tag{1.6}$$

Assuming that the metric is nondegenerate, let us define its covariant components, ω_{AB}, as elements of the matrix, inverse to (1.4)

$$\omega_{AB}\omega^{BC} = \delta^C_A. \tag{1.7}$$

Then, from (1.6) one has

$$\partial_A \omega_{BC} + \text{cycle}\,(A, B, C) = 0. \tag{1.8}$$

This locally implies the representation

$$\omega_{AB} = \partial_A V_B - \partial_B V_A, \tag{1.9}$$

with V_A being a covariant vector field.

The metric (1.4) allows us to define the fields of a contravariant reper $h^A_a(\Gamma)$ and its inverse covariant reper $h^a_A(\Gamma)$

$$\omega^{AB} = h^A_a \omega^{ab}_{(0)} h^B_b, \qquad \omega_{AB} = h^a_A \omega^{(0)}_{ab} h^b_B, \tag{1.10}$$

$$h^A_a h^a_B = \delta^A{}_B, \qquad h^a_A h^A_b = \delta^a{}_b, \qquad \omega^{(0)}_{ab} \omega^{bc}_{(0)} = \delta_a{}^c. \tag{1.11}$$

Henceforth, the small Roman indices run the $2N$-dimensional tangent phase space with the constant canonical metric $\omega^{ab}_{(0)}$ and its inverse $\omega^{(0)}_{ab}$.

The reper field, in its turn, defines in a natural way the connection of absolute parallelism

$$\Delta^D_{CA} \equiv h^D_a \partial_C h^a_A, \tag{1.12}$$

together with the corresponding covariant derivatives

$$\nabla_C V_A \equiv \partial_C V_A - \Delta^D_{CA} V_D, \qquad \nabla_C V^A \equiv \partial_C V^A + \Delta^A_{CD} V^D. \tag{1.13}$$

Covariant derivatives of the reper fields are evidently zero

$$\nabla_C h^a_A = 0, \qquad \nabla_C h^A_a = 0. \tag{1.14}$$

The Riemann curvature corresponding to the connection (1.12) is identically zero

$$\partial_D \Delta^A_{BC} - \partial_B \Delta^A_{DC} + \Delta^A_{DE} \Delta^E_{BC} - \Delta^A_{BE} \Delta^E_{DC} \equiv 0, \tag{1.15}$$

so that the parallel transport corresponding to (1.12) is integrable.

On the other hand, the commutator of the covariant derivatives (1.13)

$$[\nabla_A, \nabla_B] = -\Lambda^C_{AB} \nabla_C, \tag{1.16}$$

determines the curvature of remote parallelism

$$\Lambda^C_{AB} \equiv \Delta^C_{AB} - \Delta^C_{BA}. \tag{1.17}$$

The tensor field (1.17) satisfies identically the relations

$$(\nabla_A \Lambda^D_{BC} - \Lambda^D_{AE}\Lambda^E_{BC}) + \text{cycle}\,(A, B, C) \equiv 0 \tag{1.18}$$

that guarantee the compatability of Eqs. (1.16).

Besides, relation (1.8) imposes the condition

$$\omega_{AD}\Lambda^D_{BC} + \text{cycle}\,(A, B, C) = 0, \tag{1.19}$$

on the curvature (1.17).

Relations (1.6)–(1.19) cover the main facts of the geometry of absolute parallelism, associated with noncanonical commutation relations (1.5) at the classical level.

Let us come back to our dynamical system described by the Hamiltonian (1.1) and postulate the standard Hamiltonian equations of motion with the Poisson bracket (1.3) for it

$$\dot{\Gamma}^A = \{\Gamma^A, H\} = \omega^{AB}(\Gamma)\partial_B H(\Gamma). \tag{1.20}$$

The dot designates the time-derivative.

As the next step, we are going to find the action S to which the trajectory (1.20) would provide the unconditioned extremum

$$\frac{\delta S}{\delta \Gamma^A(t)} = \omega_{AB}(\Gamma)\dot{\Gamma}^B - \partial_A H(\Gamma). \tag{1.21}$$

First of all, one should verify the fulfillment of the integrability condition

$$\frac{\delta^2 S}{\delta \Gamma^B(t')\delta \Gamma^A(t)} = \frac{\delta^2 S}{\delta \Gamma^A(t)\delta \Gamma^B(t')}. \tag{1.22}$$

After directly calculating the functional derivative of (1.21) with respect to $\Gamma^B(t')$, one gets

$$\frac{\delta^2 S}{\delta \Gamma^B(t')\delta \Gamma^A(t)} = ((\partial_B \omega_{AC})\dot{\Gamma}^C - \partial_B\partial_A H)\delta(t-t') + \omega_{AB}\dot{\delta}(t-t'). \tag{1.23}$$

The part of (1.23) antisymmetric under the permutation $(A,t) \to (B,t')$ disappears in virtue of (1.8)

$$-(\partial_A\omega_{BC}+\partial_C\omega_{AB}\partial_B\omega_{CA})\dot{\Gamma}^C\delta(t-t')\equiv 0, \tag{1.24}$$

so that the integrability condition (1.22) is evidently fulfilled.

Let us now write (1.21) as an equation in variations

$$\delta S=\int((\delta\Gamma^A)\omega_{AB}(\Gamma)\dot{\Gamma}^B-\delta H)\,dt. \tag{1.25}$$

After making the substitution $\Gamma\to\lambda\Gamma$, with λ a parameter, and choosing the special form of the variation

$$\delta\lambda\Gamma^A=\Gamma^A d\lambda, \tag{1.26}$$

we get the differential equation with λ being an independent variable

$$\frac{d}{d\lambda}\left(S+\int H\,dt\right)=\lambda\int\Gamma^A\omega_{AB}(\lambda\Gamma)\dot{\Gamma}^B\,dt. \tag{1.27}$$

This is integrated to

$$S=\int(\Gamma^A\bar{\omega}_{AB}(\Gamma)\dot{\Gamma}^B-H(\Gamma))\,dt, \tag{1.28}$$

where the designation

$$\bar{\omega}_{AB}(\Gamma)\equiv\int_0^1\omega_{AB}(\lambda\Gamma)\lambda\,d\lambda \tag{1.29}$$

is used. Note in passing that the function (1.29) satisfies the equation

$$(\Gamma^C\partial_C+2)\bar{\omega}_{AB}=\omega_{AB}. \tag{1.30}$$

Besides, it follows from (1.9) that the function (1.29) may be locally presented as

$$\bar{\omega}_{AB}=\partial_A\bar{V}_B-\partial_B\bar{V}_A, \tag{1.31}$$

where the functions

$$\bar{V}_A(\Gamma)\equiv\int_0^1 V_A(\lambda\Gamma)\,d\lambda \tag{1.32}$$

obey the equation

$$(\Gamma^B \partial_B + 1) \bar{V}_A = V_A. \tag{1.33}$$

Finally, it follows from (1.31) that the action (1.28) locally admits the representation

$$S = W(\Gamma_{(i)}) - W(\Gamma_{(f)}) + \int (V_A \dot{\Gamma}^A - H) dt, \tag{1.34}$$

where W designates

$$W(\Gamma) \equiv \bar{V}_A(\Gamma)\Gamma^A, \tag{1.35}$$

while $\Gamma_{(i)}$ and $\Gamma_{(f)}$ are the initial and final points of the trajectory, respectively.

We conclude this section by the remark that, in accordance with (1.21), the canonical transformation generated by $G(\Gamma)$

$$\delta\Gamma^A = \{\Gamma^A, G\} = \omega^{AB}(\Gamma)\partial_B G(\Gamma) \tag{1.36}$$

results in the usual dectrement of the action (1.28)

$$\delta S = G(\Gamma_{(i)}) - G(\Gamma_{(f)}) - \int \{H, G\} dt. \tag{1.37}$$

2. Invariant Measure and Path Integral

Our further aim is to construct an invariant analog of the Liouville measure

$$D\Gamma \equiv \prod_t \prod_A d\Gamma^A(t). \tag{2.1}$$

To this end, consider the variation of the Liouville measure (2.1) under the canonical transformation (1.36)

$$\delta D\Gamma = (J-1) D\Gamma \tag{2.2}$$

where J is the Jacobian of the transformation

$$J - 1 = \delta^{(1)}(0) \int \partial_A \delta\Gamma^A dt = \delta^{(1)}(0) \int (\partial_A \omega^{AB}) \partial_B G dt. \tag{2.3}$$

Let us transform the integrand in succession and exploit relation (1.8)

$$
\begin{aligned}
(\partial_A \omega^{AB})\partial_B G &= -\omega^{AC}(\partial_A \omega_{CD})\omega^{DB}\partial_B G \\
&= \frac{1}{2}\omega^{AC}(\partial_A \omega_{CD} - \partial_C \omega_{AD})\omega^{DB}\partial_B G \\
&= \frac{1}{2}\omega^{AC}(\partial_A \omega_{DC} + \partial_C \omega_{AD})\omega^{DB}\partial_B G \\
&= -\frac{1}{2}\omega^{AC}(\partial_D \omega_{CA})\omega^{DB}\partial_B G \\
&= -\frac{1}{2}(\partial_D \ln\det \omega)\omega^{DB}\partial_B G \\
&= -\frac{1}{2}\{\ln\det \omega, G\} = -\delta\left[\frac{1}{2}\ln\det\omega\right].
\end{aligned} \tag{2.4}
$$

The quantity ω without indices under the determinant symbol denotes just the matrix ω_{AB} with the lower indices, as it is defined in (1.7).

After substituting (2.4) into (2.3), we see from (2.2) that the measure

$$
d\mu(\Gamma) \equiv \exp\left\{\frac{1}{2}\delta^{(1)}(0)\int \ln\det\omega\, dt\right\} D\Gamma \tag{2.5}
$$

is exactly invariant under the canonical transformations (1.36). Moreover, the tensorial nature of the field ω_{AB} guarantees the invariance of this measure under an arbitrary reparametrization of the phase space variables Γ. We conclude that the measure (2.5) is just the invariant analog of the standard Liouville measure sought for.

Using the action (1.28) found above and the measure (2.5), we can now define the expression for the generating functional in the form of the following path integral in the phase space of the system

$$
\begin{aligned}
Z &= \int \exp\left\{\frac{i}{\hbar}\left[S + \int J_A \Gamma^A dt\right]\right\} d\mu(\Gamma) \\
&= \int \exp\left\{\frac{i}{\hbar}\int\left[\Gamma^A \int_0^1 \omega_{AB}(\lambda\Gamma)\lambda\, d\lambda\, \dot{\Gamma}^B - H(\Gamma)\right.\right. \\
&\quad \left.\left. + \frac{1}{2}\omega_{AB}(\Gamma)C^B C^A + T_A \Gamma^A\right] dt\right\} D\Gamma DC.
\end{aligned} \tag{2.6}
$$

Here $J_A(t)$ denote components of an external source and, besides, an integration over the auxiliary Fermi variables $C^A(t)$ is defined with regards to the Gauss parametrization

$$(\mathrm{Det}\,\omega)^{1/2} \equiv \exp\left\{\frac{1}{2}\delta^{(1)}(0)\int \ln \det\omega\, dt\right\} = \int \exp\left\{\frac{i}{2\hbar}\int \omega_{AB}C^B C^A dt\right\} DC. \tag{2.7}$$

The external source J_A in (2.6) is introduced, as usual, directly to the phase variables Γ^A so that the reparametrization invariance in (2.6) only occurs on the mass shell. There exists, however, a simple way to define a source (I^a) of manifestly reparametrization-invariant variables, namely, of local components of a virtual external force

$$\Xi_a = h_a^A \frac{\delta S}{\delta \Gamma^A} = \omega_{ab}^{(0)} h_A^b \dot{\Gamma}^A - h_a^A \partial_A H \tag{2.8}$$

which corresponds to the change of the last term in the exponent in (2.6) according to

$$J_A \Gamma^A \rightarrow I^a \,\Xi_a. \tag{2.9}$$

Moreover, the variables (2.8) are most adequate as far as the calculation of the path integral using the stationary phase method is concerned, since the mass shell $\Xi = 0$ corresponds directly to the classical extremal, while the stationary phase method deals in fact with expansion of the integral in powers of Ξ. In view of the abovesaid, it is reasonable to use the components (2.8) as invariant integration variables instead of the original variables (1.2). Passing formally to the new variables in (2.6), one finds after the change (2.9)

$$Z = \int \exp\left\{\frac{i}{\hbar}\int\left[\left(\Gamma^A \int_0^1 \omega_{AB}(\lambda\Gamma)\lambda\, d\lambda - \prod^a \omega_{ad}^{(0)} h_A^d \Lambda_{BD}^A h_b^D \Phi^b\right)\dot{\Gamma}^B + \prod^a \omega_{ab}^{(0)}\dot{\Phi}^b - H(\Gamma) - \prod^a h_a^A(\nabla_B \nabla_A H) h_b^B \Phi^b + I^a \Xi_a\right] dt\right\} D\Xi D\Pi D\Phi, \tag{2.10}$$

where Γ is related to Ξ by the equation

$$\omega_{AB}(\Gamma)\dot{\Gamma}^B - \partial_A H(\Gamma) = h_A^a(\Gamma)\Xi_a. \tag{2.11}$$

In the course of transition to the new variables (2.8), the Jacobian cancels out the local measure (2.7), and, in its place, a nonlocal determinant arises, effectively presented in (2.10) by the integral over the auxiliary Bose variables Π and Φ.

The invariant variables analogous to (2.8), but in the configuration space, were used before in our works[6,7] for getting an invariant functional formulation of nonlinear chiral dynamics.

Note also that within the present formalism, we have at our disposal a connection (1.12) and covariant derivatives (1.13), so that we might write the equations

$$\sigma^A(\Gamma,\Gamma')\nabla_A\sigma^B(\Gamma,\Gamma') = \sigma^B(\Gamma,\Gamma')\sigma^A(\Gamma,\Gamma) = 0, \tag{2.12}$$

and define an effective action, considering an external source to σ, as was suggested in Refs. 8 and 9. We shall not, however, go into the details of such a construction here.

To summarize our main result in this section is the generating functional, Eq. (2.6) or Eq. (2.10). Both these expressions relate to the pure Bose case. We are going now to point out the modifications necessary to cover the supercase, when Bose and Fermi degrees of freedom are both present, as applied to Eq. (2.6), the simpler one of the two expressions.

First of all, the phase variables (1.2) in the case of mixed statistics bear the Grassmann parity

$$\varepsilon(\Gamma^A) \equiv \varepsilon_A. \tag{2.13}$$

The definition (1.3) of the Poisson bracket is now extended to

$$\{X, Y\} \equiv X\overleftarrow{\partial}_A\omega^{AB}\overrightarrow{\partial}_B Y, \tag{1.3a}$$

where $\overleftarrow{\partial}$ ($\overrightarrow{\partial}$) is the right (left) derivative.

The metric ω^{AB} is reversible even matrix subject to the generalized antisymmetry property

$$\varepsilon(\omega^{AB}) = \varepsilon_A + \varepsilon_B, \qquad \omega^{AB} = -\omega^{BA} = (-1)^{\varepsilon_A\varepsilon_B}. \tag{1.4a}$$

Relation (1.5) retains its form in terms of the brackets (1.3a).

The key condition (1.6) is now written as

$$\omega^{AD}\partial_D\omega^{BC}(-1)^{\varepsilon_A\varepsilon_C} + \text{cycle}\,(A, B, C) = 0. \tag{1.6a}$$

The matrix ω_{AB} supplied with subscripts, inverse to the super-scripted matrix

ω^{AB}, is again defined by Eq. (1.7).

Relation (1.8) and its local consequence are now extended to

$$\partial_A \omega_{BC}(-1)^{(\varepsilon_A+1)\varepsilon_C} + \text{cycle}\,(A, B, C) = 0, \tag{1.8a}$$

$$\omega_{AB} = \partial_A V_B + \partial_B V_A (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}. \tag{1.9a}$$

Now that these facts, basic for the supercase, are listed, we shall point an accurate interpretation of Eq. (2.6).

First, the statistics of the sources J_A coincides for each "A" with that of the corresponding Γ^A

$$\varepsilon(J_A) = \varepsilon_A. \tag{2.14}$$

Second, the statistics of the auxiliary fields C^A is at each "A" opposite to that of the corresponding Γ^A

$$\varepsilon(C^A) = \varepsilon_A + 1. \tag{2.15}$$

Then Eq. (2.6) is literally valid provided that the matrix ω_{AB} obeys (1.5) and (1.8a).

3. Conclusion

As stated above, the most interesting aspect of the noncanonicity of commutation relations is due to the possibility of considering a curved phase space. Once the curvature is here, one cannot define canonical coordinates in a finite region, which drastically complicates the operational quantization. These difficulties can, probably, be overcome appearing to the compensatory mechanism, like the one used by us in the operational quantization of dynamical systems subject to second-class constraints.[10–12] A possible alternative may be an efficient use of the theory of symbols. For special systems with curved phase spaces (e.g., the Lobachevskii plane, sphere, Gross-Neveu-like models), the most essential results were, to the best of our knowledge, achieved by F. A. Berezin[13] just within the theory of symbols. We are going to come back to these problems elsewhere.

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