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New developments in D-dimensional conformal quantum field theory

E.S. Fradkin[a,b,*], M.Ya. Palchik[c]

[a] *Theoretical Division, CERN, CH-1211 Geneva 23, Switzerland*
[b] *Lebedev Physical Institute, Moscow 117924, Russia*
[c] *Institute of Automation and Electrometry, Novosibirsk 630090, Russia*

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* Corresponding author. e-mail: fradkina@math.ias.edu

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Abstract

A review of recent developments in conformal quantum field theory in D-dimensional space is presented. The conformally invariant solution of the Ward identities is studied. We demonstrate the existence of D-dimensional analogues of primary and secondary fields, the central charge, and the null vectors. The Hilbert space is shown to possess a specific model-independent structure defined by the $\frac{1}{2}(D+1)(D+2)$-dimensional symmetry and the Ward identities. In particular, there exists a sector H of the Hilbert space related to an infinite family of "secondary" fields which are generated by the currents and the energy-momentum tensor. The general solution of the Ward identities in $D > 2$ defining the sector H necessarily includes the contribution of the gauge fields. We derive the conditions which single out the conformal theories of a direct (non-gauge) interaction.

We examine the class of models satisfying these conditions. It is shown that the Green functions of the current and the energy-momentum tensor in these models are uniquely determined by the Ward identities for any $D \geq 2$. The anomalous Ward identities containing contributions of c-number and operator analogues of the central charge, are discussed. Closed sets of expressions for the Green functions of secondary fields are obtained in D-dimensional space.

A family of exactly solvable conformal models in $D \geq 2$ is constructed. Each model is defined by the requirement of vanishing of a certain field Q_s, $s = 1,2 \ldots$. The fields Q_s are constructed as definite superpositions of secondary fields. After that, one requires each field Q_s to be primary. The latter is possible for specific values of scale dimensions of fundamental fields (a D-dimensional analogue of the Kac formula). The states $Q_s|0\rangle$ are analogous to null vectors. One can derive closed sets of differential equations for higher Green functions in each of the models. These results are demonstrated on examples of several exactly solvable models in $D > 2$.

The approach developed here is based on the finite-dimensional conformal symmetry for any $D \geq 2$. However the family of models under consideration does have the structure identical to that of two-dimensional conformal theories. This analogy is discussed in detail. It is shown that when $D = 2$, the above family coincides with the well-known family of models based on infinite-dimensional conformal symmetry. The analysis of this phenomenon indicates the possibility of existence of D-dimensional analogue of the Virasoro algebra.

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1. Introduction

1.1. Preliminary remarks

The present review describes a family of exactly solvable models of quantum field theory in D-dimensional space. The studies of conformal models by the authors was initiated as far back as 1970s [1,2] and approached its essential developments in the recent works [3–6].

The conformal symmetry is usually treated as a non-perturbative effect pertinent to the description of physical phenomena in the asymptotic region. Any dimensional parameters entering the bare Hamiltonian are supposed to be immaterial in this region. Regardless of the specific character of physical phenomena under discussion (the critical behaviour of statistical systems, late turbulence or interactions of elementary particles, etc.), the systems in this region may exhibit certain fundamental properties which are independent on the structure of the initial Hamiltonian.

Adopting this hypothesis, the following formulation of the problem becomes natural: one aims to find a set of axiomatic principles which would completely fix the effective interaction in the asymptotic region. It is straightforward to expect the result to be independent of the choice of initial Hamiltonian. Formulated in such manner, the problem has been repeatedly discussed both by physicists and mathematicians. As the most popular candidates to the role of the above principles, the field algebra hypothesis and the hypothesis of the scale and conformal symmetry were considered (see, for example, [7–11]). In the works [1,2,12] we have studied additional restrictions which follow from generalized Ward–Fradkin–Takakhashi identities [13,64], provided that the latter are completed by the requirement of conformal symmetry. As a result, a closed set of conditions defining a family of exactly solvable models in D-dimensional space was formulated. Each model is determined [3,6] by a certain condition on the states generated by the currents and the energy–momentum tensor. These states are analogous to the null-vectors of two-dimensional conformal theories. This fact was first demonstrated on the example of the Thirring model in late seventies, see Refs. [2,14]. A complete and detailed solution of several aspects of this approach was given in the book [15] as well as in the works [5,6].

When concerning exactly solvable models, we imply the following feature: one may derive a closed set of differential equations for any higher Green function. In addition one can deduce algebraic equations for scale dimensions of fields and massless parameters analogous to the central charge.

An important approach to obtain exactly solvable models in two-dimensional space was developed in the works [16–19]. However, the case of $D = 2$ is exceptional since the conformal group of two-dimensional space is infinite dimensional. The method developed in Refs. [16–19] does not allow a straightforward generalization until the proper D-dimensional analogue of the Virasoro algebra is found. What is essential in our approach is that the $\frac{1}{2}(D+1)(D+2)$-dimensional conformal symmetry is assumed in the space of any dimension D. For the case of $D = 2$ this symmetry is 6-parametric. Its generators

$$L_0, L_\pm, \bar{L}_0, \bar{L}_\pm \qquad (1.1)$$

compose the algebra of the group

$$\mathrm{SL}(2, R) \times \mathrm{SL}(2, R) \qquad (1.2)$$

which is the maximal finite-dimensional subalgebra of the Virasoro algebra. However, all the conformal models found in the works [16–19] may be derived in the framework of our approach (see Section 4). Note that the form of the Ward identities for the Green functions of the energy–momentum tensor is prescribed by the symmetry under the group (1.2). The Ward identities include the complete information on the commutators between the components of the energy–momentum tensor, as well as between these components and other fields. Thus, in the approach based on the symmetry (1.2) and the Ward identities for $D = 2$, the infinite-dimensional symmetry arises as an auxiliary result which is not presumed initially: adopting the formulation described here to the case of $D = 2$ implies one to act as if an infinite-dimensional symmetry were unknown.

A similar situation is likely to be realized in the case $D > 2$ as well. The structure of D-dimensional models discussed below is analogous to the structure of two-dimensional conformal models. We shall demonstrate the existence of certain analogues of primary and secondary fields, the central charge and the null vectors. Moreover, we shall show that each of the models possesses an infinite set of self-consistency conditions which provides the analogy with conformal symmetry of two-dimensional models. Hence, one can expect that a definite analogue of the Virasoro algebra should exist in D-dimensional space, the realization of the very analogue being observed in the above models.

The principal difference between the conformal models in $D \geq 3$ and two-dimensional ones consists in the following. The general solution of conformal Ward identities necessarily includes the contribution of gravitational interaction, while the conformal solution of the Ward identities for conserved currents includes the contribution of gauge interactions (see Section 2). Since in the present article we restrict ourselves solely to the discussion of the models of direct (non-gauge) interaction, the problem which arises is: how to eliminate gauge interactions from the general solution of conformally invariant Ward identities? This problem is solved in Section 2. To facilitate the understanding of these results, Section 5 contains the discussion of gauge interactions (gravitational included) in four-dimensional space. In its course, all the main results of Section 2 are reproduced on a slightly different standpoint. Moreover, a possibility of introduction of gauge interactions into a family of models under consideration is also discussed in Section 5.

For the sake of illustration, the solution of several non-trivial models is presented in Section 4. Guided by methodical considerations we restricted the discussion to the study of models which demanded technically simple calculations. The latter models were meant to serve as illustration of the principal ideas as well as the features of calculation technique developed in the paper. Besides that, the class of models considered is the one that allows for the most evident analogy between the structure of our approach and that of known two-dimensional theories.

The most physically interesting models require that the operator analogue of the central charge should be introduced (the field $P^{D-2}(x)$ of scale dimension $d_P = D - 2$, see Section 3). In particular, we believe that the three-dimensional Ising model is contained in this larger class. Such models demand a certain modification of the technique. Though up to that time the principal investigations had been already completed, we decided to refrain from surveying these new results in the present review due to the two reasons: firstly, it appeared to be methodically inexpedient – since the modified formulation mentioned above clouded the analogy with two-dimensional theories. Secondly, the analysis of the more complex models called for cumbersome calculations which were rather appropriate for the separate publications.

The present review primarily deals with the Euclidean formulation of quantum field theory [20,64]. Discussions concerning the structure of the Hilbert space of conformal theory will be based on the formulation in Minkowski space with the metric $g_{\mu\nu} = (- + \ldots +)$.

A number of questions involved is sketched rather briefly: the more detailed discussion of those is found in Ref. [15]. Especially the latter concerns the solution of conformal Ward identities for $D \geq 2$, and the analysis of two-dimensional conformal models in the framework of the approach developed in this paper. The work [15] includes also a consistent review of the main principles of conformal quantum field theory in $D \geq 2$, see also [2,21,22].

1.2. Conformal symmetry in D dimensions

The conformal group of D-dimensional space is a $\frac{1}{2}(D+1)(D+2)$-parametric group. It includes $\frac{1}{2}D(D+1)$ transformations of the Poincaré group, as well as scale transformations and special conformal transformations

$$x^\mu \xrightarrow{\lambda} \lambda x^\mu, \quad x^\mu \xrightarrow{a} \frac{x^\mu + a^\mu x^2}{1 + 2ax + a^2x^2}, \tag{1.3}$$

where $\lambda > 0$, a^μ is an arbitrary D-dimensional vector. Rather than handling special conformal transformations it proves helpful to use the transformation of conformal inversion R

$$x^\mu \xrightarrow{R} Rx = x^\mu/x^2. \tag{1.4}$$

A special conformal transformation may be derived as a sequence of three transformations: conformal inversion, translation by the vector $b^\mu = a^\mu/a^2$, conformal inversion again.

Each conformal field in D-dimensional space is characterized by its scale dimension which determines its transformation properties under scale and special conformal transformations. Let $\varphi(x)$ be a scalar conformal field of scale dimension d,

$$\varphi(x) \xrightarrow{\lambda} \lambda^d \varphi(\lambda x), \quad \varphi(x) \xrightarrow{R} (x^2)^{-d}\varphi(Rx). \tag{1.5}$$

The condition of invariance of the theory under transformations (1.5) leads to the following coordinate dependence of two- and three-point invariant Green functions (see reviews [2,21,22] and references therein):

$$\langle\varphi(x_1)\varphi(x_2)\rangle \sim (x_{12}^2)^{-d}, \tag{1.6}$$

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle = g(x_{12}^2)^{-(d_1+d_2-d_3)/2}\,(x_{13}^2)^{-(d_1-d_2+d_3)/2}\,(x_{23}^2)^{-(d_2+d_3-d_1)/2}, \tag{1.7}$$

where $\varphi_1, \varphi_2, \varphi_3$ are scalar conformal fields with dimensions d_1, d_2, d_3, and g is an arbitrary constant.

Scale dimensions are the most fundamental parameters of conformal theory. Its values determine the character of physical phenomena described by the conformal field theory. For example, the quantities measured experimentally in the statistical systems near the 2nd-order phase transition point are the critical indices. They govern the singular behaviour of the correlators of a free energy, magnetization, etc. in the critical region. The above parameters are expressed through the scale dimensions of fundamental fields with the help of known relations [11]. Thus, scale dimensions are experimentally observable in statistical systems. Viewing the conformal symmetry as an asymptotic symmetry of the theory of elementary particles, the scale dimensions control the powers of growth and decay of the effective potential in the asymptotic region. One can show (see Ref. [2,22,23] and references therein) that the positivity axiom restricts the possible values of scale dimension to the interval (in the case of the scalar field)

$$d \geq D/2 - 1 . \tag{1.8}$$

The lower bound in this inequality coincides with the dimension of a free massless field

$$d_{\mathrm{can}} = D/2 - 1 , \tag{1.9}$$

and is called the canonical dimension. The latter is fixed by the quantization rules. In the presence of interaction we face up with renormalized fields

$$\varphi_{\mathrm{ren}}(x) = z_2^{1/2}\, \varphi_{\mathrm{can}}(x) . \tag{1.10}$$

The renormalization constant is dimensional; hence, the dimension of the renormalized field takes an anomalous value $d \neq d_{\mathrm{can}}$. Under the shifting of the regularization

$$z_2 \to 0 , \tag{1.11}$$

provided that no negative-norm states are present in the theory. One has, in accordance with Eq. (1.8):

$$d > d_{\mathrm{can}} .$$

This expresses the well-known fact that the singularity of the propagator in the presence of interaction is more sharp than in the case of free theory (assuming that the states with negative norms are absent).

Besides Eqs. (1.6) and (1.7), the conformal symmetry conditions lead to several strong restrictions on the higher Green functions. In particular, one has for the Green function of four fields (see, for example, Refs. [2,22] and references therein):

$$\langle \varphi_1(x_1)\varphi_1(x_2)\varphi_2(x_3)\varphi_2(x_4)\rangle = (x_{12}^2)^{-d_1}(x_{34}^2)^{-d_2} F_{12}(\xi, \eta) , \tag{1.12}$$

where F_{12} is an arbitrary function of variables

$$\xi = x_{12}^2 x_{34}^2 / x_{13}^2 x_{24}^2 , \quad \eta = x_{12}^2 x_{34}^2 / x_{14}^2 x_{23}^2 \tag{1.13}$$

known as harmonic ratios.

Thus the problem of the construction of the exact solution of conformal theory consists in the evaluation of scale dimensions of the fields and "coupling constants" such as the parameter g in Eq. (1.7), as well as functions of harmonic ratios entering the conformally invariant representations of the type (1.12), which define higher Green functions.

Consider the operator product expansion of the pair of scalar fields $\varphi_1(x_1)\varphi_2(x_2)$ at neighbouring points. The tensor fields $\Phi_k(x)$ together with all their derivatives contribute to this expansion. Each field Φ_k is a traceless symmetric tensor

$$\Phi_k(x) = \Phi_{s_k}^{l_k}(x) = \Phi_{\mu_1 \ldots \mu_k}^{l_k}(x) , \tag{1.14}$$

where l_k is the scale dimension and s_k is the tensor rank. Under the coordinate transformations (1.3) and (1.4) the fields (1.14) transform as follows:

$$\Phi^l_{\mu_1\ldots\mu_s}(x) \xrightarrow{\lambda} \lambda^l \Phi^l_{\mu_1\ldots\mu_s}(\lambda x),\ \Phi^l_{\mu_1\ldots\mu_s}(x) \xrightarrow{R} (x^2)^{-l} g_{\mu_1\nu_1}(x)\ldots g_{\mu_s\mu_s}(x)\Phi^l_{\nu_1\ldots\nu_s}(Rx)\,, \tag{1.15}$$

where

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - 2\, x_\mu x_\nu/x^2\,. \tag{1.16}$$

Denote the mth order derivative of the field $\Phi_k(x)$ as $\Phi_k^{(m)}(x)$. Thus the operator expansion of the product $\varphi_1\varphi_2$ is written as

$$\varphi_1(x)\varphi_2(0) = \sum_{k,m} A_k^{(m)}(x)\Phi_k^{(m)}(0)\,, \tag{1.17}$$

where $A_k^{(m)}(x) \sim (x^2)^{-(d_1+d_2-l_k-m)/2}$. For example, the contribution of the scalar field $\Phi_0^l(x)$ and its derivatives into Eq. (1.17) is

$$[\Phi_0] = g_0(x^2)^{-(d_1+d_2-l)/2}\{\Phi_0(0) + a_1 x_\mu \partial_\mu \Phi(0) + a_2 x_\mu x_\nu \partial_\mu\partial_\nu\Phi(0) + a_3 x^2\Box\Phi(0) + \cdots\}\,, \tag{1.18}$$

where g_0 and a_i are some constants.

The operator equality (1.17), like any other relation between Euclidean fields, should be understood as a symbolic notation representing the asymptotic expansion of Euclidean Green functions

$$\langle\varphi_1(x)\varphi_2(0)\Phi_1(x_1)\ldots\Phi_n(x_n)\rangle|_{x\to 0} \simeq \sum_{k,m} A_k^{(m)}(x)\langle\Phi_k^{(m)}(0)\Phi_1(x_1)\ldots\Phi_n(x_n)\rangle\,, \tag{1.19}$$

where $\Phi_1\ldots\Phi_n$ are arbitrary conformal fields. The words "neighbouring points" mean that

$$x^2 \ll x_{kr}^2\,,\quad 1\le k,r\le n\,.$$

one can show [24] (see also Refs. [2,22] and references therein) that the invariance under the transformations (1.15) fixes all the coefficients a_m in the expansion series of the type (1.18) uniquely and allows one to take an explicit sum of all the terms with derivatives of the field $\Phi_k(x)$. The result has the form

$$[\Phi_k] = g_k \int \mathrm{d}y\, Q^{d_1 d_2 \tilde{l}_k}_{\mu_1\ldots\mu_{s_k}}(x,0|y)\Phi_{\mu_1\ldots\mu_{s_k}}(y)\,, \tag{1.20}$$

where g_k is the coupling constant of the field Φ_k and $Q^{d_1 d_2 \tilde{l}_k}_{\mu_1\ldots\mu_{s_k}}$ is a known function, which expression can be found in Refs. [2,10,21,22] and in the references therein. In what follows the operator expansions of the type (1.17) will be written in a symbolic form

$$\varphi_1(x)\varphi_2(0) = \sum_k [\Phi_k]\,, \tag{1.21}$$

where $[\Phi_k]$ is given by expression (1.20).

One can show [2,10] that the problem of calculation of conformally invariant Green functions (1.12) is equivalent to the problem of calculation of coupling constants and dimensions

$$g_k, l_k \tag{1.22}$$

of the fields Φ_k. The examples of such calculations (in the approximation of skeleton graphs) may be found in Refs. [12,15,58], see also references therein.

1.3. Conformal partners and amputation conditions

Let us introduce a unified notation σ for the pair of quantum numbers l, s. Each field (1.14) transforms by irreducible representation of the conformal group [2,15,21,22] of the Euclidean space. Denote this representation as T_σ. The fields $\Phi_k = \Phi_{\sigma_k}$ and $\Phi_m = \Phi_{\sigma_m}$, transforming by non-equivalent representations T_{σ_k} and T_{σ_m}, are orthogonal:

$$\langle \Phi_{\sigma_k}(x_1)\Phi_{\sigma_m}(x_2)\rangle = 0 \quad \text{if } l_k \neq l_m \text{ or } s_k \neq s_m \,. \tag{1.23}$$

This may be checked directly using the transformation laws (1.15). The pair of fields

$$\Phi_\sigma(x) = \Phi_s^l(x)\,, \quad \Phi_{\tilde\sigma}(x) = \Phi_s^{\tilde l}(x)\,, \quad \tilde l = D - l\,, \tag{1.24}$$

where

$$\sigma = (l, s)\,, \quad \tilde\sigma = (\tilde l, s) = (D - l, s) \tag{1.25}$$

is an exception. These fields transform by equivalent representations [2,21,22]

$$T_\sigma \sim T_{\tilde\sigma}\,. \tag{1.26}$$

One has for the fields (1.24) (see Ref. [15] for the details):

$$\langle \Phi_\sigma(x_1)\Phi_{\tilde\sigma}(x_2)\rangle = \mathrm{sym}\{\delta_{\mu_1\nu_1}\ldots\delta_{\mu_s\nu_s}\}\delta(x_1 - x_2)\,, \tag{1.27}$$

where the notation "sym" stands for the symmetrization and subtraction of traces performed in each group of indices $\mu_1 \ldots \mu_s$, $\nu_1 \ldots \nu_s$. Below we call fields (1.24) the conformal partners.

Equivalence condition (1.26) is expressed by the following operator equality (see Ref. [15] for more details):

$$\Phi^l_{\mu_1\ldots\mu_s}(x) = \int \mathrm{d}y\, \Delta^l_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(x - y)\Phi^{D-l}_{\nu_1\ldots\nu_s}(y)\,, \tag{1.28}$$

which will be used below in a shorthand notation

$$\Phi_\sigma(x) = \int \mathrm{d}y\, \Delta_\sigma(y)\Phi_{\tilde\sigma}(y)\,. \tag{1.29}$$

The intertwining operator Δ_σ coincides [15] with the conformally invariant propagator

$$\begin{aligned}\Delta_\sigma(x_{12}) &= \Delta_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(x_{12}) = \langle \Phi^l_{\mu_1\ldots\mu_s}(x_1)\Phi^l_{\nu_1\ldots\nu_s}(x_2)\rangle \\ &= (2\pi)^{-D/2} n(\sigma)(\tfrac{1}{2}\,x_{12}^2)^{-l}\,\mathrm{sym}\{g_{\mu_1\nu_1}(x_{12})\ldots g_{\mu_s\nu_s}(x_{12})\}\,,\end{aligned} \tag{1.30}$$

where $g_{\mu\nu}(x_{12})$ is given by expression (1.16), the notation "sym" has the same sense as in (1.27), and $n(\sigma)$ is the normalization factor.

Consider the invariant function of three fields:

$$G_\sigma^{d_1 d_2}(x_1 x_2 x_3) = \langle \varphi_1(x_1)\varphi_2(x_2)\Phi_\sigma(x_3)\rangle = g_\sigma C_\sigma^{d_1 d_2}(x_1 x_2 x_3)\,, \tag{1.31}$$

where g_σ is the coupling constant, while the invariant function $C_\sigma^{d_1 d_2}$ is calculated using Eqs. (1.5) and (1.15) and has the following form [2,21,22]:

$$C_\sigma^{d_1 d_2}(x_1 x_2 x_3) = (2\pi)^{-D/2} N(\sigma d_1 d_2)(\tfrac{1}{2}x_{12}^2)^{-(d_1+d_2-l+s)/2} \times (\tfrac{1}{2}x_{13}^2)^{-(d_1-d_2+l-s)/2}\,(\tfrac{1}{2}x_{23}^2)^{-(d_2-d_1+l-s)/2}\,\lambda^{x_3}_{\mu_1\ldots\mu_s}(x_1 x_2)\,, \tag{1.32}$$

where

$$\lambda^{x_3}_{\mu_1\ldots\mu_s}(x_1 x_2) = \lambda^{x_3}_{\mu_1}(x_1 x_2)\ldots\lambda^{x_3}_{\mu_s}(x_1 x_2) - \text{traces}, \quad \lambda^{x_3}_{\mu}(x_1 x_2) = \frac{(x_{13})_\mu}{x_{13}^2} - \frac{(x_{23})_\mu}{x_{23}^2}\,, \tag{1.33}$$

and $N(\sigma d_1 d_2)$ is the normalization factor.

When applied to the Green functions (1.30) and (1.31), the operator Eq. (1.28) means that under the suitable choice of normalization factor the following relations should hold:

$$\Delta_\sigma^{-1}(x_{12}) = \Delta_{\tilde\sigma}(x_{12})\,, \tag{1.34}$$

$$G_{\tilde\sigma}^{d_1 d_2}(x_1 x_2 x_3) = \int dx\, G_\sigma^{d_1 d_2}(x_1 x_2 x)\Delta_\sigma^{-1}(x - x_3)\,, \tag{1.35}$$

$$G_\sigma^{\tilde d_1 d_2}(x_1 x_2 x_3) = \int dx\, G_{d_1}^{-1}(x_1 - x) G_\sigma^{d_1 d_2}(x x_2 x_3)\,, \quad \tilde d_1 = D - d_1\,, \tag{1.36}$$

and in the x_2 argument by analogy. Here G_{d_1} stands for the propagator of the field φ_1:

$$G_d(x_{12}) = \langle \varphi_d(x_1)\varphi_d(x_2)\rangle = (2\pi)^{-D/2}\frac{\Gamma(d)}{\Gamma(D/2-d)}\left(\frac{1}{2}x_{12}^2\right)^{-d}. \tag{1.37}$$

In what follows, conditions (1.34)–(1.36) will be called the amputation conditions.

As one can easily check by a direct calculation, the amputation conditions will hold provided one chooses [2,15]

$$n(\sigma) = \frac{\Gamma(l+s)}{\Gamma(D/2-l)}\frac{\Gamma(D-l-1)}{\Gamma(D-l+s-1)}\,, \tag{1.38}$$

$$N(\sigma d_1 d_2) = \left\{\frac{\Gamma\left(\frac{l+d_1+d_2+s-D}{2}\right)\Gamma\left(\frac{d_1+d_2-l+s}{2}\right)\Gamma\left(\frac{l-d_1+d_2+s}{2}\right)\Gamma\left(\frac{l+d_1-d_2+s}{2}\right)}{\Gamma\left(\frac{2D-l-d_1-d_2+s}{2}\right)\Gamma\left(\frac{D+l-d_1-d_2+s}{2}\right)\Gamma\left(\frac{D-l-d_1+d_2+s}{2}\right)\Gamma\left(\frac{D-l+d_1-d_2+s}{2}\right)}\right\}^{1/2} \tag{1.39}$$

for the normalization factors in Eqs. (1.30) and (1.32). The calculations use conducted using relations (A.1)–(A.3). One can show that the Green function $G_\sigma^{d_1d_2}$ is expressed through the function $Q^{d_1d_2\sigma}$ which defines the operator product expansions of fields, (see Eq. (1.20)):

$$G_\sigma^{d_1d_2}(x_1x_2x_3) = -g_\sigma \frac{\pi}{\sin \pi(l - \frac{D}{2} + s)} \times \left\{ Q^{d_1d_2\sigma}(x_1x_2|x_3) - \int \mathrm{d}x\, \Delta_\sigma(x_3 - x) Q^{d_1d_2\tilde{\sigma}}(x_1x_2|x) \right\}. \tag{1.40}$$

The amputation condition (1.35) is the natural consequence of such representation.

Fields (1.24) have the same status. The latter means that when the field Φ_σ is present in the theory, the field $\Phi_{\tilde{\sigma}}$ must also exist, their Green functions being related by Eq. (1.29):

$$\langle \Phi_{\tilde{\sigma}}(x)\Phi_1(x_1)\dots\Phi_n(x_n)\rangle = \int \mathrm{d}y\, \Delta_\sigma^{-1}(x-y)\langle \Phi_\sigma(y)\Phi_1(x_1)\dots\Phi_n(x_n)\rangle\,, \tag{1.41}$$

where $\Phi_1 \dots \Phi_n$ are any conformal fields. One can prove [2,15] that the skeleton and bootstrap equations are invariant under the change:

$$l \to D - l \quad \text{or} \quad d_{1,2} \to D - d_{1,2}\,. \tag{1.42}$$

This symmetry is also present in the system of the renormalized Schwinger–Dyson equations [25,64] as long as its conformally invariant solution [1,2] is concerned. The transition from the fields Φ_σ, φ_d to the conformal partners $\Phi_{\tilde{\sigma}}$, $\varphi_{\tilde{d}}$ is equivalent to the transition from the formulation of the conformal theory in terms of Green functions to the formulation in terms of vertices and propagators.

However, the symmetry (1.42) is broken in the Ward identities. The latter selects one of the fields (1.24) to be a physical field. It is essential that the current j_μ and the energy- -momentum tensor $T_{\mu\nu}$ belong to the class of fields with canonical dimensions

$$l_s = D - 2 + s \tag{1.43}$$

and do not literally satisfy the equivalence conditions (1.29) (or (1.41)), since the corresponding representations of the conformal group are undecomposable [21,26].

The same is also true for the conformal partners of the fields Φ_s with the integer dimensions

$$\tilde{l}_s = D - l_s = 2 - s\,. \tag{1.44}$$

The Euclidean fields

$$j_\mu,\ T_{\mu\nu} \tag{1.45}$$

have the canonical dimensions

$$l_j = D - 1, \quad l_T = D\,. \tag{1.46}$$

Its conformal partners are the electromagnetic potential A_μ and the traceless part of the metric tensor $h_{\mu\nu}$ (in linear conformal gravity). The Euclidean fields

$$A_\mu,\ h_{\mu\nu} \tag{1.47}$$

have the dimensions (1.44)

$$l_A = 1, \quad l_h = 0\,. \tag{1.48}$$

In the next section we discuss the analogue of equivalence relations (1.29) between the fields (1.45) and (1.47). When $D = 4$, these relations coincide with the Maxwell equations and the equations of linear conformal gravity.

Let us remind that any relations between Euclidean fields are understood as if they were placed inside the averaging symbols. In what follows we examine the Euclidean Green functions

$$G^j_\mu(xx_1 \dots x_{2n}) = \langle j_\mu(x)\varphi(x_1)\dots\varphi^+(x_{2n})\rangle , \tag{1.49}$$

$$G^T_{\mu\nu}(xx_1 \dots x_m) = \langle T_{\mu\nu}(x)\varphi_1(x_1)\dots\varphi_m(x_m)\rangle , \tag{1.50}$$

$$G^A_\mu(xx_1 \dots x_{2n}) = \langle A_\mu(x)\varphi(x_1)\dots\varphi^+(x_{2n})\rangle , \tag{1.51}$$

$$G^h_{\mu\nu}(xx_1 \dots x_m) = \langle h_{\mu\nu}(x)\varphi_1(x_1)\dots\varphi_m(x_m)\rangle , \tag{1.52}$$

where Φ is a charged field of dimension d, and $\varphi_1 \dots \varphi_m$ are neutral fields of dimensions $d_1 \dots d_m$, respectively.

The Green functions (1.49) and (1.50) satisfy the conformally invariant Ward identities

$$\partial^x_\mu G^j_\mu(xx_1 \dots x_{2n}) = -\left[\sum_{k=1}^{n} \delta(x-x_k) - \sum_{k=n+1}^{2n} \delta(x-x_k)\right] G(x_1 \dots x_{2n}) , \tag{1.53}$$

$$\partial^x_\nu G^T_{\mu\nu}(xx_1 \dots x_m) = -\left[\sum_{k=1}^{n} \delta(x-x_k)\partial^{x_k}_\mu - \partial^x_\mu \sum_{k=1}^{m} \frac{d_k}{D}\delta(x-x_k)\right] G(x_1 \dots x_m) , \tag{1.54}$$

where

$$G(x_1 \dots x_{2n}) = \langle\varphi(x_1)\dots\varphi^+(x_{2n})\rangle,\ G(x_1 \dots x_m) = \langle\varphi_1(x_1)\dots\varphi_m(x_m)\rangle . \tag{1.55}$$

1.4. Conformal partial wave expansions in Minkowski space

Consider the fields φ_1, φ_2 and Φ_{σ_k} in Minkowski space. Let Q^+_σ be the positive frequency representations of the conformal group of Minkowski space. On can show that for any values of quantum numbers $\sigma_k = (l_k, s_k)$ different from Eqs. (1.43) and (1.44), the states

$$\Phi_{\sigma_k}(x)|0\rangle \subset M^+_{\sigma_k} \tag{1.56}$$

form a basis of the space $M^+_{\sigma_k}$ of the representation Q^+_σ, see Ref. [2] and, for more details, [15]. The Wightman functions

$$W_d(x_{12}) = \langle 0|\varphi(x_1)\varphi(x_2)|0\rangle,\ W_{\sigma_k}(x_{12}) = \langle 0|\Phi_{\sigma_k}(x_1)\Phi_{\sigma_k}(x_2)|0\rangle \tag{1.57}$$

represent invariant scalar products of the states (1.56). The spaces $M^+_{\sigma_k}$ of different irreducible representations are mutually orthogonal

$$\langle 0|\Phi_{\sigma_k}(x_1)\Phi_{\sigma_m}(x_2)|0\rangle = 0 \quad \text{if} \quad \sigma_k \neq \sigma_m . \tag{1.58}$$

All the states (1.56) form the basis in the Hilbert space of the conformal theory.

Consider the states $\varphi_2(x_2)\varphi_1(x_1)|0\rangle$. These states transform by an infinite direct sum of irreducible representations and can be decomposed through an infinite set of states (1.56) (see Refs. [2,15] and references therein):

$$\langle 0|\varphi_1(x_1)\varphi_2(x_2) = \sum_k A_k \int dx_3\, Q^{d_1 d_2 \tilde{\sigma}_k}(x_1 x_2|x_3)\,\langle 0|\Phi_{\sigma_k}(x_3)\,, \tag{1.59}$$

where A_k are unknown constants, $Q^{d_1 d_2 \tilde{\sigma}_k}$ is the same function as in Eq. (1.20) but written for the case of Minkowski space. Decomposition (1.59) is called the *vacuum operator decomposition* [22]. It is analogous to the operator decomposition of the Euclidean fields product, see Eq. (1.21). Taking into account Eq. (1.58), we have from (1.59)

$$\langle 0|\varphi_1(x_1)\varphi_2(x_2)\Phi_{\sigma_k}(x_3)|0\rangle = A_k \int dx\, Q^{d_1 d_2 \tilde{\sigma}_k}(x_1 x_2|x) W_{\sigma_k}(x - x_3)\,. \tag{1.60}$$

The l.h.s. of this equality is a conformally invariant Wightman function.

Any higher conformally invariant Wightman function, for example

$$W(xyx_1 \ldots x_m) = \langle 0|\varphi_1(x)\varphi_2(y)\Phi_1(x_1)\ldots\Phi_m(x_m)|0\rangle\,, \tag{1.61}$$

where $\Phi_1 \ldots \Phi_m$ are arbitrary conformal fields in Minkowski space, may be represented as an invariant scalar product of the states

$$\varphi_2(y)\varphi_1(x)|0\rangle \quad \text{and} \quad \Phi_1(x_1)\ldots\Phi_m(x_m)|0\rangle\,. \tag{1.62}$$

Using Eq. (1.59) we find the *conformal partial wave expansion of the Wightman functions* (see Refs. [2,15,22] and references therein):

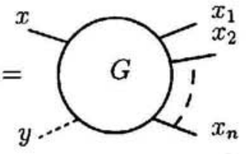
$$W(xyx_1 \ldots x_m) = \sum_k A_k \int dz\, Q^{d_1 d_2 \tilde{\sigma}_k}(xy|z)\langle 0|\Phi_k(z)\Phi_1(x_1)\ldots\Phi_m(x_m)|0\rangle \tag{1.63}$$

1.5. Conformal partial wave expansions of Euclidean green functions

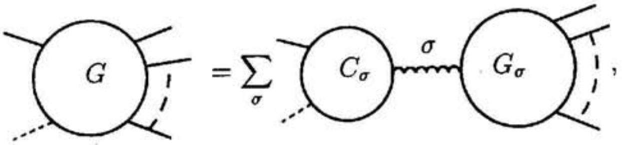
The Euclidean conformal partial wave expansion may be derived from the expansions (1.63) by analytic continuation into Euclidean coordinates [2,15], see also Refs. [10,21–24] and references therein. Let us present several results. The notations and normalizations are chosen as in Refs. [2,15], see Eqs. (1.38) and (1.39).

Let φ, χ be neutral scalar fields. The Green function

$$G(xyx_1 \ldots x_n) = \langle\, \varphi(x)\chi(y)\varphi(x_1)\cdots\varphi(x_n)\,\rangle = \tag{1.64}$$



may be written in the form of expansion which represents an Euclidean analogue of expansion (1.63)



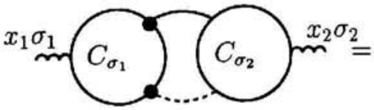
(1.65)

where

$$\sum_\sigma = \frac{1}{2\pi i} \sum_s \int_{D/2 - i\infty}^{D/2 + i\infty} dl \, \mu(\sigma) , \tag{1.66}$$

$$\mu(\sigma) = \frac{1}{4} 2^s (2\pi)^{-D/2} \frac{\Gamma(D/2 + s)}{\Gamma(s + 1)} n(\sigma) n(\tilde{\sigma}) , \tag{1.67}$$

$n(\sigma)$ is the normalizing factor (1.38), C_σ is the conformally invariant function (1.32) satisfying the amputation conditions (1.35) and (1.36), which is normalized by the condition

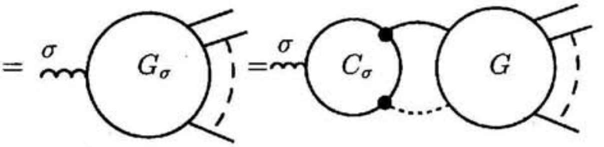


$= I_{\sigma_1\sigma_2} = \frac{1}{2} [\delta_{\sigma_1\sigma_2} \Delta_{\sigma_2}(x_{12}) + \delta_{\sigma_1\tilde{\sigma}_2} \delta(x_{12}]$. (1.68)

The dots on internal lines mean the amputation of argument, see Eq. (1.36); δ-symbols in the r.h.s. are defined by the condition

$$\sum_{\sigma'} \delta_{\sigma\sigma'} f(\sigma') = f(\sigma) .$$

Condition (1.68) holds for functions (1.31) when $g_\sigma = 1$, see Refs. [2,15]. The kernels of partial wave expansions G_σ are determined, on account of Eq. (1.68), by the relation

$G_\sigma(xx_1 \ldots x_m) =$ 

. (1.69)

On the internal lines δ-functions are placed.

To a contribution of each field Φ_m into expansion (1.21) corresponds a pole of the kernel G_σ at the point

$$\sigma = \sigma_m = (l_m, s_m) , \tag{1.70}$$

$$G^{(n)}_{\Phi_m}(xx_1 \ldots x_n) = \langle \Phi_m(x) \varphi(x_1) \ldots \varphi(x_n) \rangle = \Lambda_m \operatorname*{res}_{\sigma = \sigma_m} G_\sigma(xx_1 \ldots x_n) , \tag{1.71}$$

where Λ_m is known constant [2,21,22].

Note that due to amputation conditions, the kernels G_σ are invariant under the change $\sigma \to \tilde{\sigma}$

$$G_\sigma(xx_1 \dots x_n) = G_{\tilde{\sigma}}(xx_1 \dots x_n)\,.$$

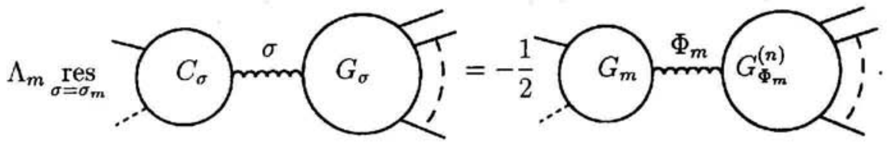
Each term $[\Phi_m]$ in the operator expansion (1.21) may be derived by the shift of the integration contour in Eq. (1.65) to the right up to the intersection of the pole in the point (1.70). Under this, one should use the symmetry of the integration contour in Eq. (1.71), the integrand under the changes $\sigma \to \tilde{\sigma}$ and the representations of functions C_σ in the form of Eq. (1.40). As the result, we have (see Refs. [2,21,22] for more details)

$$G(xyx_1 \dots x_n) \simeq \sum_{l_m, s_m} \left(- \frac{2\pi g_{\sigma_m}}{\sin \pi(l_m - D/2 + s_m)} \right) \int dz\, Q^{d\ \Delta\tilde{\sigma}_m}(xy|z) G^{(n)}_{\Phi_m}(zx_1 \dots x_n)\,, \tag{1.72}$$

where g_{σ_m} is the coupling constant:

$$\langle \varphi(x_1)\chi(x_2)\Phi_m(x_3)\rangle = g_{\sigma_m} C_{\sigma_m}(x_1x_2x_3)\,.$$

The contribution of each pole into partial wave expansion may be written as [2,21]



(1.73)

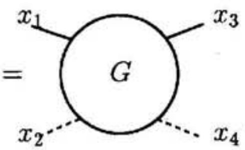
The r.h.s. of this formula contains the Green functions

$$G_m = \langle \varphi(x_1)\chi(x_2)\Phi_m(x_3)\rangle\,, \quad G^{(n)}_\Phi = \langle \Phi_m(x)\varphi(x_1) \dots \varphi(x_n)\rangle\,,$$

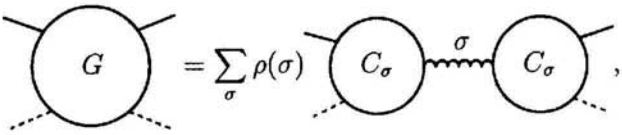
with the inner line corresponding to an inverse propagator Δ_m^{-1} of the Φ_m field

$$\Delta_m(x_{12}) = \langle \Phi_m(x_1)\Phi_m(x_2)\rangle\,.$$

The partial wave expansion of the function including four fields

$G(x_1x_2x_3x_4) = \langle\, \varphi(x_1)\chi(x_2)\varphi(x_3)\chi(x_4)\,\rangle =$ 

may be written as



(1.74)

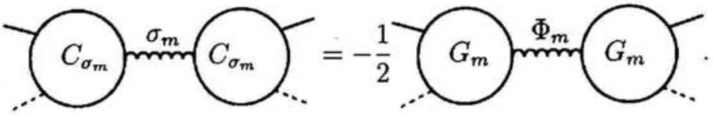
where the function $\rho(\sigma)$ is defined by

$$G_\sigma(xx_3x_4) = \rho(\sigma)C_\sigma(xx_3x_4)\,.$$

The function $\rho(\sigma)$ satisfies the equation

$$\rho(\sigma)=\rho(\tilde{\sigma})$$

and has the poles in the points (1.70). Eq. (1.73) in this case can be rewritten in the form

$\mu(\sigma_m)\left[\operatorname*{res}_{\sigma=\sigma_m}\rho(\sigma)\right]$ 

. (1.75)

Consider the partial wave expansion of Green functions which include a single vector or tensor field besides the scalar ones. To proceed we need explicit formulas for the other pair of conformally invariant three-point functions:

$$\langle \Phi^l_s \varphi \Phi^{l'}_\mu \rangle \quad \text{and} \quad \langle \Phi^l_s \varphi \Phi^{l'}_{\mu\nu} \rangle\,.$$

The Green function which includes the tensor $\Phi^l_{\mu_1\dots\mu_s}$ and the conformal vector $\Phi^{l'}_\mu$, has the form:

$$\begin{aligned}&\langle \Phi^l_{\mu_1\dots\mu_s}(x_1)\varphi_d(x_2)\Phi^{l'}_\mu(x_3)\rangle\\ &\quad=\Bigg\{A\lambda^{x_3}_\mu(x_1x_2)\lambda^{x_1}_{\mu_1\dots\mu_s}(x_2x_3)\\ &\qquad+B\frac{1}{x^2_{13}}\left[\sum_{k=1}^{s} g_{\mu\mu_k}(x_{13})\lambda^{x_1}_{\mu_1\dots\hat{\mu}_k\dots\mu_s}(x_2x_3)-\text{traces}\right]\Bigg\}\Delta^{ll'}_1(x_1x_2x_3)\,,\end{aligned}\tag{1.76}$$

where A and B are unknown constants, $\hat{\mu}_k$ means the omission of the index,

$$\Delta^{ll'}_1(x_1x_2x_3)=(x^2_{12})^{-(l+d-l'-s+1)/2}\,(x^2_{13})^{-(l+l'-d-s-1)/2}\,(x^2_{23})^{-(l'+d-l+s-1)/2}\,.\tag{1.77}$$

To derive Eq. (1.76), Eqs. (1.5) and (1.15) were used. The expression in curly braces provides an example of typical tensor structures. It does not depend on field dimensions. We will use the notation $\{A, B\}$ for such structures:

$$\langle \Phi^l_{\mu_1\dots\mu_s}(x_1)\varphi(x_2)\Phi^{l'}_\mu(x_3)\rangle=\{A,B\}\Delta^{ll'}_1(x_1x_2x_3).\tag{1.78}$$

As another example, consider the Green function that includes the tensors $\Phi^l_{\mu_1\dots\mu_s}$ and $\Phi^{l'}_{\mu\nu}$. It contains three terms

$$\begin{aligned}&\langle \Phi^l_{\mu_1\dots\mu_s}(x_1)\varphi(x_2)\Phi^{l'}_{\mu\nu}(x_3)\rangle\\ &\quad=\Bigg\{A\lambda^{x_3}_{\mu\nu}(x_1x_2)\lambda^{x_1}_{\mu_1\dots\mu_s}(x_2x_3)+B\Bigg[\lambda^{x_3}_\mu(x_1x_2)\frac{1}{x^2_{13}}\left(\sum_{k=1}^{s} g_{\nu\mu_k}(x_{13})\lambda^{x_1}_{\mu_1\dots\hat{\mu}_k\dots\mu_s}(x_2x_3)-\text{traces}\right)\\ &\qquad+(\mu\leftrightarrow\nu)-\text{trace in }\mu,\nu\Bigg]\\ &\qquad+C\left[\frac{1}{(x^2_{13})^2}\sum_{k,r=1}^{s} g_{\mu\mu_k}(x_{13})g_{\nu\mu_r}(x_{13})\lambda^{x_1}_{\mu_1\dots\hat{\mu}_k\dots\hat{\mu}_r\dots\mu_s}(x_2x_3)-\text{traces}\right]\Bigg\}\Delta^{ll'}_2(x_1x_2x_3)\,,\end{aligned}\tag{1.79}$$

where A, B, C are unknown constants,

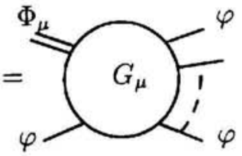
$$\Delta_2^{ll'}(x_1x_2x_3) = (x_{12}^2)^{-(l+d-l'-s+2)/2}\,(x_{13}^2)^{-(l+l'-d-s-2)/2}\,(x_{23}^2)^{-(l'+d-l+s-2)/2}\,. \tag{1.80}$$

It is convenient to introduce the notation analogous to Eq. (1.78)

$$\langle \Phi^l_{\mu_1\ldots\mu_s}(x_1)\varphi(x_2)\Phi^{l'}_{\mu\nu}(x_3)\rangle = \{A, B, C\}\Delta_2^{ll'}(x_1x_2x_3)\,. \tag{1.81}$$

Consider partial wave expansion of the Green function

$$G_\mu(xyx_1\ldots x_n) = \langle \Phi^{l_1}_\mu(x)\varphi(y)\varphi(x_1)\ldots\varphi(x_n)\rangle =$$

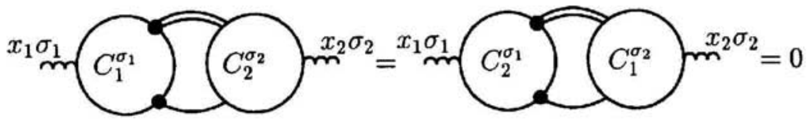


(1.82)

including the vector field $\Phi^{l_1}_\mu$ of dimension l_1. In this case the partial wave expansion is made up by a pair of terms of the type (1.65), since there are two independent invariant functions

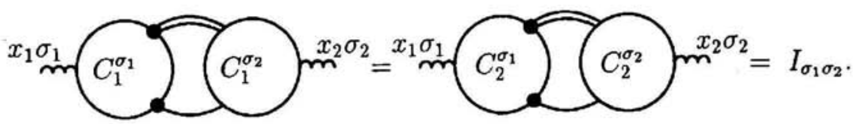
$$C_1^\sigma \doteq C^\sigma_{1\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \{A_1^{(1)}, A_2^{(1)}\}\Delta_1^{ldl_1}(x_1x_2x_3)\,,$$
$$C_2^\sigma = C^\sigma_{2\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \{A_1^{(2)}, A_2^{(2)}\}\Delta_1^{ldl_1}(x_1x_2x_3)\,. \tag{1.83}$$

It is useful to choose coefficients $A^{(i)}_{1,2}$, $i = 1, 2$ in such a way as to make the functions (1.83) mutually orthogonal



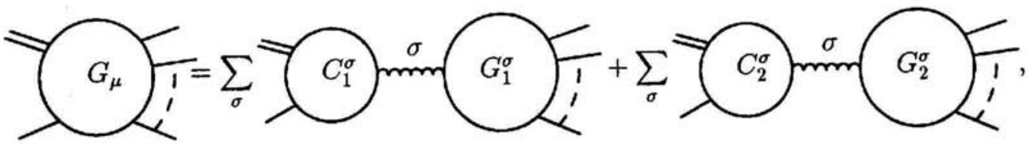
(1.84)

and normalized by the condition (1.68).



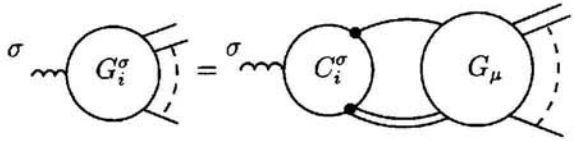
(1.85)

Then the partial wave expansion for the Green function (1.82) may be written in the form [2,15]



(1.86)

where the kernels $G^{\sigma}_{1,2}$ are determined, according to Eqs. (1.84) and (1.85), by the equations

$G^{\sigma}_{i}(xx_1\ldots x_n) =$ 

$, i = 1, 2.$ (1.87)

Each pole of kernels $G^{\sigma}_{1,2}$ corresponds to a field which contributes to operator product expansion $\Phi_\mu(x)\varphi(0)$.

There are two mutually orthogonal sets of fields

$$\{\Phi_{1m}\} \quad \text{and} \quad \{\Phi_{2m'}\} \tag{1.88}$$

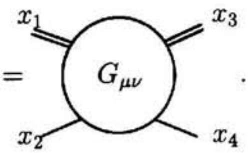
contributing to this operator expansion

$$\Phi_\mu(x)\varphi(0) = \sum_m\,[\Phi_{1,m}] + \sum_{m'}\,[\Phi_{2,m'}]\,. \tag{1.89}$$

The Green functions of the fields $\Phi_{1,m}$ and $\Phi_{2,m'}$ are determined by the equations

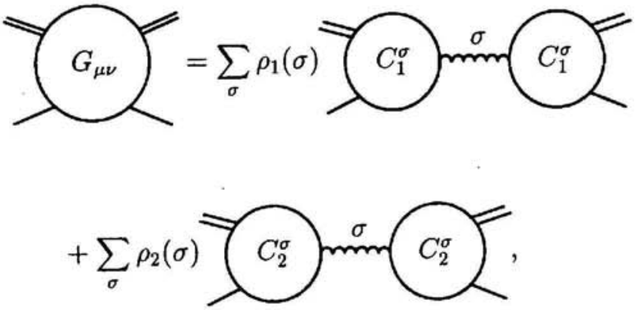
$$\langle \Phi_{i,m}(x)\varphi(x_1)\ldots\varphi(x_n)\rangle = \Lambda_{i,m}\,\underset{\sigma=\sigma_m}{\text{res}}\;G^{\sigma}_{i}\,(xx_1\ldots x_n)\,,\quad i=1,2\,. \tag{1.90}$$

The orthogonality of the fields $\Phi_{1,m}$ and $\Phi_{2,m'}$ may be proved by an analysis of the expansion for 4-point Green function:

$G_{\mu\nu}(x_1x_2x_3x_4) = \langle\, \Phi^{l_1}_{\mu}(x_1)\varphi(x_2)\Phi^{l_1}_{\nu}(x_3)\varphi(x_4)\,\rangle =$ 

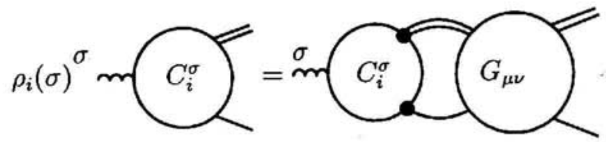
. (1.91)

Let us put its expansion in the form



(1.92)

where the functions $\rho_{1,2}(\sigma)$ are determined by the equations



$$, \quad i = 1,2. \tag{1.93}$$

Note that the diagonality condition for the expansion (1.92) together with the orthonormality conditions (1.84) and (1.85) fix the coefficients $A^{(i)}_{1,2}$ in Eq. (1.83) up to SO(2) transformation. Thus, the choice of combinations of independent invariant structures in Eq. (1.83) is possible only on the base of analysis of the Green function (1.91). The meaning of this result [2,3,15] may be easily understood, considering the Wightman function

$$\langle 0|\varphi(x_2)\, \Phi^{l_1}_\mu(x_1)\, \Phi^{l_1}_\nu(x_3)\, \varphi(x_4)|0\rangle\,, \tag{1.94}$$

which represents an invariant scalar product of the states

$$\Phi^{l_1}_\mu(x_1)\varphi(x_2)|0\rangle\,, \quad \Phi^{l_1}_\nu(x_3)\varphi(x_4)|0\rangle\,. \tag{1.95}$$

Apparently, *all* the fields, contributing into the vacuum operator expansion of these states, are represented by the sets (1.88). If among these fields there is a pair of fields $\tilde{\Phi}_{1,m}$ and $\tilde{\Phi}_{2,m'}$ with identical dimensions and tensor ranks, one can always choose such their combinations $\Phi_{1,m} = \alpha\tilde{\Phi}_{1,m} + \beta\tilde{\Phi}_{2,m'}$ and $\Phi_{2,m'} = \gamma\tilde{\Phi}_{1,m} + \delta\tilde{\Phi}_{2,m'}$, that the states

$$\Phi_{1,m}(x_1)|0\rangle \quad \text{and} \quad \Phi_{2,m'}(x_2)|0\rangle \tag{1.96}$$

will be orthogonal to each other:

$$\langle 0|\Phi_{1,m}(x_1)\, \Phi_{2,m'}(x_2)|0\rangle = 0\,. \tag{1.97}$$

Thus *the states* (1.95) *represent a direct sum of mutually orthogonal subspaces*:

$$H_1 \oplus H_2 \tag{1.98}$$

which belong to the total Hilbert space of the theory. The states $\Phi_{1,m}(x)|0\rangle$ span the basis of H_1 space

$$\Phi_{1,m}(x)|0\rangle \subset H_1 \quad \text{for all } m\,. \tag{1.99}$$

In analogy,

$$\Phi_{2,m'}(x)|0\rangle \subset H_2 \quad \text{for all } m'\,. \tag{1.100}$$

Correspondingly, in Euclidean version of the operator product expansion, the orthogonality of H_1 and H_2 manifests itself in the diagonal form of the partial wave expansion (1.92), with the Euclidean Green functions $\langle \Phi_{1,m}\, \Phi_{2,m'}\rangle$ being equal to zero

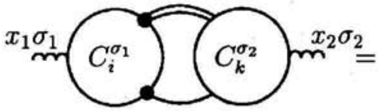
$$\langle \Phi_{1,m}(x_1)\Phi_{2,m'}(x_2)\rangle = 0$$

even when quantum numbers of fields $\Phi_{1,m}$ and $\Phi_{2,m'}$ coincide.

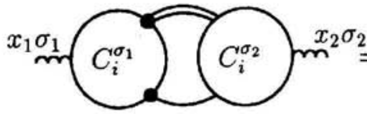
By analogy, one can consider partial wave expansions for the Green functions which include the tensor field $\Phi^{l_1}_{\mu\nu}$. It contains three different terms, since there exist three independent invariant functions (see Eq. (1.81)):

$$C_i^\sigma = C^\sigma_{i\ \mu\nu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \{A_1^{(i)}, A_2^{(i)}, A_3^{(i)}\}\Delta_2^{ldl_1}(x_1x_2x_3)\,, \tag{1.101}$$

where $i = 1, 2, 3$. It is useful to choose the coefficients $A^{(i)}_{1,2,3}$ from the condition



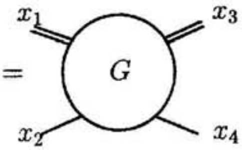
$$= 0 \quad \text{for } i \neq k, \tag{1.102}$$



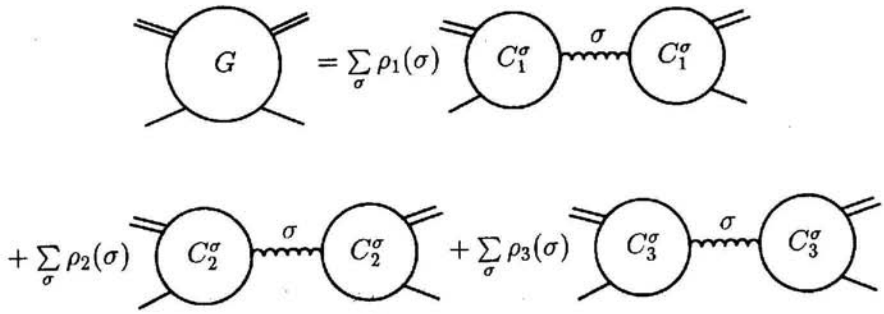
$$= I_{\sigma_1\sigma_2}, \quad i = 1, 2, 3. \tag{1.103}$$

This does not yet fix the $A^{(i)}_{1,2,3}$ coefficients uniquely. Let us demand the expansion for the Green function

$$G_{\mu\nu,\rho\sigma}(x_1x_2x_3x_4) = \langle\, \Phi_{\mu\nu}(x_1)\varphi(x_2)\Phi_{\rho\sigma}(x_3)\varphi(x_4)\,\rangle = \tag{1.104}$$



to be diagonal [3,15]



(1.105)

This fixes the choice of orthonormal basis of functions C_i^σ, $i = 1, 2, 3$, up to a SO(3) transformation. To make the choice of these functions unique, additional physical arguments are necessary, see Section 2.

In the general case, *there exist three mutually orthogonal sets of fields* [3,15]

$$\{\Phi_{1,m}\}\,,\quad \{\Phi_{2,r}\}\,,\quad \{\Phi_{3,k}\}\,, \tag{1.106}$$

contributing to the operator product expansion $\Phi^{l_1}_{\mu\nu}\varphi$:

$$\Phi^{l_1}_{\mu\nu}(x)\varphi(0) = \sum_m [\Phi_{1,m}] + \sum_r [\Phi_{2,r}] + \sum_k [\Phi_{3,k}]\,. \tag{1.107}$$

The orthogonality conditions

$$\langle \Phi_{1,m}(x_1)\,\Phi_{2,r}(x_2)\rangle = \langle \Phi_{1,m}(x_1)\Phi_{3,k}(x_2)\rangle = \langle \Phi_{2,r}(x_1)\Phi_{3,k}(x_2)\rangle = 0 \tag{1.108}$$

hold even when quantum number of two fields from different sets coincide. The states

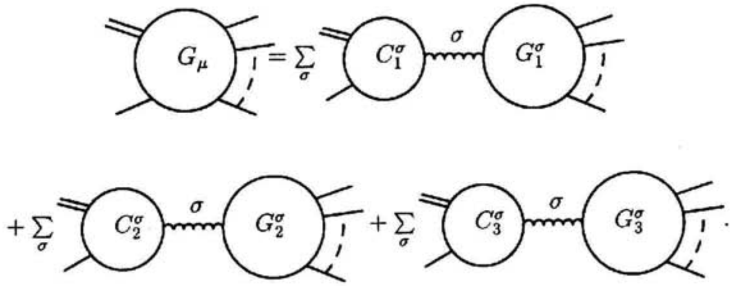
$$\Phi^{l_1}_{\mu\nu}(x_1)\varphi(x_2)|0\rangle \tag{1.109}$$

span the *sector of a Hilbert space which can be represented as a direct sum of three orthogonal subspaces*:

$$H_1 \oplus H_2 \oplus H_3, \tag{1.110}$$

$$\Phi_{1,m}(x)|0\rangle \subset H_1 \text{ for all } m, \quad \Phi_{2,r}(x)|0\rangle \subset H_2 \text{ for all } r, \quad \Phi_{3,k}(x)|0\rangle \subset H_3 \text{ for all } k. \tag{1.111}$$

The expansion of higher Green functions could be discussed no sooner than all the three independent sets of functions (1.101) are found. Let us stress that the requirement of diagonality of expansion (1.105) should be treated as one of the conditions that fix the form of these functions. Resultantly, we have [3,15]:



(1.112)

It is essential that the expansions of Green functions

$$\langle \Phi^{l_1}_{\mu\nu}\varphi\Phi_1\Phi_2 \ldots \Phi_n\rangle \tag{1.113}$$

with any number of fields $\Phi_1 \ldots \Phi_n$ of any tensor structure, also have the form (1.112). The latter is quite apparent from the above analysis. Besides that, one can show that this property is the consequence [2,12,15] of exact solution to renormalized Schwinger–Dyson system.

Finally, let us consider several consequences of operator product expansions for the Green function

$$\langle \varphi(x_1)\chi(x_2)\Phi_1(x_3)\Phi_2(x_4)\rangle\,, \tag{1.114}$$

where Φ_1, Φ_2 are any fields. Its asymptotic behaviour in each of the two regions

$$1.\ x_{12} \to 0, \quad \text{i.e. } x_{12}^2 \ll x_{13}^2, x_{14}^2 , \tag{1.115}$$

$$2.\ x_{34} \to 0, \quad \text{i.e. } x_{34}^2 \ll x_{13}^2, x_{14}^2 , \tag{1.116}$$

is determined by one of the operator produce expansions

$$\varphi(x_1)\chi(x_2)|_{x_{12}\to 0} \simeq \sum_m [\Phi_m] , \tag{1.117}$$

$$\Phi_1(x_3)\Phi_2(x_4)|_{x_{34}\to 0} \simeq \sum_k [\Phi'_k] . \tag{1.118}$$

On the other side, two sets of fields

$$\{\Phi_r\} \quad \text{and} \quad \{\Phi'_k\} \tag{1.119}$$

do not generally coincide. Each field $\Phi(x)$, contributing to the asymptotic region (1.115) should also contribute to the asymptotics (1.116). The latter is caused by the orthogonality property of conformal fields

$$\langle \Phi_r(x)\Phi'_k(x')\rangle \neq 0 \quad \text{only if } l_r = l'_k,\ s_r = s'_k .$$

Hence two sets of fields (1.119) should have an intersection $\{\tilde{\Phi}\}$ which consists of the fields $\tilde{\Phi}_r$ and $\tilde{\Phi}'_k$ with the same quantum numbers:

$$\{\tilde{\Phi}\} \in \{\Phi_r\}, \quad \{\tilde{\Phi}\} \in \{\Phi'_k\} . \tag{1.120}$$

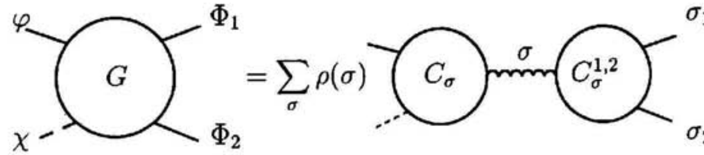
If such an intersection is empty, then the Green function (1.114) is zero. The states of a Hilbert space

$$\varphi(x_1)\chi(x_2)|0\rangle , \quad \Phi_1(x_3)\Phi_2(x_4)|0\rangle$$

are orthogonal:

$$\langle 0|\varphi(x_1)\chi(x_2)\Phi_1(x_3)\Phi_2(x_4)|0\rangle = 0 .$$

This statement can be formulated in terms of partial wave expansions in the following manner: the poles of kernel $\rho(\sigma)$ of the expansion



(1.121)

correspond to the fields $\tilde{\Phi}$ that belong to an intersection of sets (1.119).

The most interesting consequences of this statement may be obtained from the analysis of Green functions for the energy–momentum tensor or the current. Examining the Green functions

$$\langle T_{\mu\nu}\varphi T_{\rho\sigma}\varphi^+\rangle \quad \text{or} \quad \langle j_\mu\varphi j_\nu\varphi^+\rangle ,$$

one can find the operator product expansions

$$T_{\mu\nu}(x_1)\varphi(x_2) \quad \text{or} \quad j_\mu(x_1)\varphi(x_2)\,. \tag{1.122}$$

Suppose that the fields contributing to these expansions are found. Then one can expect that the latter contribute, as well, to the operator product expansion of fundamental fields

$$\varphi(x_1)\chi(x_2)\,, \tag{1.123}$$

since the Green functions

$$\langle\varphi\chi\varphi^+ j_\mu\rangle \quad \text{and} \quad \langle\varphi\chi\varphi^+ T_{\mu\nu}\rangle$$

are non-zero. Thus *from the analysis of the conformally invariant solution of Ward identities one can find some of the operator contributions into operator product expansions of fundamental fields.* This is done in Sections 2–4.

2. Conformally invariant solution of the Ward identities

2.1. Definition of conserved currents and energy–momentum tensor in Euclidean conformal field theory

As already mentioned in Section 1.3 to the current, energy–momentum tensor and their conformal partners, undecomposable representations of the conformal group correspond. The latter belong to the so-called representations in exceptional points (1.43), (1.44), studied in Refs. [21,26].

Denote any undecomposable representation as Q. Let M be a space of such representation. The characteristic property of these representations is that in the space M there exists a subspace which is invariant under the action of group transformations:

$$M_0 \subset M\,. \tag{2.1}$$

In the case of current and energy–momentum tensor the subspace M_0 consists of transversal Euclidean conformal fields j^{tr}_μ, $T^{\text{tr}}_{\mu\nu}$

$$\partial_\mu j^{\text{tr}}_\mu(x) = 0\,, \quad \partial_\nu T^{\text{tr}}_{\mu\nu}(x) = 0\,. \tag{2.2}$$

Indeed, consider the transformation laws of the fields $j_\mu(x)$ and $T_{\mu\nu}(x)$ with respect to the conformal inversion (see Eqs. (1.15), (1.43)):

$$j_\mu(x) \overset{R}{\rightarrow} j'_\mu(x) = \frac{1}{(x^2)^{D-1}}\, g_{\mu\nu}(x)\, j_\nu(Rx)\,, \tag{2.3}$$

$$T_{\mu\nu}(x) \overset{R}{\rightarrow} T'_{\mu\nu}(x) = \frac{1}{(x^2)^D}\, g_{\mu\rho}(x)\, g_{\nu\sigma}(x)\, T_{\rho\sigma}(Rx)\,. \tag{2.4}$$

As may be easily demonstrated by a direct check, after transformations (2.3) and (2.4), the fields

$$j^{\text{tr}}_\mu(x)\,, \quad T^{\text{tr}}_{\mu\nu}(x) \tag{2.5}$$

beget the fields j'^{tr}_μ, $T'^{\text{tr}}_{\mu\nu}$ that are also transversal:

$$\partial^x_\mu j_\mu(x) = 0 \overset{R}{\rightarrow} \partial^x_\mu j'^{\text{tr}}_\mu(x) = 0\,, \quad \partial^x_\nu T_{\mu\nu}(x) = 0 \overset{R}{\rightarrow} \partial^x_\nu T'^{\text{tr}}_{\mu\nu}(x) = 0\,. \tag{2.6}$$

An analogous result is valid for the conformal partners (1.47). The invariant subspace M_0 in this case consists of the longitudinal fields

$$A_\mu^{\text{long}}(x) = \partial_\mu \varphi_0(x)\,, \tag{2.7}$$

$$h_{\mu\nu}^{\text{long}}(x) = \partial_\mu h_\nu(x) + \partial_\nu h_\mu(x) - \frac{2}{D}\,\delta_{\mu\nu}\partial_\lambda h_\lambda(x)\,, \tag{2.8}$$

where $\varphi_0(x)$ is the scalar fields of dimension $d_A = 0$, $h_\nu(x)$ is the conformal vector of dimension $d_h = -1$. As may be easily checked, under the action of conformal inversion

$$A_\mu(x) \xrightarrow{R} A'_\mu(x) = (1/x^2) g_{\mu\nu}(x)\, A_\nu(Rx)\,, \tag{2.9}$$

$$h_{\mu\nu}(x) \xrightarrow{R} h'_{\mu\nu}(x) = g_{\mu\rho}(x)\, g_{\nu\sigma}(x)\, h_{\rho\sigma}(Rx) \tag{2.10}$$

the conformal fields (2.7) and (2.8) transform to longitudinal fields

$$A'^{\text{long}}_\mu(x) = \partial_\mu^x\, \varphi'_0(x)\,, \tag{2.11}$$

$$h'^{\text{long}}_{\mu\nu}(x) = \partial_\mu^x\, h'_\nu(x) + \partial_\nu h'_\mu(x) - \frac{2}{D}\,\delta_{\mu\nu}\partial_\lambda h'_\lambda(x), \tag{2.12}$$

where

$$\varphi'_0(x) = \varphi_0(Rx)\,, \quad h'_\mu(x) = x^2 g_{\mu\nu}(x) h_\nu(Rx)\,. \tag{2.13}$$

To prove the results (2.6), (2.11) and (2.12), we have employed the relation

$$\partial_\mu^x = (1/x^2)\, g_{\mu\nu}(x)\partial_\nu^{Rx}\,. \tag{2.14}$$

More detailed calculations may be found in Ref. [15]. Thus in the case of fields A_μ, $h_{\mu\nu}$ the invariant subspace M_0 consists of longitudinal fields (2.7) and (2.8), respectively.

As it is known from the group theory, any undecomposable representation may be coupled to a pair of irreducible representations:

$$Q_0,\quad \tilde{Q}\,. \tag{2.15}$$

The representation Q_0 acts in the invariant subspace M_0, while the other, in the quotient space

$$\tilde{M} = M/M_0\,. \tag{2.16}$$

Correspondingly, one could consider a pair of different conformal fields, the first being transformed by the representation Q_0, and the second, by the representation $\tilde{Q}$. Hence, there exist two types of conformal fields with canonical dimensions (1.43), and two types of conformal partners with dimensions (1.44). This means that Euclidean conformal field theory comprises two types of currents and two types of potentials:

$$j_\mu^{\text{tr}}, \tilde{j}_\mu \quad \text{and} \quad \tilde{A}_\mu, A_\mu^{\text{long}}\,. \tag{2.17}$$

Each of these fields transforms by an irreducible representation of the conformal group. Similarly, one has a pair of types of irreducible conformal fields for each of $T_{\mu\nu}$ and $h_{\mu\nu}$:

$$T^{\rm tr}_{\mu\nu},\ \tilde{T}_{\mu\nu} \quad \text{and} \quad \tilde{h}_{\mu\nu},\ h^{\rm long}_{\mu\nu}\,. \tag{2.18}$$

Furthermore, one can consider the fields j_μ, $T_{\mu\nu}$ or A_μ, $h_{\mu\nu}$ which transform by direct sums of irreducible representations (2.15):

$$Q_0 \oplus \tilde{Q}\,. \tag{2.19}$$

It is important that for the fields with canonical dimension the conditions of equivalence of representations are different from Eq. (1.29). The latter are substituted by the conditions of partial equivalence [21,26] which manifest themselves independently among the representations of the type Q_0, and among the representations of the type $\tilde{Q}$. This means that the relations of type (1.29) hold independently both in the transversal and in longitudinal sectors.

From the above arguments it follows that different possible definitions of the conserved currents and the energy–momentum tensor in the conformal theory may be given. A more comprehensive discussion of these definitions is given in Ref. [15] (see also the next sections of the present article).

2.2. The Green functions of the current

For simplicity we shall discuss the case of an Abelian theory in the space of dimension $D \geq 3$. All the results obtained below are easily generalized to the case of non-Abelian theories.

Introduce the notation Q_j for an irreducible representation defined by the transformation law (2.3) and acting in the space M_j. According to Eq. (2.15), there exists a pair of irreducible representations

$$\tilde{Q}_j \quad \text{and} \quad Q^{\rm tr}_j \tag{2.20}$$

corresponding to irreducible conformal fields

$$\tilde{j}_\mu(x) \quad \text{and} \quad j^{\rm tr}_\mu(x)\,, \quad \partial^x_\mu j^{\rm tr}_\mu(x) = 0\,. \tag{2.21}$$

The space of irreducible representations (2.20) will be denoted as

$$\tilde{M}_j = M_j/M^{\rm tr}_j, \quad M^{\rm tr}_j \subset M_j\,. \tag{2.22}$$

By analogy, Q_A denotes an undecomposable representation defined by the transformation law (2.9) and acting in the space M_A. For a pair of irreducible representations

$$Q^{\rm long}_A \quad \text{and} \quad \tilde{Q}_A \tag{2.23}$$

the pair of irreducible conformal fields

$$A^{\rm long}_\mu(x) = \partial_\mu \varphi_0(x)\,, \quad \tilde{A}_\mu(x) \tag{2.24}$$

correspond. The notation

$$M^{\rm long}_A \subset M_A\,, \quad \tilde{M}_A = M_A/M^{\rm long}_A \tag{2.25}$$

will stand for the spaces of irreducible representations (2.23).

The representations (2.20) and (2.23) are pairwise equivalent [21,26], see also Refs. [27–29]:

$$\tilde{Q}_j \sim Q_A^{\text{long}}, \quad Q_j^{\text{tr}} \sim \tilde{Q}_A . \tag{2.26}$$

The equivalence conditions are expressed by the operator relations

$$A_\mu^{\text{long}}(x) \sim \int \mathrm{d}y\, D_{\mu\nu}^{\text{long}}(x-y)\tilde{j}_\nu(y), \quad j_\mu^{\text{tr}}(x) = \int \mathrm{d}y\, D_{\mu\nu}^{\text{tr}}(x-y)\tilde{A}_\nu(y), \tag{2.27}$$

where the kernels $D_{\mu\nu}^{\text{long}}$ and $D_{\mu\nu}^{\text{tr}}$ coincide with the invariant propagators

$$D_{\mu\nu}^{\text{long}}(x_{12}) = \langle A_\mu^{\text{long}}(x_1) A_\nu^{\text{long}}(x_2)\rangle = C_A g_{\mu\nu}(x_{12})\left(\tfrac{1}{2}x_{12}^2\right)^{-1} = C_A \partial_\mu^{x_1}\partial_\nu^{x_1} \ln x_{12}^2 , \tag{2.28}$$

C_A is the normalization constant, and

$$D_{\mu\nu}^{\text{tr}}(x_{12}) = \langle j_\mu^{\text{tr}}(x_1) j_\nu^{\text{tr}}(x_2)\rangle, \quad \partial_\nu^{x_1} D_{\mu\nu}^{\text{tr}}(x_{12}) = 0 . \tag{2.29}$$

The explicit form of the expression of the conformally invariant propagator $D_{\mu\nu}^{\text{tr}}$ depends on the dimension of the space. Introduce the regularized conformal fields j_μ^ε and A_μ^ε with the dimensions

$$l_j^\varepsilon = D - 1 + \varepsilon , \quad l_A^\varepsilon = D - l_j^\varepsilon = 1 - \varepsilon . \tag{2.30}$$

The regularized propagator of the current

$$D_{\mu\nu}^{j^\varepsilon}(x_{12}) = \langle j_\mu^\varepsilon(x_1) j_\nu^\varepsilon(x_2)\rangle = \tilde{C}_j \frac{1}{\left(\frac{1}{2}x_{12}^2\right)^{D-1+\varepsilon}} g_{\mu\nu}(x_{12}), \tag{2.31}$$

where $\tilde{C}_j$ is the normalization constant is divergent when $\varepsilon \to 0$ for even values of D:

$$D_{\mu\nu}^{j^\varepsilon}(x_{12}) \sim (\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\frac{1}{\left(\frac{1}{2}x_{12}^2\right)^{D-2+\varepsilon}} + \mathrm{O}(1) . \tag{2.32}$$

The factor $(x_{12}^2)^{-D+2-\varepsilon}$ is singular for even $D \geq 4$. Thus we set

$$D_{\mu\nu}^{\text{tr}}(x_{12}) = \begin{cases} (\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\dfrac{1}{(x_{12}^2)^{D-2}}, & D\text{-odd}, \\ \lim_{\varepsilon\to 0} \varepsilon D_{\mu\nu}^{j^\varepsilon}(x_{12}) \sim (\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\Box^{(D-4)/2}\,\delta(x_{12}), & D\text{-even} . \end{cases} \tag{2.33}$$

Here we utilized the relation [30]

$$\left.\frac{1}{(x^2)^{D/2+k+\varepsilon}}\right|_{\varepsilon\to 0} \simeq -\frac{1}{\varepsilon}\frac{\pi^{D/2}}{\Gamma(D/2+k)}\frac{4^{-k}}{\Gamma(k+1)}\Box^k\delta(x) . \tag{2.34}$$

Thus, for even $D \geq 4$ one has a pair of invariant functions $D_{\mu\nu}^{j^\varepsilon}$ and $D_{\mu\nu}^{\text{tr}}$. We shall show in Section 6 that these functions are related to a pair of irreducible representations $\tilde{Q}_j$ and Q_j^{tr} and define the propagators of irreducible fields $\tilde{j}_\mu$ and j_μ^{tr}. In the case of odd D the situation is analogous, and will not be discussed here.

Note that when $D = 4$, Eq. (2.27) coincide with the equations for electromagnetic field in the α-gauge:

$$(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\tilde{A}_\nu(x) = j_\mu^{\text{tr}}(x), \quad \tilde{A}_\mu^{\text{long}}(x) = \alpha\partial_\mu^x \int \mathrm{d}y \ln(x-y)^2 \partial_\nu \tilde{j}_\nu(y) . \tag{2.35}$$

Let us stress that in conformal theory these equations are the consequences [27–29] of the equivalence conditions for irreducible representations (2.26).

The usual equivalence conditions may be written in any of the forms:

$$\Phi_\sigma(x) = \int dy\, \Delta_\sigma(x-y)\Phi_{\tilde\sigma}(y) \quad \text{or} \quad \Phi_{\tilde\sigma} = \int dy\, \Delta_{\tilde\sigma}(x-y)\Phi_\sigma(y)\,. \tag{2.36}$$

Unlike this, Eq. (2.27) cannot be inverted since the kernels $D^{\rm long}_{\mu\nu}$, $D^{\rm tr}_{\mu\nu}$ are degenerate. The group-theoretic reason behind this consists in the fact that the elements of the spaces $M^j/M^{\rm tr}_j$ and $M_A/M^{\rm long}_A$ are the equivalence classes. Each equivalence class $\{\tilde j_\mu\}$,

$$\{\tilde j_\mu\} \subset M_j/M^{\rm tr}_j \tag{2.37}$$

includes a set of functions with different transversal parts. In particular, if the field $\tilde j_\mu \subset \{\tilde j_\mu\}$, then the field $\tilde j'_\mu = \tilde j_\mu + \tilde j^{\rm tr}_\mu$ also belongs to the same class

$$\tilde j_\mu \subset \{\tilde j_\mu\} \to \tilde j'_\mu = \tilde j_\mu + \tilde j^{\rm tr}_\mu \subset \{\tilde j_\mu\}\,, \tag{2.38}$$

where $\tilde j^{\rm tr}_\mu$ is an arbitrary transversal field. Analogously, each equivalence class

$$\{\tilde A_\mu\} \subset M_A/M^{\rm long}_A \tag{2.39}$$

consists of the set of functions with different longitudinal parts:

$$\text{if } \tilde A_\mu \subset \{\tilde A_\mu\}, \quad \text{then} \quad \tilde A_\mu + \tilde A^{\rm long}_\mu \subset \{\tilde A_\mu\}\,, \tag{2.40}$$

where $\tilde A^{\rm long}_\mu$ is an arbitrary longitudinal field. Thus any representatives of the equivalence classes $\{\tilde j_\mu\}$ and $\{\tilde A_\mu\}$ can enter the corresponding r.h.s. of Eq. (2.27), see Section 6 for more details.

As an example let us consider the Green function of the current

$$G^j_\mu(x_1x_2x_3x_4) = \langle \varphi(x_1)\varphi^+(x_2)\chi(x_3)\, j_\mu(x_4)\rangle\,, \tag{2.41}$$

where $\chi(x)$ is a neutral scalar field of dimension Δ. The general conformally invariant solution of the Ward identity

$$\partial^{x_4}_\mu G^j_\mu(x_1x_2x_3x_4) = -[\delta(x_{14}) - \delta(x_{24})]\langle \varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle \tag{2.42}$$

can be written as [2,15]

$$G^j_\mu(x_1x_2x_3x_4) = K^{x_4}_\mu(x_1x_2)\langle \varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle + G^{\rm tr}_\mu(x_1x_2x_3x_4)\,, \tag{2.43}$$

where $G^{\rm tr}_\mu$ is an arbitrary transversal conformally invariant function

$$\partial^{x_4}_\mu G^{\rm tr}_\mu(x_1x_2x_3x_4) = 0\,, \tag{2.44}$$

and the function K_μ has the form

$$K^{x_4}_\mu(x_1x_2) = \frac{1}{2}\pi^{-D/2}\,\Gamma\!\left(\frac{D}{2}\right)\left(\frac{x^2_{12}}{x^2_{14}x^2_{24}}\right)^{(D-2)/2}\lambda^{x_4}_\mu(x_1x_2)\,, \tag{2.45}$$

$$\partial^{x_4}_\mu K^{x_4}_\mu(x_1x_2) = -\delta(x_{14}) + \delta(x_{24})\,.$$

The set of functions differing by their longitudinal parts may be viewed as a definite equivalence class (2.37). Introduce the conformally invariant function [5]

$$G_\mu^{A^{\rm long}}(x_1x_2x_3x_4) = \langle \varphi(x_1)\varphi^+(x_2)\chi(x_3)A_\mu^{\rm long}(x_4)\rangle = g_A\lambda_\mu^{x_4}(x_1x_2)\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle$$
$$= \frac{1}{2}g_A\partial_\mu^{x_4}\ln\frac{x_{14}^2}{x_{24}^2}\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle\,, \tag{2.46}$$

where g_A is the coupling constant. According to the first equality in Eq. (2.27) we have

$$G_\mu^{A^{\rm long}}(x_1x_2x_3x_4) = \int dx\, D_{\mu\nu}^{\rm long}(x_4-x)G_\nu^j(x_1x_2x_3x)\,. \tag{2.47}$$

On the other hand, the set of transversal conformally invariant functions $G_\mu^{\rm tr}$, see Eq. (2.44), forms a space of irreducible representation $Q_j^{\rm tr}$

$$G_\mu^{\rm tr} \subset M_j^{\rm tr}\,. \tag{2.48}$$

Due to the second equality in Eq. (2.27) one has

$$G_\mu^{\rm tr}(x_1x_2x_3x_4) = \int dx\, D_{\mu\nu}^{\rm tr}(x_4-x)\,\tilde{G}_\nu^A(x_1x_2x_3x)\,, \tag{2.49}$$

where

$$\tilde{G}_\nu^A(x_1x_2x_3x) = \langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\tilde{A}_\nu(x)\rangle \tag{2.50}$$

is an arbitrary conformally invariant function of the field $\tilde{A}_\mu(x)$.

It is natural that the general conformally invariant solution of the Ward identities includes the contribution of electromagnetic interaction described by the field $\tilde{A}_\mu$ .What is problematic is how this contribution could be extracted explicitly. The latter is especially important when one concerns the models neglecting electromagnetic interaction. This problem is a complicated task in the case of conformally invariant theory. The longitudinal part of the first term in Eq. (2.43) is not conformally invariant. The invariance could be achieved only after a certain transversal (also non-invariant) correction to the longitudinal part is added. The latter is defined up to an arbitrary conformally invariant function which could be added to the second term in Eq. (2.43). So the question on the electromagnetic contribution into the first term remains open. Similarly, extracting the contribution of the gravitational interaction poses the problem in the case of Green functions of the energy–momentum tensor, see Section 2.5.

The mathematical origin of this problem is concealed in the fact that the representations due to the transformation laws (2.3) are undecomposable. Hence the first step should consist in the transition to the direct sum of irreducible representations (2.15):

$$Q \to \tilde{Q} \oplus Q_0\,. \tag{2.51}$$

The representations of the type Q_0 are related to the contributions of gauge fields, while the representations of the type $\tilde{Q}$ are related to the contributions of the matter fields. However, the representations $\tilde{Q}$ are defined in the space of equivalence classes. Thus, from the mathematical viewpoint, the problem of extracting of the unique solution to the Ward identities (in the absence of gauge interactions) is equivalent to the formulation of definite prescription which would uniquely fix the choice of representative in each equivalence class.

The above prescription could be formulated on the ground of the following arguments. Represent the Euclidean current $j_\mu(x)$ which transforms by the direct sum of representations (2.51), as a sum of two terms

$$j_\mu(x) = \tilde{j}_\mu(x) + j_\mu^{\rm tr}(x)\,. \tag{2.52}$$

This decomposition is conformally invariant. It becomes unique if the following condition is satisfied: the states of the Hilbert space generated by the fields $\tilde{j}_\mu$ and $j_\mu^{\rm tr}$ are mutually orthogonal. The physical meaning of the first term is discussed in detail in Section 6.1. The second term in Eq. (2.52) corresponds to the contribution of electromagnetic interaction and is related to the field $\tilde{A}_\mu$ due to the second of Eq. (2.27). One can show that the above requirement of the orthogonality fixes the choice of representative in each of the equivalence classes $\{\tilde{j}_\mu\}$ and $\{\tilde{A}_\mu\}$ uniquely, i.e. each representation $\tilde{Q}_j, \tilde{Q}_A$ becomes realized in the space of functions. Simultaneously, Eq. (2.27) become invertible. A more detailed discussion of this construction may be found in Section 6, see also [15] as well as the next sections of this paper. Let us emphasize that all the above is valid for the space of any dimension $D \geq 3$. The case $D = 2$ is exceptional and will be discussed separately in the end of this section.

To put forth the program described above, let us consider the operator product expansion $j_\mu(x_1)\varphi(x_2)$. According to Eq. (2.52), it includes two types of contributions

$$j_\mu(x_1)\varphi(x_2) = \tilde{j}_\mu(x_1)\varphi(x_2) + j_\mu^{\rm tr}(x_1)\varphi(x_2)\,, \tag{2.53}$$

where

$$\tilde{j}_\mu(x_1)\varphi(x_2) = \sum_k [P_k^j]\,, \quad j_\mu^{\rm tr}(x_1)\varphi(x_2) = \sum_k [R_k^j]\,. \tag{2.54}$$

Let us require that two sets of fields

$$\{P_k^j\} \quad \text{and} \quad \{R_k^j\} \tag{2.55}$$

should be mutually orthogonal

$$\langle P_k^j R_m^j \rangle = 0 \quad \text{for all } k, m \tag{2.56}$$

even if their quantum numbers (dimensions and tensor ranks) coincide. This ensures the orthogonality of the states (see the end of Section 1):

$$\tilde{j}_\mu(x_1)\varphi(x_2)|0\rangle \quad \text{and} \quad j_\mu^{\rm tr}(x_1)\varphi(x_2)|0\rangle\,, \tag{2.57}$$

$$\langle \varphi(x_1)\varphi^+(x_2)\,\tilde{j}_\mu(x_3)\,j_\mu^{\rm tr}(x_4)\rangle = 0\,, \tag{2.58}$$

as well as the unambiguity of the decomposition into the sum (2.52) for the solution of the Ward identities. Below we demonstrate, in particular, that the expression (2.43) for the Green function G_μ^j satisfies conditions (2.56) and (2.58).

2.3. The solution of the Ward identities for the Green functions of irreducible conformal current

All the fields (2.55) that arise in the operator expansion $j_\mu(x_1)\varphi(x_2)$ can be found analysing the Green functions $\langle j_\mu \varphi j_\nu \varphi^+ \rangle$. Consider its connected part

$$G^j_{\mu\nu}(x_1x_2x_3x_4) = \langle j_\mu(x_1)\varphi(x_2)\, j_\nu(x_3)\varphi^+(x_4)\rangle_{\text{conn}} . \tag{2.59}$$

The Ward identity for the latter has the form

$$\partial^{x_1}_\mu G^j_{\mu\nu}(x_1x_2x_3x_4) = -[\delta(x_{12}) - \delta(x_{14})]\langle\varphi(x_2)\varphi^+(x_4)\, j_\nu(x_3)\rangle , \tag{2.60}$$

where

$$\langle\varphi(x_1)\varphi^+(x_2)\, j_\nu(x_3)\rangle = K^{x_3}_\nu(x_1x_2)\langle\varphi(x_1)\varphi^+(x_2)\rangle , \tag{2.61}$$

and $K^{x_3}_\nu(x_1x_2)$ is the function (2.45).

In accordance to Eqs. (2.52) and (2.58), represent the Green function (2.59) as a sum of the pair of terms

$$G^j_{\mu\nu}(x_1x_2x_3x_4) = \tilde{G}^j_{\mu\nu}(x_1x_2x_3x_4) + G^{\text{tr}}_{\mu\nu}(x_1x_2x_3x_4) , \tag{2.62}$$

where $G^{\text{tr}}_{\mu\nu} = \langle j^{\text{tr}}_\mu \varphi j^{\text{tr}}_\nu \varphi\rangle$ is the transverse function:

$$\partial^{x_1}_\mu G^{\text{tr}}_{\mu\nu}(x_1x_2x_3x_4) = \partial^{x_3}_\nu G^{\text{tr}}_{\mu\nu}(x_1x_2x_3x_4) = 0 . \tag{2.63}$$

The first term in Eq. (2.62) is uniquely determined from the Ward identity and only contains the contributions of the fields P^j_k. As shown in Refs. [2,3], see also [15] and below, it has the form

$$\tilde{G}^j_{\mu\nu}(x_1x_2x_3x_4) = K^{x_1}_\mu(x_2x_4)K^{x_3}_\nu(x_2x_4)\langle\varphi(x_1)\varphi^+(x_2)\rangle . \tag{2.64}$$

The second term in Eq. (2.62) only contains the contributions of the fields R^j_k.

To derive these results and to formulate condition (2.56) in terms of higher Green functions, it proves useful to apply conformal partial wave expansion. Let us start with the Green function (2.59). Its expansion has the form (1.92) after one sets $\Phi^{l'}_\mu(x) = j_\mu(x)$ in this formula. The two terms in Eq. (1.92) can be identified with the two terms in Eq. (2.62) if the transversal functions are chosen for C^σ_2

$$C^\sigma_2 = C^{l,\text{tr}}_{\mu,\mu_1\dots\mu_s}(x_1x_2x_3) = \langle\Phi^l_{\mu_1\dots\mu_s}(x_1)\varphi(x_2)\, j^{\text{tr}}_\mu(x_3)\rangle$$
$$\sim \left\{ s\frac{D-2-l+d+s}{l-d}, 1\right\} \Delta^l_j(x_1x_2x_3) , \tag{2.65}$$

where

$$\Delta^l_j(x_1x_2x_3) = (\tfrac{1}{2}x^2_{12})^{-(l+d-s-D+2)/2}\,(\tfrac{1}{2}x^2_{13})^{-(l-d-s+D-2)/2}\,(\tfrac{1}{2}x^2_{23})^{-(d+s-l+D-2)/2} . \tag{2.66}$$

The notation (1.78) has been used in Eq. (2.65). One can show [2,12] that

$$\partial^{x_3}_\mu C^{l,\text{tr}}_{\mu,\mu_1\dots\mu_s}(x_1x_2x_3) = 0 \text{ for all } l . \tag{2.67}$$

Such a choice of the functions C^σ_2 guarantees the transversality condition (2.63) to hold identically. Conditions (2.56) and (2.58) also turn out to be fulfilled due to Eqs. (1.97) and (1.84). Thus the poles of the kernels $\rho_1(\sigma)$ and $\rho_2(\sigma)$ in the expansion (1.92) determine the contributions of the fields P^j_k and R^j_k, respectively.

The total set of the normalized functions C_1^σ and C_2^σ is determined by the orthogonality conditions (1.84) and (1.85), the transversality condition, and the diagonality condition of the expansion (1.92). The more detailed calculations are presented in Ref. [2], see also [3,15]. It is useful to represent the result in the following form. Introduce the functions

$$\tilde{C}_1^\sigma = \tilde{C}^l_{1\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \langle \Phi^l_{\mu_1\ldots\mu_s}(x_1)\tilde{\varphi}(x_2)A_\mu^{\mathrm{long}}(x_3)\rangle ,$$
$$\tilde{C}_2^\sigma = \tilde{C}^l_{2\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \langle \Phi^l_{\mu_1\ldots\mu_s}(x_1)\tilde{\varphi}(x_2)\tilde{A}_\mu(x_3)\rangle ,$$

where $\tilde{\varphi}$ is the conformal partner of the field φ. These functions can be derived from the functions C_1^σ and C_2^σ through the amputation by the arguments x_2, x_3. Under the normalization (1.84), (1.85), these functions read

$$C_1^\sigma = C^l_{1\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = (2\pi)^{-D/2}N_1(\sigma,d)\{(D-2+l-d-s),1\}\Delta^l_j(x_1x_2x_3)\,, \tag{2.68}$$
$$\tilde{C}_1^\sigma = \tilde{C}^l_{1\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = (2\pi)^{-D/2}\,2^{(s-1)/2}\,\tilde{N}_1(\sigma,d)\{(l+d-D-s),1\}\tilde{\Delta}^l_j(x_1x_2x_3)\,, \tag{2.69}$$

where

$$N_1(\sigma,d) = \frac{2^{(s+3)/2}}{(d+s-l)(D-2+l-d+s)} \times\left\{\frac{\Gamma\left(\frac{l+d+s-D}{2}\right)\Gamma\left(\frac{D+l-d+s}{2}\right)\Gamma\left(\frac{D-l+d+s}{2}\right)\Gamma\left(\frac{l+d+s}{2}\right)}{\Gamma\left(\frac{d+s-l}{2}\right)\Gamma\left(\frac{2D-l-d+s}{2}\right)\Gamma\left(\frac{l-d+s}{2}\right)\Gamma\left(\frac{D-l-d+s}{2}\right)}\right\}^{1/2}, \tag{2.70}$$

$$\tilde{N}_1(\sigma,d) = \left\{\frac{\Gamma\left(\frac{l-d+s+D}{2}\right)\Gamma\left(\frac{l+d+s-D}{2}\right)\Gamma\left(\frac{l-d+s}{2}\right)\Gamma\left(\frac{D-l-d+s}{2}\right)}{\Gamma\left(\frac{2D-l-d+s}{2}\right)\Gamma\left(\frac{d+s-l}{2}\right)\Gamma\left(\frac{D-l+d+s}{2}\right)\Gamma\left(\frac{l+d+s}{2}\right)}\right\}^{1/2}, \tag{2.71}$$

$$\tilde{\Delta}^l_j(x_1x_2x_3) = (\tfrac{1}{2}x_{12}^2)^{-(l-d-s+D)/2}\,(\tfrac{1}{2}x_{13}^2)^{-(l+d-s-D)/2}\,(\tfrac{1}{2}x_{23}^2)^{-(l+d-s-D)/2}, \tag{2.72)}$$
$$C_2^\sigma \sim C^{l,\mathrm{tr}}_{2\mu,\mu_1\ldots\mu_s}(x_1x_2x_3)\,, \tag{2.73}$$
$$\tilde{C}_2^\sigma \sim \left\{-s\frac{3D-l-d-4+s}{2D-l-d-2},1\right\}\tilde{\Delta}^l_j(x_1x_2x_3)\,. \tag{2.74}$$

The expressions for the renormalization factors in the last pair of functions are presented in Refs. [2,3,12]. Their explicit form is irrelevant for what follows.

The higher Green functions of the fields P^j_k, R^j_k

$$\langle P^j_k(x_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle\,,\quad \langle R^j_k(x_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle \tag{2.75}$$

are given by the expressions (1.87) for $i=1,2$, respectively, after one substitutes the functions (2.69) and (2.74) into them. Below we consider the models with no regard to electromagnetic interaction, having the property that $G^{\mathrm{tr}}_{\mu\nu}(x_1x_2x_3x_4)=0$, or

$$\langle R^j_k(x_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle = 0\,. \tag{2.76}$$

In virtue of Eq. (1.87) with $i = 1, 2$, the condition of vanishing of the fields R_k^j can be written in the form

$$\int dy_1\, dy_2\, \tilde{C}^{l}_{2\mu,\mu_1\ldots\mu_s}(x_1 y_1 y_2)\langle j_\mu(y_2)\varphi(y_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle = 0\,. \tag{2.77}$$

Condition (2.77) will hereinafter be referred to as the *irreducibility condition of the conformal current, or the condition of absence of the electromagnetic interaction.* Note that Eq. (2.77) select out *the class of theories* in which *all the Green functions of the current are uniquely determined by the Ward identities*; see also Section 6 for details.

The Green functions of the fields P_k^j are determined by the Ward identities. As an example, let us consider the Green functions (2.41) and (2.59). The solution of the Ward identities satisfying the conditions (2.77) reads [2,3,12,15]:

$$\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)j_\mu(x_4)\rangle = K_\mu^{x_4}(x_1x_2)\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle\,, \tag{2.78}$$

$$\langle j_\mu(x_1)\varphi(x_2)j_\nu(x_3)\varphi^+(x_4)\rangle = K_\mu^{x_1}(x_2x_4)K_\nu^{x_3}(x_2x_4)\,\langle\varphi(x_1)\varphi^+(x_2)\rangle\,. \tag{2.79}$$

Conditions (2.77) for these functions mean that the kernels $\rho_2^\chi(\sigma)$ and $\rho_2^j(\sigma)$ of the second term in the partial expansion of the type (1.92) should vanish for both of these functions. As may easily be demonstrated by a direct calculation (with the help of the integral relations (A.2) and (A.4)),

$$\rho_2^\chi(\sigma) = \rho_2^j(\sigma) = 0\,; \tag{2.80}$$

see Refs. [2,15] for more details. So the fields R_k do not contribute to Eqs. (2.78) and (2.79). One can find the expressions for the kernels $\rho_1^\chi(\sigma)$ and $\rho_1^j(\sigma)$ corresponding to these functions [2,3,15]

$$\rho_1^\chi(\sigma) = \frac{1}{\sqrt{2}}\frac{(2\pi)^D}{\Gamma^2(D/2)}\frac{1}{(d+s-l)}(D-2+l-d+s)^{-1}\frac{\Gamma\left(\frac{D-\Delta}{2}\right)}{\Gamma\left(\frac{\Delta}{2}\right)}$$
$$\times[N(\sigma d\Delta)N_1(\sigma,d)]^{-1}\frac{\Gamma\left(\frac{l+d+s-D}{2}\right)\Gamma\left(\frac{l-d+\Delta+s}{2}\right)}{\Gamma\left(\frac{2D-l-d+s}{2}\right)\Gamma\left(\frac{D-l+d-\Delta+s}{2}\right)}\,, \tag{2.81}$$

where Δ is the dimension of the field χ, $N(\sigma d\Delta)$ is the normalization factor (1.39),

$$\rho_1^j(\sigma) = (-1)^{s+1}\frac{(2\pi)^{D_{2}}}{\Gamma^2(D/2)}(N_1(\sigma,d))^{-2}\frac{1}{(d+s-l)}$$
$$\times(D-2+l-d+s)^{-1}\frac{\Gamma\left(\frac{l+d+s-D}{2}\right)\Gamma\left(\frac{D-2+l-d+s}{2}\right)}{\Gamma\left(\frac{d-l+s+2}{2}\right)\Gamma\left(\frac{2D-l-d+s}{2}\right)}\,. \tag{2.82}$$

In the course of calculations we have used the relations (A.2) and (A.4). The quantum number of the fields P_k^j are determined by the poles of the partial wave expansion, see, for example, Eq. (1.75).

It follows from Eqs. (2.81) and (2.82) that there exists an infinite set of fieldsP_s^j of rank s and dimension

$$l_s = d + s\,, \tag{2.83}$$

which contribute to the operator product expansions $j_\mu\varphi$ and $\varphi\chi$:

$$j_\mu(x_1)\varphi(x_2) = \sum_s [P_s^j]\,, \quad \varphi(x_1)\chi(x_2) = \sum_s [P_s^j]\,. \tag{2.84}$$

The invariant three-point Green functions of the fields P_s^j have the form [2,15]:

$$\langle P_s^j(x_1)\varphi^+(x_2)\,j_\mu(x_3)\rangle = g_j^s\,(\tfrac{1}{2}x_{23}^2)^{-(D-2)/2}\,\overleftrightarrow{\partial}{}_\mu^{x_3}\,[(\tfrac{1}{2}x_{13}^2)^{-(D-2)/2}\,\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2)]$$
$$\times(\tfrac{1}{2}x_{12}^2)^{(D-2)/2}\,\langle\varphi(x_1)\varphi^+(x_2)\rangle\,, \tag{2.85}$$

where g_j^s are known coupling constants. These Green functions may be shown to satisfy [2,3,12,15] the anomalous Ward identities

$$\partial_\mu^{x_3}\langle P^j_{\mu_1\ldots\mu_s}(x_1)\varphi^+(x_2)\,j_\mu(x_3)\rangle = g_j^s(-2)^{-s+1}\,\Gamma(d)\,\frac{\Gamma(s)}{\Gamma(D/2+s-1)}$$
$$\times\sum_{k=1}^{s}\frac{1}{\Gamma(k)\Gamma(d+s-k)}\,[\partial^{x_3}_{\mu_1}\ldots\partial^{x_3}_{\mu_k}\,\delta(x_{13})\partial^{x_1}_{\mu_{k+1}}\ldots\partial^{x_1}_{\mu_k} + (\ldots) - \text{traces}]\langle\varphi(x_1)\varphi^+(x_2)\rangle\,, \tag{2.86}$$

where (…) stands for the sum of terms which arise after the transmutations of indices which enter asymmetrically.

Higher Green functions of the fields P_s^j may be evaluated from the Ward identities using Eq. (1.87) for $i = 1$. The explicit form of these expressions will be presented in the next section.

2.4. Green functions of the energy–momentum tensor and conditions of absence of gravitational interaction

Consider the Green functions of the energy-momentum tensor (1.50). Let Q_T be an undecomposable representation defined by the transformation law (2.4) and M_T the space of this representation. Introduce two types of irreducible tensors

$$\tilde{T}_{\mu\nu}(x) \quad\text{and}\quad T^{\mathrm{tr}}_{\mu\nu}(x),\quad \partial_\mu T^{\mathrm{tr}}_{\mu\nu}(x) = 0\,, \tag{2.87}$$

transforming by the irreducible representations

$$\tilde{Q}_T \quad\text{and}\quad Q_T^{\mathrm{tr}}\,, \tag{2.88}$$

respectively. The representation Q_T^{tr} is defined by the transformation law (2.4) on the invariant subspace

$$M_T^{\mathrm{tr}} \subset M_T \tag{2.89}$$

of transversal tensors $T^{\mathrm{tr}}_{\mu\nu} \subset M^{\mathrm{tr}}$. The representation $\tilde{Q}_T$ acts in the quotient space

$$\tilde{M}_T = M_T/M_T^{\mathrm{tr}}\,. \tag{2.90}$$

Denote the space of metric fields $h_{\mu\nu}$ as M_h. The space of longitudinal fields (2.8) which is invariant under (2.10) will be denoted as M_h^{long}. Introduce the quotient space

$$\tilde{M}_h = M/M_h^{\text{long}} . \tag{2.91}$$

Consider the irreducible representations

$$Q_h^{\text{long}} \quad \text{and} \quad \tilde{Q}_h \tag{2.92}$$

acting in the spaces

$$M_h^{\text{long}} \quad \text{and} \quad \tilde{M}_h , \tag{2.93}$$

respectively. Let us introduce two types of irreducible fields

$$h_{\mu\nu}^{\text{long}} \quad \text{and} \quad \tilde{h}_{\mu\nu} , \tag{2.94}$$

transforming by irreducible representations (2.92).

Consider the fields

$$T_{\mu\nu}(x) = \tilde{T}_{\mu\nu}(x) + T_{\mu\nu}^{\text{tr}}(x), \quad h_{\mu\nu}(x) = h_{\mu\nu}^{\text{long}}(x) + \tilde{h}_{\mu\nu}(x), \tag{2.95}$$

which transform by the direct sums of irreducible representations

$$\tilde{Q}_T \oplus Q_T^{\text{tr}} , \quad Q_h^{\text{long}} \oplus \tilde{Q}_h . \tag{2.96}$$

Each of the Green functions (1.50) and (1.52) may be represented as the sum of two terms (2.95):

$$G_{\mu\nu}^T(xx_1 \ldots x_m) = \tilde{G}_{\mu\nu}^T(xx_1 \ldots x_m) + G_{\mu\nu}^{T\,\text{tr}}(xx_1 \ldots x_m) , \tag{2.97}$$

$$G_{\mu\nu}^h(xx_1 \ldots x_m) = G_{\mu\nu}^{h\,\text{long}}(xx_1 \ldots x_m) + \tilde{G}_{\mu\nu}^h(xx_1 \ldots x_m) , \tag{2.98}$$

where $G_{\mu\nu}^{T\,\text{tr}}$ is the conformally invariant transversal function

$$\partial_\mu^x G_{\mu\nu}^{T\,\text{tr}}(xx_1 \ldots x_m) = 0 , \tag{2.99}$$

while $G_{\mu\nu}^{h\,\text{long}}$ is the conformally invariant longitudinal function

$$G_{\mu\nu}^{h\,\text{long}}(xx_1 \ldots x_m) = \partial_\mu^x G_\nu^h(xx_1 \ldots x_m) + \partial_\nu^x G_\mu^h(xx_1 \ldots x_m) - \frac{2}{D}\delta_{\mu\nu}\partial_\lambda^x G_\lambda^h(xx_1 \ldots x_m) , \tag{2.100}$$

where

$$G_\mu^h(xx_1 \ldots x_m) = \langle h_\mu(x)\varphi_1(x_1) \ldots \varphi_m(x_m)\rangle \tag{2.101}$$

and $h_\mu(x)$ is the conformal vector of dimension $d_h = -1$.

Note that the decompositions (2.97) and (2.98) are not unique since both representations $\tilde{Q}_T$ and $\tilde{Q}_h$ are defined in the space of equivalence classes $\tilde{M}_T$ and $\tilde{M}_h$, respectively, see Section 6 and [15] for more details. Similar to the case of current we shall formulate the prescription which fixes decompositions (2.95) uniquely. This prescription will be given in terms of certain constraints on the Green functions $G_{\mu\nu}^T$ which allow one to conduct an unambiguous evaluation of the Green functions of $\tilde{G}_{\mu\nu}^T$ from the Ward identities, and to relate $G_{\mu\nu}^{T\,\text{tr}}$ to gravitational degrees of freedom.

Let us demand that the groups of the states of the Hilbert space generated by the conformal fields $\tilde{T}_{\mu\nu}(x)$ and $T_{\mu\nu}^{\text{tr}}(x)$ should be mutually orthogonal. The fields $T_{\mu\nu}^{\text{tr}}$ and $\tilde{h}_{\mu\nu}$ are related to gravitational degrees of freedom. It is convenient to formulate the above orthogonality condition [4,15] in terms of the states generated by operator product expansion $T_{\mu\nu}\varphi$. Represent the latter expansion in the form:

$$T_{\mu\nu}(x_1)\varphi(x_2) = \tilde{T}_{\mu\nu}(x_1)\varphi(x_2) + T_{\mu\nu}^{\text{tr}}(x_1)\varphi(x_2) , \tag{2.102}$$

where

$$\tilde{T}_{\mu\nu}(x_1)\varphi(x_2) = \sum_k [P_k^T], \quad T^{\mathrm{tr}}_{\mu\nu}(x_1)\varphi(x_2) = \sum_k [R_k^T]. \tag{2.103}$$

We require the two sets of fields

$$\{P_k^T\} \quad \text{and} \quad \{R_k^T\} \tag{2.104}$$

to be mutually orthogonal

$$\langle 0|P_k^T(x_1)\, R_{k'}^T(x_2)|0\rangle = 0 \quad \text{for all } k, k', \tag{2.105}$$

even when the quantum numbers (dimension and ranks) of some of these fields coincide. The latter guarantees the orthogonality of the states

$$\tilde{T}_{\mu\nu}(x_1)\varphi(x_2)|0\rangle \quad \text{and} \quad T^{\mathrm{tr}}_{\mu\nu}(x_1)\varphi(x_2)|0\rangle, \tag{2.106}$$

which is expressed by the condition

$$\langle \tilde{T}_{\mu\nu}(x_1)\varphi(x_2)T^{\mathrm{tr}}_{\rho\sigma}(x_3)\varphi(x_4)\rangle = 0. \tag{2.107}$$

The above remarks on the role of the fields (2.87) and (2.95) are based on the following arguments. Note first that the irreducible representations (2.88) and (2.92) are partially equivalent [21,26], see also Ref. [31]:

$$\tilde{Q}_T \sim Q_h^{\mathrm{long}}, \quad Q_T^{\mathrm{tr}} \sim \tilde{Q}_h. \tag{2.108}$$

The equivalence conditions are expressed by the following relations between independent irreducible components of (2.97) and (2.98)

$$G^{h\,\mathrm{long}}_{\mu\nu}(xx_1\ldots x_m) \sim \int \mathrm{d}y\, D^h_{\mu\nu,\rho\sigma}(x-y)\,\tilde{G}^T_{\rho\sigma}(yx_1\ldots x_m), \tag{2.109}$$

where $D^h_{\mu\nu,\rho\sigma}$ is the conformally invariant propagator of the longitudinal field $h^{\mathrm{long}}_{\mu\nu}$

$$\begin{aligned} D^h_{\mu\nu,\rho\sigma}(x_{12}) &= \langle h^{\mathrm{long}}_{\mu\nu}(x_1)\, h^{\mathrm{long}}_{\rho\sigma}(x_2)\rangle \\ &= C_h\left[g_{\mu\rho}(x_{12})g_{\nu\sigma}(x_{12}) + g_{\mu\sigma}(x_{12})g_{\nu\rho}(x_{12}) - \frac{2}{D}\,\delta_{\mu\nu}\delta_{\rho\sigma}\right] \end{aligned} \tag{2.110}$$

where C_h is a constant. Expression (2.110) may be represented in the form [15,31]

$$D^h_{\mu\nu,\rho\sigma}(x_{12}) = \partial^{x_1}_\mu D_{\nu,\rho\sigma}(x_{12}) + \partial^{x_1}_\nu D_{\mu,\rho\sigma}(x_{12}) - \frac{2}{D}\,\delta_{\mu\nu}\partial^{x_1}_\lambda D_{\lambda,\rho\sigma}(x_{12}), \tag{2.111}$$

where

$$D^h_{\mu,\rho\sigma}(x) = \frac{1}{2}\, C_h\left[x_\sigma g_{\mu\rho}(x) + x_\rho g_{\mu\sigma}(x) + \frac{2}{D}\,\delta_{\rho\sigma}x_\mu\right]. \tag{2.112}$$

From Eqs. (2.109) and (2.111) the representation of $\tilde{G}^{\mathrm{long}}_{\mu\nu}$ in the form (2.100) follows.

The second condition of equivalence is given by the relation

$$G^{T,\mathrm{tr}}_{\mu\nu}(xx_1\dots x_m) = \int \mathrm{d}y\, D^{T,\mathrm{tr}}_{\mu\nu,\rho\sigma}(x-y)\,\tilde{G}^h_{\rho\sigma}(yx_1\dots x_m)\,, \tag{2.113}$$

where $D^{T,\mathrm{tr}}_{\mu\nu,\rho\sigma}$ is the transversal propagator. The general conformally invariant expression for the propagator of the energy-momentum tensor has the form

$$\begin{aligned} D^T_{\mu\nu,\rho\sigma}(x_{12}) &= \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle \\ &= \tilde{C}_T\left\{g_{\mu\rho}(x_{12})g_{\nu\sigma}(x_{12}) + g_{\mu\sigma}(x_{12})g_{\nu\rho}(x_{12}) - \frac{2}{D}\delta_{\mu\nu}\delta_{\rho\sigma}\right\}\frac{1}{(x^2_{12})^D}\,. \end{aligned} \tag{2.114}$$

In the space of odd dimension this expression is given by a well-defined function and may be represented as [15,31]

$$D^{T,\mathrm{tr}}_{\mu\rho,\rho\sigma}(x_{12}) \sim H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^{x_1})\frac{1}{(x^2_{12})^{D-2}}\,, \tag{2.115}$$

where

$$\begin{aligned} H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x) = \Bigg\{&\frac{(D-2)}{(D-1)}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma - \frac{1}{2}(\delta_{\mu\rho}\partial_\nu\partial_\sigma + \delta_{\mu\sigma}\partial_\nu\partial_\rho + \delta_{\nu\rho}\partial_\mu\partial_\sigma + \delta_{\nu\sigma}\partial_\mu\partial_\rho)\Box \\ &+ \frac{1}{(D-1)}(\delta_{\mu\nu}\partial_\rho\partial_\sigma\Box + \delta_{\rho\sigma}\partial_\mu\partial_\nu\Box) \\ &+ \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\Box^2 - \frac{1}{(D-1)}\delta_{\mu\nu}\delta_{\rho\sigma}\Box^2\Bigg\}\,, \end{aligned} \tag{2.116}$$

$$H^{\mathrm{tr}}_{\mu\nu,\rho\sigma} = H^{\mathrm{tr}}_{\nu\mu,\rho\sigma} = H^{\mathrm{tr}}_{\rho\sigma,\mu\nu}\,,\quad H^{\mathrm{tr}}_{\mu\mu,\rho\sigma} = 0\,.$$

$$H^{\mathrm{tr}}_{\mu\nu,\lambda\tau}(\partial^x)H^{\mathrm{tr}}_{\lambda\tau,\rho\sigma}(\partial^x) = \Box^2 H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x)\,, \tag{2.117}$$

$$\partial^x_\mu H^{\mathrm{tr}}_{\mu\nu,\rho\tau}(\partial^x) = 0\,. \tag{2.118}$$

In the even-dimensional space the expression (2.114) diverges due to the singularity of the factor $(x^2_{12})^{-D}$. Let us redefine this propagator as follows. Introduce the conformally invariant regularization by an addition of a small anomalous correction to the dimension l_T of the field $T_{\mu\nu}$:

$$l_T = D \;\to\; l^\varepsilon_T = D + \varepsilon\,. \tag{2.119}$$

The regularized propagator $D^{T^\varepsilon}_{\mu\nu,\rho\sigma}$ results from Eq. (2.114) after the substitution of the factor $(x^2_{12})^{-D-\varepsilon}$ for the factor $(x^2_{12})^{-D}$. Define a new propagator for the space of even dimension $D \geq 4$ by

$$D^{T,\mathrm{tr}}_{\mu\nu,\rho\sigma}(x_{12}) = \lim_{\varepsilon\to 0}\varepsilon D^{T^\varepsilon}_{\mu\nu,\rho\sigma}(x_{12})\,. \tag{2.120}$$

Resolving the ambiguity with the help of relation (2.34) one gets

$$D^{T,\mathrm{tr}}_{\mu\nu,\rho\sigma}(x_{12}) \sim H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^{x_1})\Box^{(D-4)/2}\delta(x_{12})\,,\quad \text{for even } D \geq 4\,. \tag{2.121}$$

Thus, there are two invariant kernel $D^{T^c}_{\mu\nu,\rho\sigma}$ and $D^{\mathrm{tr}}_{\mu\nu,\rho\sigma}$ for even $D \geq 4$. In Section 6 we show that these kernels are related to a pair of irreducible representations $\tilde{Q}_T$ and Q^{tr}_T, and define the propagators of irreducible fields $\tilde{T}_{\mu\nu}$ and $T^{\mathrm{tr}}_{\mu\nu}$. In the case of odd D the situation is analogous, and will not be discussed here.

It is convenient to introduce the projection operators P^{tr} and P^{long} onto the transversal and longitudinal sectors. Owing to Eq. (2.117) the latter is done in a manner which is natural to conformal theory, by setting

$$P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x) = \frac{1}{\Box^2} H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x), \tag{2.122}$$

$$P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x) + P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x) = \frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \frac{2}{D}\delta_{\mu\nu}\delta_{\rho\sigma}\right). \tag{2.123}$$

One can easily check that thus defined operator $P^{\mathrm{long}}_{\mu\nu,\rho\sigma}$ has the following properties:

$$P^{\mathrm{long}}_{\mu\nu,\lambda\tau}(\partial^x)\, P^{\mathrm{long}}_{\lambda\tau,\rho\sigma}(\partial^x) = P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x), \tag{2.124}$$

$$P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x) = \partial^x_\mu P_{\nu,\rho\sigma}(\partial^x) + \partial^x_\nu P_{\mu,\rho\sigma}(\partial^x) - \frac{2}{D}\delta_{\mu\nu}\partial^x_\lambda P_{\lambda,\rho\sigma}(\partial^x), \tag{2.125}$$

where

$$P_{\mu,\rho\sigma}(\partial^x) = -\frac{1}{2}\left[\frac{D-2}{D-1}\partial_\mu\partial_\rho\partial_\sigma\frac{1}{\Box^2} - (\delta_{\mu\rho}\partial_\sigma + \delta_{\mu\sigma}\partial_\rho)\frac{1}{\Box} + \frac{1}{D-1}\delta_{\rho\sigma}\partial_\mu\frac{1}{\Box}\right]. \tag{2.126}$$

Furthermore, one can explicitly check that the longitudinal propagator (2.110) satisfies the relation

$$P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^{x_1})\, D^h_{\rho\sigma,\lambda\tau}(x_{12}) = D^h_{\mu\nu,\lambda\tau}(x_{12}). \tag{2.127}$$

As follows from Eqs. (2.115), (2.121) and (2.122) the transversal propagator of the energy–momentum tensor satisfies the relation

$$P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^{x_1}) D^{T,\mathrm{tr}}_{\rho\sigma,\lambda\tau}(x_{12}) = D^{T,\mathrm{tr}}_{\mu\nu,\lambda\tau}. \tag{2.128}$$

Using these relations one finds from Eqs. (2.109) and (2.113)

$$P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x)\, G^{T\,\mathrm{tr}}_{\rho\sigma}(xx_1 \dots x_m) = G^{T\,\mathrm{tr}}_{\mu\nu}(xx_1 \dots x_m), \tag{2.129}$$

$$P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x)\, G^{h\,\mathrm{long}}_{\rho\sigma}(xx_1 \dots x_m) = G^{h\,\mathrm{long}}_{\mu\nu}(xx_1 \dots x_m). \tag{2.130}$$

Note that the remaining pair of irreducible functions, $\tilde{G}^T$ and $\tilde{G}^h$ do not satisfy similar relations. Each of these functions has both the transversal and the longitudinal components, and only the whole sums possess the property of conformal invariance. As shown in Ref. [15], the requirement of conformal invariance allows to reconstruct the transversal part of the function $\tilde{G}^T_{\mu\nu}$ uniquely from the longitudinal part which is known from the Ward identities provided that one chooses a certain realization of the representation $\tilde{Q}_T$. The choice of the realization in this case is imposed by the orthogonality condition (2.107). As shown in Section 6, see also Ref. [15], the latter allows to separate out the contribution of the gravitational interaction into the Green function (1.50) in an explicit manner; see below.

A resume consists in the following. The general solution of the Ward identities (1.54) represents a sum of the two conformally invariant terms (2.97). The first one, $\tilde{G}^T_{\mu\nu}$ is uniquely defined by the Ward identity and the requirement of the conformal symmetry. The second term is transverse, and may be expressed through the Green function of the metric field by Eq. (2.113). In the space of even dimension this equation takes the form:

$$G^{T\,\mathrm{tr}}_{\mu\nu}(xx_1 \dots x_m) = \Box_x^{D/2} P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x)\langle h_{\rho\sigma}(x)\varphi_1(x_1)\dots\varphi_m(x_m)\rangle\,. \tag{2.131}$$

In four-dimensional space it coincides [31] with the equation of linear conformal gravity:

$$\Box^2 h_{\mu\nu}(x) - \Box(\partial_\mu\partial_\sigma h_{\nu\sigma}(x) + \partial_\nu\partial_\sigma h_{\mu\sigma}(x)) + \tfrac{2}{3}\partial_\mu\partial_\nu\partial_\sigma\partial_\lambda h_{\sigma\lambda}(x) + \tfrac{1}{3}\delta_{\mu\nu}\Box\partial_\sigma\partial_\lambda h_{\sigma\lambda}(x) = T^{\mathrm{tr}}_{\mu\nu}(x)\,. \tag{2.131a}$$

The longitudinal part $G^{h\,\mathrm{long}}_{\mu\nu}$ of the Green function $G^h_{\mu\nu} = \langle h_{\mu\nu}\varphi_1 \dots \varphi_m\rangle$ does not contribute to Eq. (2.131). It is determined from Eq. (2.109) and may be calculated directly from the Ward identities:

$$G^{h\,\mathrm{long}}_{\mu\nu}(xx_1 \dots x_m) = -2\int \mathrm{d}y\, D_{\mu\nu,\rho}(x-y)\partial^y_\sigma\langle T_{\rho\sigma}(y)\varphi_1(x_1)\dots\varphi_m(x_m)\rangle\,, \tag{2.132}$$

where $D_{\mu\nu,\rho}(x)$ is the function (2.112)

$$D_{\mu\nu,\rho}(x-y) = P_{\rho,\lambda\tau}(\partial^x)D^h_{\mu\nu,\lambda\tau}(x-y)\,.$$

Thus, the functions

$$\tilde{G}^T_{\mu\nu} \quad \text{and} \quad G^{h\,\mathrm{long}}_{\mu\nu} \tag{2.133}$$

are determined by the Ward identities, the function $\tilde{G}^h_{\mu\nu}$ remains arbitrary, and the function $G^{T\,\mathrm{tr}}_{\mu\nu}$ is expressed through it by Eq. (2.131) (or a similar one for odd D). To this pair of functions (2.133) a pair of equivalent irreducible representations $\tilde{Q}_T \sim Q^{\mathrm{long}}_h$ corresponds.

According to Ref. [15], see also Section 6, the Green functions (2.133) describe the contribution of matter fields into energy–momentum tensor, while the function $\tilde{G}^h_{\mu\nu}$, as well as the transversal function $G^{T\,\mathrm{tr}}_{\mu\nu}$ which expresses through it, are related to gravitational interaction. To this pair of functions, $G^{T\,\mathrm{tr}}_{\mu\nu}$ and $\tilde{G}^h_{\mu\nu}$, another pair of equivalent irreducible representations $\tilde{Q}^{\mathrm{tr}} \sim \tilde{Q}^h$ corresponds.

Due to this, the theories which are free of gravitation interaction are selected by the following condition: *the energy–momentum tensor transforms by the irreducible representation* $\tilde{Q}^T$. *Its Green functions* coincide with $\tilde{G}^T_{\mu\nu}$:

$$\langle T_{\mu\nu}(x)\varphi_1(x_1)\dots\varphi_m(x_m)\rangle = \tilde{G}^T_{\mu\nu}(xx_1 \dots x_m)\,, \tag{2.134}$$

and, in virtue of the above arguments, *are uniquely determined by the Ward identities.*

2.5. The algorithm of solution of Ward identities in D-dimensional space

Condition (2.105) or (2.107) select out irreducible contributions of the energy–momentum tensor into the Green functions (1.50). Let us formulate these conditions in terms of conformal partial wave expansions. Consider the Green function

$$G^T_{\mu\nu\rho\sigma}(x_1x_2x_3x_4) = \langle T_{\mu\nu}(x_1)\varphi(x_2)T_{\rho\sigma}(x_3)\varphi(x_4)\rangle_{\text{conn}}\,. \tag{2.135}$$

On account of Eq. (2.107), the latter may be represented as

$$G^T_{\mu\nu\rho\sigma}(x_1x_2x_3x_4) = \tilde{G}^T_{\mu\nu\rho\sigma}(x_1x_2x_3x_4) + G^{T\,\text{tr}}_{\mu\nu\rho\sigma}(x_1x_2x_3x_4)\,, \tag{2.136}$$

where

$$\tilde{G}^T_{\mu\nu\rho\sigma} = \langle \tilde{T}_{\mu\nu}\varphi\tilde{T}_{\rho\sigma}\varphi\rangle\,, \quad G^{T\,\text{tr}}_{\mu\nu\rho\sigma} = \langle T^{\text{tr}}_{\mu\nu}\varphi T^{\text{tr}}_{\rho\sigma}\varphi\rangle\,. \tag{2.137}$$

The partial wave expansion of the Green function (2.135) has the form (1.105), if one sets $\Phi^{l'}_{\mu\nu}(x) = T_{\mu\nu}(x)$ in that formula. The invariant functions $C^\sigma_i \equiv C^\sigma_{i\mu\nu}$ entering this expansion may be chosen so that the last term in Eq. (1.105) becomes transverse. The general expression for the functions $C^\sigma_{i\mu\nu}$ has the form (1.101) after one sets $l_1 = D$. We have

$$C^\sigma_{i\mu\nu}(x_1x_2x_3) = \langle \Phi^l_{\mu_1\ldots\mu_s}(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = \{A^{(i)}_1, A^{(i)}_2, A^{(i)}_3\}\Delta^l_T(x_1x_2x_3)\,, \tag{2.138}$$

where $A^{(i)}_k$ are the constants; i, $k = 1,2,3$,

$$\Delta^l_T(x_1x_2x_3) = (\tfrac{1}{2}x^2_{12})^{-(l+d-s-D+2)/2}\,(\tfrac{1}{2}x^2_{13})^{-(l-d-s+D-2)/2}\,(\tfrac{1}{2}x^2_{23})^{-(d+s-l+D-2)/2}\,. \tag{2.139}$$

The coefficients $A^{(i)}_k$ are determined by the orthogonality conditions (1.102) and (1.103) up to an SO(3) transformation. Using this ambiguity, one can make the function $C^\sigma_{3\mu\nu}(x_2x_2x_3)$ transversal:

$$C^\sigma_{3\mu\nu}(x_1x_2x_3) = \langle \Phi_\sigma(x_1)\varphi(x_2)T^{\text{tr}}_{\mu\nu}(x_3)\rangle\,, \; \partial^{x_3}_\mu\, C^\sigma_{3\mu\nu}(x_1x_2x_3) = 0\,. \tag{2.140}$$

One can show [3,15] that this condition leads to the following equations for the coefficients $A^{(3)}_k$, $k = 1,2,3$:

$$\frac{1}{D}[(D-1)(l-d)+s]A^{(3)}_1 - s\frac{D-2}{D}(D-l+d+s)A^{(3)}_2 = 0\,,$$

$$-\frac{1}{D}A^{(3)}_1 + \left(l-d-\frac{2s}{D}\right)A^{(3)}_2 + (s-1)(l-d-s-D+2)A^{(3)}_3 = 0\,. \tag{2.141}$$

Note that these equations have non-empty solutions only for $s \geq 2$, when all three coefficients $A^{(3)}_k$ are non-zero. When $s = 0$, 1, no transversal function exists, see Eqs. (1.79) and (1.81). Under this choice of $C^\sigma_{3\mu\nu}$ the partial wave expansions for the Green functions (2.137) take the form, see Eq. (1.105):

$$\tilde{G}_{\mu\nu\rho\sigma} = \sum_{i=1}^{2}\sum_\sigma \int dx\, dy\, \rho_i(\sigma)\, C^\sigma_{i\mu\nu}(xx_1x_2)\Delta^{-1}_\sigma(x-y)\, C^\sigma_{i\rho\sigma}(yx_3x_4)\,, \tag{2.142}$$

$$G^{\text{tr}}_{\mu\nu\rho\sigma} = \sum_\sigma \int dx\, dy\, \rho_3(\sigma)\, C^\sigma_{3\mu\nu}(xx_1x_2)\Delta^{-1}_\sigma(x-y)\, C^\sigma_{3\rho\sigma}(yx_3x_4)\,, \tag{2.143}$$

One can show [3,15] that the function $\tilde{G}_{\mu\nu\rho\sigma}$ is determined uniquely from the Ward identity. Substituting expression (2.142) into this identity and using the orthogonality conditions (1.102) and (1.103), one can evaluate the kernels $\rho_1(\sigma)$ and $\rho_2(\sigma)$ as well as the coefficients $A^{(1)}_k$ and $A^{(2)}_k$, on which the functions $C^\sigma_{1\mu\nu}$ and $C^\sigma_{2\mu\nu}$ depend. The problem consists in the diagonalization of the partial wave expansion of the general solution of the Ward identity (similar to the procedure conducted in Section 3 for the Green function $G^{j}_{\mu\nu}$). It is discussed to a greater extent in Ref. [15], see also Ref. [3]. In principle, this problem is not much complicated, though it calls for quite cumbersome calculations to be published separately.

The fields $P^T_k(x)$ are determined by the poles of the kernels $\rho_1(\sigma)$ and $\rho_2(\sigma)$, while the field $R^T_k(x)$, by the poles of $\rho_3(\sigma)$. From what has been said it follows that the Green functions of the fields P^T_k are calculated from the Ward identities, while the Green functions of the fields R^T_k are determined by the metric field $\tilde{h}_{\mu\nu}$.

The higher Green functions of the fields $P^T_k(x)$ and R^T_k are expressed through the residues of the kernels

$$G^\sigma_1 = G^\sigma_1(x_1 \ldots x_m)\,, \quad G^\sigma_2 = G^\sigma_2(x_1 \ldots x_m)\,, \quad G^\sigma_3 = G^\sigma_3(x_1 \ldots x_m) \tag{2.144}$$

of conformal partial wave expansions (1.112) for the Green functions $\langle T_{\mu\nu}\varphi_1 \ldots \varphi_m\rangle$. The first two terms in Eq. (1.112) coincide with the function $\tilde{G}^T_{\mu\nu}$, see Eq. (2.97), while the third term, with the function $G^{T\,\mathrm{tr}}_{\mu\nu}$. To calculate the kernels (2.144) it proves helpful to introduce the functions $\tilde{C}^\sigma_{i\mu\nu}$ amputed by two arguments. The latter enter into orthogonality conditions (1.102) and (1.103) and into the equations of the type (1.87),

$$\tilde{C}^\sigma_{i\mu\nu}(x_1x_2x_3) \sim \langle \Phi^l_s(x_1)\tilde{\varphi}(x_2)h_{\mu\nu}(x_3)\rangle_i\,, \tag{2.145}$$

where $\tilde{\varphi}$ is the conformal partner of the field φ. In particular, one has for the kernel G^σ_3

$$\begin{aligned} G^\sigma_3(x_1 \ldots x_m) &= \int \mathrm{d}x\, \mathrm{d}y\, \tilde{C}^\sigma_{3\mu\nu}(x_1xy)\langle T^{\mathrm{tr}}_{\mu\nu}(y)\varphi_1(x)\varphi_2(x_2)\ldots\varphi_m(x_m)\rangle \\ &= \int \mathrm{d}x\, \mathrm{d}y\, \tilde{C}^\sigma_{3\mu\nu}(x_1xy)\langle T_{\mu\nu}(y)\varphi_1(x)\varphi_2(x_2)\ldots\varphi_m(x_m)\rangle\,. \end{aligned} \tag{2.146}$$

The last equality is the consequence of orthogonality of the function $\tilde{C}^\sigma_{3\mu\nu}$ to the functions $C^\sigma_{1\mu\nu}$ and $C^\sigma_{2\mu\nu}$:

$$\int \mathrm{d}x\, \mathrm{d}y\, \tilde{C}^\sigma_{3\mu\nu}(x_1xy)\, C^{\sigma'}_{1\mu\nu}(x_2xy) = \int \mathrm{d}x\, \mathrm{d}y\, \tilde{C}^\sigma_{3\mu\nu}(x_1xy)\, C^{\sigma'}_{2\mu\nu}(x_2xy) = 0 \tag{2.147}$$

for all σ, σ'.

The irreducibility condition (2.134) means that the kernels $\rho_3(\sigma)$ and G^σ_3 vanish

$$\rho_3(\sigma) = 0, \quad G^\sigma_3(x_1 \ldots x_m) = 0\,. \tag{2.148}$$

Taking into account Eq. (2.146), one can rewrite these conditions as the following equations on the Green functions of the energy–momentum tensor:

$$\int \mathrm{d}x\, \mathrm{d}y\, \tilde{C}^\sigma_{3\mu\nu}(x_1xy)\langle T_{\mu\nu}(y)\varphi_1(x)\varphi_2(x_2)\ldots\rangle = 0\,. \tag{2.149}$$

In the models where these conditions are fulfilled, all the Green functions of the energy–momentum tensor are uniquely determined by the Ward identities.

The Green functions of the fields $P_k^T(x)$ in such models are calculated from the equations of the type (1.71)

$$\langle P_k^T(x_1)\varphi_2(x_2)\dots\varphi_m(x_m)\rangle = \Lambda_k^T \operatorname*{res}_{\sigma=\sigma_k} \int dx\, dy\, \tilde{C}^{\sigma}_{\mu\nu}(x_1xy)\langle T_{\mu\nu}(y)\varphi_1(x)\dots\varphi_m(x_m)\rangle \tag{2.150}$$

where $\tilde{C}^{\sigma}_{\mu\nu}$ is the function (2.145) orthogonal to the transversal functions $C^{\sigma}_{3\mu\nu}$:

$$\int dx\, dy\, \tilde{C}^{\sigma}_{\mu\nu}(x_1xy)\, C^{\sigma'}_{3\mu\nu}(x_2xy) = 0 \quad \text{for all } \sigma, \sigma'. \tag{2.151}$$

As will be shown in the next section, the only fields which may exist in the models satisfying conditions (2.149) are the fields P_k^T with the quantum numbers $\sigma_k = (d+s, s)$. We will consider an infinite set of such fields

$$\{P_s^{l_s}\} \quad \text{where } l_s = d + s\,. \tag{2.152}$$

2.6. Conformal Ward identities in two-dimensional field theory

Two-dimensional space is specific by the property that both the current and energy–momentum tensor are irreducible fields. When $D = 2$, there is no problem in the decoupling of Euclidean transversal field $\tilde{T}^{\text{tr}}_{\mu\nu}(x)$, just because this field is zero. Gravitational interaction in this case is trivial and has no influence on the dynamics of matter fields. The representation of conformal group[1], given by the transformation law

$$T_{\mu\nu}(x) \xrightarrow{R} T'_{\mu\nu}(x) = \frac{1}{(x^2)^2}\, g_{\mu\rho}(x) g_{\nu\sigma}(x) T_{\rho\sigma}(Rx) \tag{2.153}$$

is irreducible. The energy–momentum tensor, being the traceless symmetric tensor, has two independent components

$$T_{11} + T_{22} = 0, \quad T_{12} = T_{21}\,. \tag{2.154}$$

The transversality condition

$$\partial_\mu T^{\text{tr}}_{\mu\nu}(x) = 0$$

is equivalent to a pair of equations on these components, having the unique solution

$$T^{\text{tr}}_{\mu\nu}(x) = 0\,.$$

The projection operator introduced above to utilize the decoupling of the subspace M_T^{tr} also vanishes for $D = 2$, while the longitudinal projector $P^{\text{long}}_{\mu\nu,\rho\sigma}(\partial/\partial x)$ is unity

$$P^{\text{tr}}_{\mu\nu,\rho\sigma}\left(\frac{\partial}{\partial x}\right) = 0, \quad P^{\text{long}}_{\mu\nu,\rho\sigma}\left(\frac{\partial}{\partial x}\right) = I_{\mu\nu,\rho\sigma} \quad \text{for } D = 2\,.$$

[1] The six-parameter conformal group is assumed.

Thus any traceless symmetric tensor $T_{\mu\nu}(x)$ is longitudinal

$$T_{\mu\nu}(x) = P^{\text{long}}_{\mu\nu,\rho\sigma}\left(\frac{\partial}{\partial x}\right) T_{\rho\sigma}(x) = T^{\text{long}}_{\mu\nu}(x)$$

and may be represented in the form

$$T_{\mu\nu}(x) = \partial_\mu \tilde{T}_\nu(x) + \partial_\nu \tilde{T}_\mu(x) - \delta_{\mu\nu}\partial_\lambda \tilde{T}_\lambda(x)\,, \tag{2.155}$$

$$\tilde{T}_\mu = \frac{\partial v}{\Box} T_{\mu\nu}(x) = \frac{1}{\Box} T_\mu(x)\,, \tag{2.156}$$

where $T_\mu(x) = \partial_\nu T_{\mu\nu}(x)$ is the conformal vector of scale dimension $D+1$. Thus the irreducible representation, given by the transformation law (2.153), is the analogue of the representation $\tilde{Q}_T$, which corresponds for $D > 2$ to the models where the gravitation is neglected.

From the above it is clear that the Green functions

$$\langle T_{\mu\nu}(x)\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle, \quad \langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle \tag{2.157}$$

are uniquely determined by Ward identities. For the case of $D > 2$ this property is proved for the conformal theories satisfying the condition (2.151), which fixes the realization of the representation $\tilde{Q}_T$ and simultaneously drops down the gravitational interaction. We have already mentioned the similarity between such theories and two-dimensional models. In the next sections we expand this analogy to a greater extent. For this reason, in the present section we keep the component form of Ward identities (though, the complex variables are more useful).

The conformally invariant solution to Ward identities is given by Eqs. (2.155) and (2.156). Consider the Ward identities for the Green functions (2.157) for $D = 2$:

$$\partial^x_\nu\langle T_{\mu\nu}(x)\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle = -\left[\sum_{k=1}^m \delta(x-x_k)\partial^{x_k}_\mu - \frac{1}{2}\partial^x_\mu \sum_{k=1}^m d_k\delta(x-x_k)\right] \times \langle\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle\,, \tag{2.158}$$

where d_k are scale dimensions of the scalar fields φ_k, $k = 1,\ldots,m$. The r.h.s. represents the Green functions for the vector $T_\mu(x)$

$$\langle T_\mu(x)\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle$$

Using Eqs. (2.155) and (2.156), and the relation

$$\frac{1}{\Box}\delta(x) = -\frac{1}{4\pi}\ln x^2\,, \tag{2.159}$$

we find

$$\langle T_{\mu\nu}(x)\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle = \frac{1}{2\pi}\left\{\sum_{k=1}^m \frac{1}{(x-x_k)^2}\left[(x-x_k)_\mu\partial^{x_k}_\nu + (x-x_k)_\nu\partial^{x_k}_\mu - \delta_{\mu\nu}(x-x_k)_\lambda\partial^{x_k}_\lambda\right] - \sum_{k=1}^m \frac{d_k}{(x-x_k)^2} g_{\mu\nu}(x-x_k)\right\}\langle\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle\,. \tag{2.160}$$

The anomalous Ward identity considered in the next section takes the form for $D = 2$

$$\partial_\mu^{x_1}\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle$$

$$= -\left\{\delta(x_{13})\partial_\nu^{x_3} + \delta(x_{14})\partial_\nu^{x_4} + \delta(x_{12})\partial_\nu^{x_2} - \frac{d}{2}\partial_\nu^{x_1}[\delta(x_{13}) + \delta(x_{14})]\right\}$$

$$\times \langle T_{\rho\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle + \partial_\rho^{x_1}\delta(x_{12})\langle T_{\nu\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle$$

$$+ \partial_\sigma^{x_1}\delta(x_{12})\langle T_{\nu\rho}(x_2)\varphi(x_3)\varphi(x_4)\rangle - \delta_{\rho\sigma}\partial_\lambda^{x_1}\,\delta(x_{12})\langle T_{\nu\lambda}(x_2)\varphi(x_3)\varphi(x_4)\rangle$$

$$-\frac{1}{24\pi}C\left\{\partial_\nu^{x_1}(\partial_\rho^{x_1}\partial_\sigma^{x_1} - \frac{1}{2}\delta_{\rho\sigma}\Box_{x_1})\delta(x_{12})\right.$$

$$\left.-\frac{1}{4}(\delta_{\nu\rho}\partial_\sigma^{x_1} + \delta_{\nu\sigma}\partial_\rho^{x_1} - \delta_{\rho\sigma}\partial_\nu^{x_1})\Box_{x_1}\delta(x_{12})\right\}\langle\varphi(x_3)\,\varphi(x_4)\rangle\,, \tag{2.161}$$

where C is the central charge. Its solution may be also easily written down using Eqs. (2.155), (2.156) and (2.159).

For the sake of convenience in juxtaposition of some of D-dimensional theory results with those of known two-dimensional models, let us list several formulas concerning the transition to complex variables for $D = 2$

$$x^{\pm} = x^1 \pm ix^2, \quad \partial_{\pm} = \frac{1}{2}(\partial_1 \mp i\partial_2)\,. \tag{2.162}$$

Any traceless symmetric tensor in two-dimensional space has two independent components. Define the complex components of the tensor $V_{\mu_1\ldots\mu_s}$ by the relations

$$V_{\pm} = 2^{s-2}\left(V_{\underbrace{11\ldots1}_{s}} \mp iV_{\underbrace{11\ldots12}_{s-1}}\right). \tag{2.163}$$

The contraction of a pair of traceless symmetric tensors V and W has the form

$$2^{s-2}V_{\mu_1\ldots\mu_s}W_{\mu_1\ldots\mu_s} = V_+W_- + V_-W_+\,. \tag{2.164}$$

In particular, we will use complex components of the tensor fields P_s, with dimensions $d + s$

$$P_s^T(x) = P^T_{\mu_1\ldots\mu_s}(x)\,.$$

They read

$$P_s^T(x) = P_{\pm}^{d+s}(x) = 2^{s-2}\left(P^{d+s}_{\underbrace{11\ldots1}_{s}} \mp iP^{d+s}_{\underbrace{11\ldots12}_{s-1}}\right). \tag{2.165}$$

Denote the components of the tensor

$$\lambda_s^{x_1}(x_2x_3) = \lambda_{\mu_1 \dots \mu_s}^{x_1}(x_2x_3)$$

as $\lambda_{s\pm}^{x_1}(x_2x_3)$. One can show, that

$$\lambda_{s\pm}^{x_1}(x_2x_3) = \frac{1}{2^{s-1}}\,[\lambda_{\pm}^{x_1}(x_2x_3)]^s = \frac{1}{2^{s-1}}\left(\frac{x_{32}^{\pm}}{x_{12}^{\pm}x_{13}^{\pm}}\right)^s \tag{2.166}$$

where

$$\lambda_{\pm}^{x_1}(x_2x_3) = \lambda_1^{x_1}(x_2x_3) \mp \mathrm{i}\lambda_2^{x_1}(x_2x_3) = x_{32}^{\pm}/x_{12}^{\pm}x_{13}^{\pm}\,. \tag{2.167}$$

The components $(1/x^2)g_{\pm}(x)$ of the tensor $(1/x^2)g_{\mu\nu}(x)$ are

$$\frac{1}{x^2}\,g_{\pm}(x) = \frac{1}{x^2}\,[g_{11}(x) \mp \mathrm{i}g_{12}(x)] = -\frac{1}{(x^{\pm})^2}\,.$$

Let us rewrite the Ward identities using the complex variables. Introduce $T_{\pm}$ components of the energy-momentum tensor

$$T_{\pm}(x) = T_{11}(x) \mp \mathrm{i}T_{12}(x)\,. \tag{2.168}$$

The Ward identities have the form

$$\partial_{\mp}^{x}\langle T_{\pm}(x)\varphi(x_1)\dots\varphi_m(x_m)\rangle$$
$$= -\left\{\sum_{k=1}^{m}\ \delta(x-x_k)\partial_{\pm}^{x_k} - \frac{1}{2}\,\partial_{\pm}^{x}\sum_{k=1}^{m}\ d_k\delta(x-x_k)\right\}\langle\varphi_1(x_1)\dots\varphi_m(x_m)\rangle\,, \tag{2.169}$$

$$\partial_{\mp}^{x_1}\langle T_{\pm}(x_1)T_{\pm}(x_2)\varphi(x_3)\varphi(x_4)\rangle$$
$$\begin{aligned}= &-\left\{\delta(x_{13})\partial_{\pm}^{x_3} + \delta(x_{14})\partial_{\pm}^{x_4} - \frac{d}{2}\,\partial_{\pm}^{x_1}\,[\delta(x_{13}) + \delta(x_{14})]\right\}\langle T_{\pm}(x_2)\varphi(x_3)\varphi(x_4)\rangle \\ &- [\delta(x_{12})\partial_{\pm}^{x_2} - 2\partial_{\pm}^{x_1}\delta(x_{12})]\,\langle T_{\pm}(x_2)\varphi(x_3)\varphi(x_4)\rangle \\ &- \frac{C}{12\pi}\,\partial_{\pm}^{x_1}\,\partial_{\pm}^{x_1}\,\partial_{\pm}^{x_1}\delta(x_{12})\langle\varphi(x_3)\varphi(x_4)\rangle\,.\end{aligned} \tag{2.170}$$

Eqs. (2.155) and (2.156) take the form

$$T_{\pm}(x) = 4\partial_{\pm}\tilde{T}_{\pm}(x), \quad \Box\tilde{T}_{\pm}(x) = \partial_{\mp}T_{\pm}(x)\,,$$

where

$$\tilde{T}_{\pm}(x) = \tfrac{1}{2}\,(\tilde{T}_1(x) \mp \mathrm{i}\tilde{T}_2(x))\,.$$

The solution of the Ward identities (2.169) and (2.170) reads, in complex variables

$$\langle T_+(x)\varphi_1(x_1)\dots\varphi_m(x_m)\rangle = \frac{1}{2\pi}\left\{\sum_{k=1}^{m}\frac{d_k}{(x^\pm - x_k^\pm)^2} + \sum_{k=1}^{m}\frac{2}{(x^\pm - x_k^\pm)}\partial^{x_k}_{\pm}\right\}\langle\varphi_1(x_1)\dots\varphi_m(x_m)\rangle\,, \tag{2.171}$$

$$\langle T_\pm(x_1)T_\pm(x_2)\varphi(x_3)\varphi(x_4)\rangle = \frac{1}{2\pi}\left\{\frac{4}{(x_{12}^\pm)^2} + \frac{d}{(x_{13}^\pm)^2} + \frac{d}{(x_{14}^\pm)^2} + 2\sum_{k=2}^{4}\frac{1}{x_{1k}^\pm}\partial^{x_k}_{\pm}\right\} \times\langle T_\pm(x_2)\varphi(x_3)\varphi(x_4)\rangle + \frac{C}{8\pi^2}\frac{1}{(x_{12}^\pm)^4}\langle\varphi(x_1)\varphi(x_2)\rangle\,. \tag{2.172}$$

2.7. *Ward identities for the propagators of irreducible fields*[2] $\tilde{j}_\mu$ *and* $\tilde{T}_{\mu\nu}$

The conformally invariant propagators should be defined according to the assumptions on the contractions of Euclidean fields

$$\int \mathrm{d}x\, A_\mu(x)j_\mu(x),\quad \int \mathrm{d}x\, h_{\mu\nu}(x)T_{\mu\nu}(x)\,. \tag{2.173}$$

In particular, such contractions enter to the equivalence relations (2.27), (2.109) and (2.113). The definition of the propagators depends on the choice of representations (2.15) of the conformal group, which are connected to the fields A_μ, j_μ and $h_{\mu\nu}, T_{\mu\nu}$. Consider the models satisfying conditions (2.77) and (2.149). Only irreducible components

$$\tilde{j}_\mu(x),\ A_\mu^{\mathrm{long}}(x),\quad \tilde{T}_{\mu\nu}(x),\ h_{\mu\nu}^{\mathrm{long}}(x) \tag{2.174}$$

are non-zero in such models, while the components $\tilde{A}_\mu(x)$ and $\tilde{h}_{\mu\nu}$ vanish:

$$\tilde{A}_\mu(x) = \tilde{h}_{\mu\nu}(x) = 0\,.$$

The propagators of the current and the energy–momentum tensor in these models may be chosen from the requirement of finiteness for the contractions

$$\int \mathrm{d}x\,\mathrm{d}y\, A_\mu^{\mathrm{long}}(x)D^j_{\mu\nu}(x-y)A_\nu^{\mathrm{long}}(y),\quad \int \mathrm{d}x\,\mathrm{d}y\, h_{\mu\nu}^{\mathrm{long}}(x)D^T_{\mu\nu\rho\sigma}(x-y)h_{\rho\sigma}^{\mathrm{long}}(y)\,. \tag{2.175}$$

As follows from Eqs. (2.31) and (2.114), the latter property depends on the dimension of the space. Here we restrict ourselves to the case of even $D \geq 2$. A somewhat modified recipe may be proposed for the case of odd D. It will be discussed in the other paper.

Now consider the propagators of the current. The first contraction is finite, provided that one utilizes the expression (2.32) for $\varepsilon = 0$. The propagator (2.32), though divergent, is formally transversal for $\varepsilon = 0$. Its contraction with longitudinal fields enters the first expression (2.175). As

[2] The total conformal propagators of current and energy-momentum tensor, see Eqs. (2.52) and (2.95), are discussed in Section 6.

the result, the integral in Eq. (2.175) has an ambiguity $0 \times \infty$. To resolve it, let us make use of the regularization (2.30). As $\varepsilon \to 0$, we get from Eq. (2.32):

$$\partial_\mu^{x_1} D^{j^\varepsilon}_{\mu\nu}(x_{12})|_{\varepsilon\to 0} = -\varepsilon(D-2)^{-1}\tilde{C}_j\,\partial_\nu^{x_1}\left(\tfrac{1}{2}x_{12}^2\right)^{-D+1-\varepsilon}. \tag{2.176}$$

It is seen that in the limit $\varepsilon = 0$ propagator (2.30) contains a finite longitudinal part. Using relation (2.34) for calculation of the limit (2.176) we get the following Ward identity:

$$\partial_\mu^{x_1}\langle j_\mu(x_1)j_\nu(x_2)\rangle = C_j\,\partial_\nu^{x_1}\Box^{(D-2)/2}\,\delta(x_{12}), \tag{2.177}$$

where C_j is the independent parameter analogous to the central charge, and

$$\tilde{C}_j = \tfrac{1}{2}\pi^{-D/2}\,\Gamma(D)\Gamma(D/2)\,C_j.$$

Note that one cannot use the limiting expression (2.177) for the calculation of the contraction (2.175). The latter would break down the conformal invariance. One should calculate the integral of the regularized expression, taking the limit $\varepsilon = 0$ only after that. Let us stress that such a definition of the propagator is admissible in the theories which are free of electromagnetic interaction, where $\tilde{A}_\mu(x) = 0$.

The condition of equivalence of representations $\tilde{Q}_j$ and Q_A^{long} may be written in any of the two forms

$$\tilde{j}_\mu(x) = \int \mathrm{d}y\, D^j_{\mu\nu}(x-y)A^{\text{long}}_\nu(y), \quad A^{\text{long}}_\mu(x) = \int \mathrm{d}y\, D^A_{\mu\nu}(x-y)\tilde{j}_\nu(y). \tag{2.178}$$

The conformally invariant propagators $D_{\mu\nu} = \langle \tilde{j}_\mu \tilde{j}_\nu\rangle$ and $D^A_{\mu\nu} = \langle A_\mu A_\nu\rangle$ satisfy

$$\langle j_\mu(x_1)A_\nu(x_2)\rangle = \int \mathrm{d}x\, D^j_{\mu\rho}(x_1-x)D^A_{\rho\nu}(x-x_2) = \delta_{\mu\nu}\delta(x_{12}). \tag{2.179}$$

The integrals in Eqs. (2.178) and (2.179) are calculated with the help of regularization (2.30). The normalization constant C_A in Eq. (2.28) is calculated from Eq. (2.179) and is equal to

$$C_A = \frac{1}{2}(-1)^{D/2+1}(2\pi)^{-D/2}\,\Gamma(D)\frac{1}{\tilde{C}_j} = (-1)^{D/2+1}\frac{(4\pi)^{-D/2}}{\Gamma(D/2)}\frac{1}{C_j}. \tag{2.180}$$

In the course of calculation we have used the integral relation (A.1).

It follows from Eqs. (2.178) and (2.179) that in the class of models under discussion one can pass from the Green functions of the current to the Green functions of the field A_μ^{long}. This results in a number of technical facilities, see Section 4. The Green functions which include several fields A_μ^{long} satisfy the following Ward identities [5]:

$$\begin{aligned}
&C_j\Box_x^{(D-2)/2}\,\partial_\mu^x\langle\varphi(y_1)\varphi^+(y_2)A_\mu(x)A_{\mu_1}(x_1)\ldots A_{\mu_k}(x_k)\rangle\\
&\quad= \partial_\mu^x\langle\varphi(y_1)\varphi^+(y_2)j_\mu(x)A_{\mu_1}(x_1)\ldots A_{\mu_k}(x_k)\rangle\\
&\quad= -[\delta(x-y_1)-\delta(x-y_2)]\,\langle\varphi(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots A_{\mu_k}(x_k)\rangle\\
&\qquad+\sum_{r=1}^{k}\partial_{\mu_r}^{x_r}\delta(x-x_r)\,\langle\varphi(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots\hat{A}_{\mu_r}\ldots A_{\mu_k}(x_k)\rangle
\end{aligned} \tag{2.181}$$

where the notation $\hat{A}_{\mu_r}$ means that the field $A_{\mu_r}(x_r)$ is dropped. Here and below A_μ is assumed to be a longitudinal field

$$A_\mu(x) \equiv A_\mu^{\text{long}}(x)\,. \tag{2.182}$$

The solution to the Ward identity has the form [5]

$$\langle \varphi(x_1)\varphi^+(x_2)A_{\mu_1}(x_1)\dots A_{\mu_k}(x_k)\rangle = (g_A)^k\,\lambda_{\mu_1}^{x_1}(y_1\,y_2)\dots\lambda_{\mu_k}^{x_1}(y_1 y_2)\,\langle\varphi(y_1)\varphi^+(y_2)\rangle\,, \tag{2.183}$$

where

$$g_A = 2C_A = 2(-1)^{D/2+1}\frac{(4\pi)^{-D/2}}{\Gamma(D/2)}\frac{1}{C_j}\,. \tag{2.184}$$

Consider the propagator of the energy-momentum tensor. The second contraction (2.175) can be made meaningful introducing the dimensional regularization (2.119) and $l_h \to l_h^\varepsilon = D - l_T^\varepsilon = -\varepsilon$. All the above statements concerning the propagator of the current, as well as relations (2.176)–(2.180) are easily generalized to the case of the fields $\hat{T}_{\mu\nu}$, $h_{\mu\nu}^{\text{long}}$. In particular, one has for the regularized propagator:

$$\partial_\nu^{x_1} D_{\mu\nu,\rho\sigma}^{T^\varepsilon}(x_{12}) = \varepsilon\tilde{C}_T[(D-1+\varepsilon)(D+1-\varepsilon)]^{-1}\left\{\partial_\mu\partial_\nu\partial_\rho - \frac{(D-1+\varepsilon)}{2(D+2\varepsilon)}(\delta_{\mu\rho}\partial_\sigma + \delta_{\mu\sigma}\partial_\rho)\Box \right.$$
$$\left. - \frac{1}{(D+\varepsilon)^2}\,\delta_{\rho\sigma}\partial_\mu\Box\right\}(x_{12}^2)^{-D+1-\varepsilon}\,. \tag{2.185}$$

Transition to the limit $\varepsilon = 0$ with the help of relation (2.34) will result in the following Ward identity:

$$\partial_\nu^{x_1}\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle$$
$$= C_T\left\{\partial_\mu\partial_\rho\partial_\sigma - \frac{D-1}{2D}(\delta_{\mu\rho}\partial_\sigma + \delta_{\mu\sigma}\partial_\rho)\Box - \frac{1}{D^2}\,\delta_{\rho\sigma}\partial_\mu\Box\right\}\Box^{(D-2)/2}\,\delta(x_{12})\,, \tag{2.186}$$

where C_T is the analogue of the central charge of two-dimensional conformal models.

When using partial wave expansions, both formulations, either in terms of the fields $T_{\mu\nu}$ or in terms of $h_{\mu\nu}^{\text{long}}$, are equivalent. However, this transition is possible no sooner than the Ward identities for the Green functions of two or more $T_{\mu\nu}$ fields are solved, due to the non-Abelian character of the latter.

3. Hilbert space of conformal field theory in D dimensions

Let M be a Hilbert space. The requirement of conformal symmetry together with several assumptions of quite general character leads to rigid constraints on its structure. Any field model may be formulated in terms of a certain consistent conditions imposed on the states of the Hilbert space. In this section we study possible types of these conditions and consider the class of conformal models issued by the latter. We demonstrate the existence of the subspace H of the Hilbert space

$$H \subset M$$

in which the conditions fixing each model are formulated.

3.1. Model-independent assumptions. Secondary fields

Consider a theory described in terms of a definite (finite) sets of fields. The latter fields will be treated as fundamental. In the Lagrangian approach they are the very fields the Lagrangian is constructed from. In our approach no model-dependent assumptions are made. The number of these fields as well as their spin-tensor structure is supposed to be given. For the simplicity, let us discuss the theory including one or two fundamental scalar fields

$$\varphi(x),\ \chi(x)$$

having scale dimensions d, Δ. The $\varphi(x)$ field may be either charged or neutral, and $\chi(x)$ is the neutral field.

Then we make two model-independent assumptions:

1. *The existence of the field algebra*:

$$\Phi_i(x_1)\Phi_k(x_2)=\sum_m\,[\Phi_m]\,. \tag{3.1}$$

   It may be shown [2,12,15,21,22] that the latter result, being indifferent to a choice of bare interactions, is a general consequence of renormalized Schwinger–Dyson system.

2. It is supposed that the *field algebra includes the energy–momentum tensor and the conserved currents* (in the presence of internal symmetries). The latter satisfy the following conservation laws (in Minkowski space):

$$\partial_\mu T_{\mu\nu}(x)=0,\quad \partial_\mu j_\mu(x)=0 \tag{3.2}$$

   and have the canonical dimensions $l_T=D$, $l_j=D-1$.

The generators of the conformal group are expressed through the components of the energy–momentum tensor. If a symmetry higher than the conformal one will appear to be present in the model, then its generators will also be representable in terms of energy–momentum tensor moments. Analogously, the internal symmetry generators are expressed in terms of local currents. For the simplicity, the Abelian symmetry will be considered here (though the most interesting models arise in non-Abelian case). Let us stress that no model-dependent assumptions on the structure of either the energy-momentum tensor or the currents in terms of fundamental fields are made. Being the "local symmetry generators" of the theory, the current and energy–momentum tensor define the transformation properties of the fields. Equal-time commutators of their components with the fields

$$\delta(x^0-y^0)\,[T_{0\nu}(x),\varphi(y)],\quad \delta(x^0-y^0)\,[j_0(x),\varphi(y)],\quad \delta(x^0-y^0)\,[T_{0\nu}(x),\chi(y)] \tag{3.3}$$

are considered to be given. Moreover, the algebra of the conformal group fixes the equal-time commutators of energy–momentum tensor components up to gradient terms. An internal symmetry algebra fixes the commutators of currents components (the only terms admissible in an Abelian case are the gradient terms). Thus the equal-time generators

$$[T_{0\mu}(x_1),T_{\rho\sigma}(x_2)],\quad [j_0(x_1),j_\mu(x_2)] \tag{3.4}$$

are considered to be known up to gradient terms. Their choice defines different types of models (see below).

These two statements, supplied with the requirement of conformal symmetry, would result in a quite specific structure of a Hilbert space resembling that of two-dimensional conformal theories. There exists a special subspace H in a total Hilbert space, which is begot by the energy–momentum tensor and the current. We call this subspace the *dynamical sector* (see Sections 3.3 and 3.4).

The next step consists in a formulation of the dynamical principle that fixes a model. As we show below, under a definite choice of anomalous (gradient) terms in commutators (3.4) the dynamical sector includes special states, which may be set to zero with no contradiction to Ward identities. These states are analogous to null vectors of two-dimensional conformal theories. Each of them defines an exactly solvable model. The simplest models of scalar fields discussed in Refs. [4–6], belong to this class of models, see also Section 4.

This program is conducted in three steps. At first, the total Hilbert space of the conformal theory is constructed in the framework of the first statement. As described in Section 1, the Hilbert space can be represented as an infinite direct sum of mutually orthogonal subspaces M_k^+

$$M = \sum_k \oplus M_k^+ \,. \tag{3.5}$$

Each subspace M_k^+ is spanned by the states [2,15]

$$\Phi_k(x)|0\rangle \,, \tag{3.6}$$

where $\Phi_k(x)$ is any field entering the algebra (3.1). The states (3.6) for each k form a basis of the space of irreducible representations of the conformal group. From this fact the orthogonality of subspaces M_k^+ follows

$$\langle 0|\Phi_m(x_1)\Phi_k(x_2)|0\rangle = 0 \quad \text{for } m \neq k \,. \tag{3.7}$$

The subspace of states

$$\varphi(x)|0\rangle, \quad \chi(x)|0\rangle \tag{3.8}$$

also enter the sum (3.5). Thus the set of states (3.6) for all k may be considered as a basis of a total Hilbert space M. All the states of fields φ and χ of the type

$$\varphi(x_1)\varphi(x_2)|0\rangle, \quad \varphi(x_1)\chi(x_2)\varphi(x_3)|0\rangle, \quad \varphi(x_1)\varphi(x_2)\varphi(x_3)|0\rangle, \; \ldots \tag{3.9}$$

as well as the states

$$\Phi_m(x_1)\varphi(x_2)|0\rangle, \quad \Phi_m(x_1)\Phi_k(x_2)|0\rangle \tag{3.10}$$

are decomposed into this basis owing to the statement (3.1) on the conformal field algebra, see Eq. (1.59).

At the next step the second statement, that is, the existence of the energy–momentum tensor and the current, is studied. The principal result consists in the fact that the *choice of commutators* (3.3) *and* (3.4) *together with the conditions of conformal symmetry makes possible to find the states* *of the type*

$$j_\mu(x_1)\varphi(x_2)|0\rangle, \quad j_\mu(x_1)j_\nu(x_2)\varphi(x_3)|0\rangle, \ \ldots, \tag{3.11}$$

$$T_{\mu\nu}(x_1)\varphi(x_2)|0\rangle, \quad T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\varphi(x_3)|0\rangle, \quad T_{\mu\nu}(x_1)\chi(x_2)|0\rangle, \ \ldots, \tag{3.12}$$

and to obtain their expansion in the basis (3.6). This result is justified rigorously under the following *additional constraint: a current and energy–momentum tensor in Euclidean space transform by irreducible representations of the conformal group.* This constraint singles out a class of models of direct (non-gauge) interaction of the matter fields for $D \geq 2$ and is expressed by Conditions (2.77) and (2.149) (for $D = 2$ the latter hold identically). As shown in the Section 2, the Green functions

$$\langle j_\mu\varphi \ldots \varphi^+\rangle, \quad \langle j_\mu j_\nu\varphi \ldots \varphi^+\rangle, \quad \langle T_{\mu\nu}\varphi \ldots \chi\rangle, \quad \langle T_{\mu\nu}T_{\rho\sigma}\varphi \ldots \chi\rangle, \ \ldots, \tag{3.13}$$

and, therefore the states (3.11) and (3.12) are uniquely determined by the Ward identities, provided that Conditions (2.77) and (2.149) hold.

The expansion of the states (3.11) and (3.12) in the basis (3.6) is formulated in terms of Euclidean operator product expansions. In particular, when applied to the states

$$j_\mu(x_1)\varphi(x_2)|0\rangle, \quad T_{\mu\nu}(x_1)\varphi(x_2)|0\rangle$$

these expansions read (see Section 2 and [2,3,15] for more detail)

$$j_\mu(x_2)\varphi(x_1) = \sum_s \, [P_s^j]\,, \tag{3.14}$$

$$T_{\mu\nu}(x_2)\varphi(x_1) = \sum_s \, [P_s^T]\,, \tag{3.15}$$

where P_s^j and P_s^T are symmetric traceless tensor fields of the rank s with the scale dimensions

$$d_s^j = d_s^T = d + s\,.$$

This result is the consequence of the Ward identities, see below. In what follows we use the unified notation

$$P_s(x) = \{P_s^j(x), P_s^T(x)\} = P_{\mu_1 \ldots \mu_s}(x), \quad d_s = d + s\,. \tag{3.16}$$

The fields P_s^j and P_s^T may be either orthogonal to each other (i.e. $\langle P_s^j(x_1)P_s^T(x_2)\rangle = 0$) or not, depending on the model. When $s = 0$ both these fields coincide with the fundamental field

$$P_s(x)|_{s=0} = \varphi(x)\,. \tag{3.17}$$

The fields P_s^T exist only for a definite type of gradient terms in commutators (3.4); see below.

The fields P_s^T and P_s^j have the transformation properties similar to those of secondary fields of two-dimensional conformal theories. The part way evidence to this is the presence of non-zero Green functions

$$\langle P_s^j\varphi^+ j_\mu\rangle, \quad \langle P_s^T\varphi^+ T_{\mu\nu}\rangle, \quad s \neq 0\,,$$

which satisfy anomalous Ward identities. The Green functions $\langle P_s^j\varphi^+ j_\mu\rangle$ as well as the Ward identities for there functions are given above, see Eqs. (2.85) and (2.86), and they are discussed in more detail in Refs. [2,3,15], where the explicit expressions for the Green functions $\langle P_s^T\varphi^+ T_{\mu\nu}\rangle$

may be also found. The commutators of fields P_s^j with the component j_0 are expressed through the fields P_s^j in the usual manner and have, beyond that, anomalous contributions of the fields $P_{s'}^j(x)$, where $s' = 0, 1, \dots, s-1$. The corresponding Ward identities also have anomalous contributions of these fields. In Section 4 we give the examples of such anomalous Ward identities.

The commutators of fields P_s^T with energy–momentum tensor and current components include anomalous operator terms

$$\delta(x^0 - y^0)[T_{0\mu}(x), P_s^T(y)]$$
$$= \mathrm{i}\delta^{(D)}(x-y)\partial_\mu P_s^T(y) + \sum_{m=1}^{s} \hat{H}_\mu^{s,m}(\vec{\partial}_x, \partial^y)\delta^{(D)}(x-y)P_{s-m}^T(y), \quad \mu = 0, 1, \dots, D-1\,. \tag{3.18}$$

The analogous terms enter the commutator $[j_0(x), P_s^j(y)]$. The $\hat{H}_\mu^{s,m}$ operators in Eq. (3.18) are the differential operators of rank $s+1$

$$\vec{\partial}_x = \frac{\partial}{\partial x^i}, \quad i = 1, 2, \dots, D-1; \quad \partial^y = \frac{\partial}{\partial y^\mu}, \quad \mu = 0, 1, \dots, D-1\,.$$

The ∂^y derivatives act on the argument of a field $P_{s-m}^T(y)$ only. The $\hat{H}$ operators are the sums of terms of the type

$$(\vec{\partial}_x)^{m-r+1}\,\delta(x-y)\,(\partial^y)^{s-m+r}, \quad r = 0, 1, \dots, m\,,$$

which, in principle, could be calculated for any class of models. In the same manner, *conformal Ward identities for the Green functions*

$$G_{\mu\nu,\mu_1\dots\mu_s}^{P_s}(xx_1\dots x_m) = \langle T_{\mu\nu}(x)P_{\mu_1\dots\mu_s}^T(x_1)\varphi(x_2)\dots\varphi_m(x_m)\rangle \tag{3.19}$$

satisfy the anomalous Ward identities

$$\partial_\nu^x G_{\mu\nu,\mu_1\dots\mu_s}^{P_s}(xx_1\dots x_m) = -\left[\sum_{k=1}^{m}\delta(x-x_k)\partial_\mu^{x_k} - \sum_{k=2}^{m}\frac{d}{D}\partial_\mu^x\delta(x-x_k)\right.$$
$$+ H_{s,0}^T(\partial^x, \delta(x-x_1); \partial^{x_1})\langle P_s(x_1)\varphi(x_1)\dots\varphi(x_m)\rangle$$
$$+ \sum_{k=1}^{s} H_{s,k}^T(\partial^x, \delta(x-x_1), \partial^{x_1})\langle P_{s-k}(x_1)\varphi(x_1)\dots\varphi(x_m)\rangle\,, \tag{3.20}$$

where $H_{s,k}^T$ are the differential operators which consist of the terms

$$(\partial^x)^{k+1-r}\,\delta(x-x_1)\,(\partial^{x_1})^r, \quad r = 0, 1, \dots, k\,.$$

The explicit form of these differential operators may be derived from the Ward identities for the conformally invariant Green functions

$$\langle T_{\mu\nu}(x)P_s(x_1)P_{s'}(x_2)\rangle, \quad s' = 0, 1, \dots, s-1\,. \tag{3.21}$$

These expressions are in general very cumbersome. As an example consider the Green functions $\langle T_{\mu\nu}\varphi P_s\rangle$. The corresponding formulae were found explicitly in Refs. [3,15]. They read:

$$\langle P_s(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = g_s^T\,(x_{12}^2)^{-(D-2)/2}\Big\{(x_{23}^2)^{-(D-2)/2}$$
$$\times\left[\overleftrightarrow{\partial}_\mu^{x_3}\overleftrightarrow{\partial}_\nu^{x_3} - \frac{2}{(D-2)}(\overleftarrow{\partial}_\mu^{x_3}\overrightarrow{\partial}_\nu^{x_3} + \overleftarrow{\partial}_\nu^{x_3}\overrightarrow{\partial}_\mu^{x_3}) - \text{trace}\right]$$
$$\times(x_{13}^2)^{-(D-2)/2}\,\lambda_{\mu_1\ldots\mu_s}^{x_1}(x_3x_2)\Big\}\,\langle\varphi(x_1)\varphi(x_2)\rangle + \cdots\,, \tag{3.22}$$

where $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$, g_s^T is the coupling constant, and dots stand for quasilocal terms, see Refs. [3,15]. The Green functions (3.22) satisfy the Ward identities

$$\partial_\nu^{x_3}\langle P_{\mu_1\ldots\mu_s}(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = H_{s,s}^T(\partial^{x_3}, \delta(x_{13}), \partial^{x_1})\,\langle\varphi(x_1)\varphi(x_2)\rangle\,. \tag{3.23}$$

The general expression for the operators $H_{s,s}^T$ for $s\geq 2$ depends on two parameters [3,15]. One has for $s=1$;

$$H_{1,1}^T \sim \frac{D-2}{D}\,\partial_{\mu_1}^{x_3}\partial_\mu^{x_3}\delta(x_{13}) + \delta_{\mu\mu_1}\Box_{x_3}\delta(x_{13})$$
$$-\frac{D}{d}\left[\partial_{\mu_1}^{x_3}\delta(x_{13})\partial_\mu^{x_1} + \delta_{\mu\mu_1}\partial_\nu^{x_3}\delta(x_{13})\partial_\nu^{x_1} - \frac{2}{D}\partial_\mu^{x_3}\delta(x_{13})\partial_{\mu_1}^{x_1}\right]. \tag{3.24}$$

The fields having anomalous commutators of the type (3.18) will be called *secondary fields* generated by the *primary field* $\varphi(x)$. A complete set of secondary fields will be studied below. Let us remark that the origin and transformation properties of these fields are analogous to those of the secondary fields in two-dimensional conformal theories.

Note also that no field other than fields (2.15) can enter the operator expansions (2.14) and (2.15). Indeed, suppose that some field $\Phi_0 = \Phi_s^{l_0}$ with $l_0 \neq d+s$ is present in the expansion (2.15). Then there exists a non-vanishing Green function $\langle\Phi_0\varphi T_{\mu\nu}\rangle$. The Ward identity for this function has the form:

$$\partial_\mu^{x_3}\langle\Phi_0(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = \{-\delta(x_{13})\partial_\nu^{x_1} - \delta(x_{23})\partial_\nu^{x_2} + \cdots\}\,\langle\Phi_0(x_1)\varphi(x_2)\rangle = 0\,. \tag{3.25}$$

In the last equality the orthogonality condition

$$\langle\Phi_0(x_1)\varphi(x_2)\rangle = 0 \quad \text{if } l_0\neq d,\ s\neq 0 \tag{3.26}$$

has been used. Consequently, the Green function $\langle\Phi_0\varphi T\rangle$ is either transversal, or vanishes. However, the transversal functions cannot exist in theories satisfying the condition (2.149). Hence it follows that

$$\langle\Phi_0(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = 0 \quad \text{if } l_0\neq d,\ s\neq 0\,. \tag{3.27}$$

The fields P_s^T is an exception since its Green functions satisfy anomalous Ward identities. The r.h. sides of the latter contain the terms $\sim \langle\varphi(x_1)\varphi(x_2)\rangle$ besides the "usual" terms accounted for in Eq. (3.25). Though, such anomalous terms are admissible only for the fields with dimensions $d+s$, see Ref. [3,15] for details.

3.2. Green functions of secondary fields

Consider the higher Green functions of the fields P_s^j. In the theories satisfying the conditions (2.77) the latter are defined by Eq. (1.87):

$$\langle P_s^j(x_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle$$
$$= \Lambda_s^j \operatorname*{res}_{l=d\pm s} \int \mathrm{d}x\,\mathrm{d}y\,\tilde{C}^l_{1\mu,\mu_1\dots\mu_s}(x_1xy)\langle j_\mu(y)\varphi(x)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle\,, \tag{3.28}$$

where the functions $\tilde{C}^l_{1\mu,\mu_1\dots\mu_s}$ are given by the expressions (2.69) and (2.72), and may be represented as

$$\tilde{C}^l_{1\mu,\mu_1\dots\mu_s}(x_1x_2x_3) = \langle\Phi^l_{\mu_1\dots\mu_s}(x_1)\tilde{\varphi}(x_2)A_\mu^{\mathrm{long}}(x_3)\rangle = \partial_\mu^{x_3}\tilde{C}^l_{\mu_1\dots\mu_s}(x_1x_2x_3)\,, \tag{3.29}$$

where

$$\tilde{C}^l_{\mu_1\dots\mu_s}(x_1x_2x_3) = (2\pi)^{-D/2}\,2^{(s-1)/2}\,\tilde{N}_1(\sigma,d)\,\lambda^{x_1}_{\mu_1\dots\mu_s}(x_3x_2)\tilde{\Delta}^l_j(x_1x_2x_3)\,. \tag{3.30}$$

Substitute Eq. (3.29) into Eq. (3.28) and bring it into the form

$$\langle P_s^j(x_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle$$
$$= -\Lambda_s^j \operatorname*{res}_{l=d\pm s} \int \mathrm{d}x\,\mathrm{d}y\,\tilde{C}^l_{\mu_1\dots\mu_s}(x_1xy)\partial_\mu^y\langle j_\mu(y)\varphi(x)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle\,. \tag{3.31}$$

The integral in the r.h.s. is taken as follows. Applying the Ward identity, represent integrand expression as a sum of terms, each term containing the factor $\delta(y-x_i)$. The term containing $\delta(x-y)$ does not contribute since it is multiplied by the power of the difference $(x-y)$. Resultantly, the integral over y evaluates due to the factors $\delta(y-x_i)$. As for the integral over x, note that the function $\tilde{\Delta}^l_j(x_1xy)$ entering Eq. (3.30) contains the factor $[(x_1-x)]^{-(D+l-d-s)/2}$, see Eq. (2.72). This factor is singular in the limit $l\to d+s$. Represent the expression (3.30) as the sum of terms containing this factor and then thread the symbol $\mathrm{res}_{l=d+s}$ through the integral sign. Calculating the pole of the integrand, we obtain the sum of terms each containing derivatives of $\delta(x_1-x)$. After evaluation of the integral over x, Eq. (3.31) may be rewritten in the form [5,15]

$$\langle P^j_{\mu_1\dots\mu_s}(x_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle = \hat{P}^j_{\mu_1\dots\mu_s}(x,\partial_{x_1})\,\langle\varphi(x_1)\dots\varphi^+(x_{2n})\rangle\,, \tag{3.32}$$

where $\hat{P}^j_{\mu_1\dots\mu_s}(x,\partial_{x_1})$ is known polynomial of rank s in derivative the ∂_{x_1}, its coefficients being dependent on the differences (x_1-x_k), $k=2,\dots,2n$. As an example we display its expression for $s=1$:

$$\hat{P}^j_\mu(x,\partial_{x_1}) \sim \left[\partial_\mu^{x_1} - 2d\sum_{k=2}^{n}\frac{(x_{1k})_\mu}{x_{1k}^2} + 2d\sum_{k=n+1}^{2n}\frac{(x_{1k})_\mu}{x_{1k}^2}\right]. \tag{3.33}$$

Next section contains expressions in more complicated cases.

The Green functions of the fields P_s^T are defined by Eq. (2.150) upon the substitution $\sigma_k = (d+s, s)$. The functions $\tilde{C}^\sigma_{\mu\nu}$ entering Eq. (2.150) are orthogonal to the functions $C^\sigma_{3\mu\nu}$, see Eq. (2.151), and can be represented in the form (see Refs. [3,6,15] for more details)

$$\tilde{C}^\sigma_{\mu\nu} = \tilde{C}^l_{\mu\nu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \langle \Phi^l_{\mu_1\ldots\mu_s}(x_1)\tilde{\varphi}(x_2)\tilde{h}^{\mathrm{long}}_{\mu\nu}(x_3)\rangle$$
$$= \partial^{x_3}_\mu B^l_{\nu,\mu_1\ldots\mu_s}(x_1x_2x_3) + \partial^{x_3}_\nu B^l_{\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) - \frac{2}{D}\delta_{\mu\nu}\partial^{x_3}_\lambda B^l_{\lambda,\mu_1\ldots\mu_s}(x_1x_2x_3)\,, \tag{3.34}$$

where $B^l_{\mu,\mu_1\ldots\mu_s}$ is the conformally invariant function

$$B^l_{\mu,\mu_1\ldots\mu_s}(x_1x_2x_3)$$
$$= \langle \Phi^l_{\mu_1\ldots\mu_s}(x_1)\tilde{\varphi}(x_2)h_\mu(x_3)\rangle$$
$$\sim \left\{\lambda^{x_3}_\mu(x_1x_2)\,\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2) + \tilde{\alpha}_s\frac{1}{x_{13}^2}\left[\sum_{k=1}^{s} g_{\mu\mu_k}(x_{13})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_3x_2) - \text{traces}\right]\right\}$$
$$\times\,\tilde{\Delta}^l_h(x_1x_2x_3)\,, \tag{3.35}$$

$\tilde{\varphi}$ is the conformal partner of the field φ, h_μ is the conformal vector of dimension $d_h = -1$, $\tilde{\alpha}_s$ is the constant,

$$\tilde{\Delta}^l_h(x_1x_2x_3) = (x_{12}^2)^{-(l-d-s+D+2)/2}\,(x_{13}^2)^{-(l-d-s+D-2)/2}\,(x_{23}^2)^{-(l+d-s-D-2)/2}\,. \tag{3.36}$$

Substitute Eq. (3.34) into Eq. (2.150) and bring it to the form

$$\langle P^T_{\mu_1\ldots\mu_s}(x_1)\varphi(x_2)\ldots\varphi(x_m)\rangle$$
$$= -2\Lambda^T_s \operatorname*{res}_{l=d+s}\int \mathrm{d}x\,\mathrm{d}y\,B^l_\mu(x_1xy)\partial^y_\nu\langle T_{\mu\nu}(y)\varphi(x)\varphi(x_2)\ldots\varphi(x_m)\rangle\,. \tag{3.37}$$

The integral in the r.h.s. is calculated in the same manner as the integral (3.31). One can show that the resulting expression takes the form

$$\langle P^T_{\mu_1\ldots\mu_s}(x_1)\varphi(x_2)\ldots\varphi(x_m)\rangle = \hat{P}^T_{\mu_1\ldots\mu_s}(x,\partial_{x_1})\,\langle\varphi(x_1)\ldots\varphi(x_m)\rangle\,, \tag{3.38}$$

where $\hat{P}^T_{\mu_1\ldots\mu_s}(x,\partial_{x_1})$ is known polynomial of rank $s+1$ in derivatives. Its expression is presented in the fourth section for the case $s=1, m=4$.

3.3. Dynamical sector of the Hilbert space

Consider the states

$$j_\mu(x_1)j_\nu(x_2)\varphi(x_3)\,|\,0\rangle,\quad T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\varphi(x_3)\,|\,0\rangle\,. \tag{3.39}$$

According to Eq. (3.1), the basis vectors of a Hilbert space may be found from expansion of the states

$$j_\mu(x_1)P^j_s(x_2)\,|\,0\rangle,\quad T_{\mu\nu}(x_1)P^T_s(x_2)\,|\,0\rangle\,. \tag{3.40}$$

It is possible to bring the result into the form of operator expansions[3]

$$j_\mu(x_1)P^j_s(x_2) = \sum_{s'} [P^j_{s'}] + \sum_{s'} [P^{j,s}_{s'}], \tag{3.41}$$

$$T_{\mu\nu}(x_1)P^T_s(x_2) = \sum_{s'} [P^T_{s'}] + \sum_{s'} [P^{T,s}_{s'}], \tag{3.42}$$

where

$$P^s_{s'}(x) = \{P^{j,s}_{s'}, P^{T,s}_{s'}\} = P^s_{\mu_1 \ldots \mu_{s'}}(x) \tag{3.43}$$

is a new family of symmetric traceless tensor fields with dimensions

$$d^s_{s'} = d + s'.$$

The fields $P^s_{s'}$ have the same properties as the P_s fields, see Eqs. (3.18) and (3.20).

Repeating the above steps again and again, we end up with a *family of fields*

$$P_s, P^{s_1}_s, P^{s_1 s_2}_s, \ldots, P^{s_1 \ldots s_k}_s, \ldots, \quad s_1 + s_2 + \cdots + s_k \leq s \tag{3.44}$$

spanning the basis of a subspace of total Hilbert space, to which all the states (3.11) *and* (3.12) *belong.* The dynamical principle which governs the effective interaction, is formulated in this subspace, called below the *dynamical sector of a Hilbert space.* The states

$$T_{\mu\nu}(x_1)\chi(x_2)\,|\,0\rangle, \quad T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\chi(x_3)\,|\,0\rangle, \ldots$$

also belong to this sector.

Since the dynamical sector is generated by energy-momentum tensor and the current, an introduction of any consistent model-fixing condition on the states of this sector may be regarded as a means of specification of an *effective Hamiltonian.* For a number of simplest models the constraint on the states of the dynamical sector can be obtained directly from the initial Lagrangian [4,15]. For this purpose one introduces the conformally invariant regularization. The renormalization constants z_1, z_2, z_3 (in models with two fields) remain finite as long as the regularization is kept up. In the renormalized Schwinger–Dyson equations the term $z_1\gamma$ is held, leading to the greater transparency of the equations [4,15].

The above discussion is a subject of extensive studies in Refs. [2–6,15]. The fields P_s were first found in our works [12,32] as a consequence of conformally invariant solution to Ward identities and later they were discussed in Refs. [2,33,34].

Note that the fields (3.44) have the properties analogous to those of the secondary fields in two-dimensional theory. In the next section we show that for $D = 2$ the fields (3.44) literally represent covariant combinations of secondary fields. Adopting the terminology of two-dimensional theories [16], these fields can be viewed as the *conformal family* of fields generated by a *primary field* $\varphi(x)$.

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[3] These expressions are written in a formal style. In fact, for each pair of values $s, s' > s$ there may exist several fields $P^s_{s'}$ orthogonal with each other. The number of those depends on s, s'.

The dynamical sector might be completely fixed only when the anomalous contributions to commutators (3.4) are given. The assignments of definite values to commutators (3.3) and (3.4) may be thought as the way by which the quantization rules are taken into account effectively, leading to a consistent definition of renormalized Schwinger–Dyson system, see Refs. [5,15]. For this purpose it proves necessary to define operator product expansions

$$j_\mu(x_1)j_\nu(x_2), \quad T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2), \quad j_\mu(x_1)T_{\rho\sigma}(x_2) . \tag{3.45}$$

It is apparent that the contributions to commutators (3.4) are solely due to C-number terms of these expansions, or to operator terms comprising the fields having integer dimensions (which cannot exceed dimensions of a current or energy–momentum tensor) and a definite tensor structure (different for spaces with even and odd dimensions). In Refs. [3,4,15] the following anomalous contributions are considered:

$$j_\mu(x_1)j_\nu(x_2) = [C_j] + [P_j] + [T_{\lambda\tau}] + \cdots, \tag{3.46}$$

$$T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) = [C_T] + [P_T] + [T_{\lambda\tau}] + \cdots, \tag{3.47}$$

$$T_{\mu\nu}(x_1)j_\rho(x_2) = [j_\sigma] + \cdots, \tag{3.48}$$

where $[C_j]$ and $[C_T]$ are the C-number contributions to expansions. The constants C_j and C_T define the normalization of Euclidean Green functions

$$\langle j_\mu(x_1)j_\nu(x_2)\rangle, \quad \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle . \tag{3.49}$$

In the class of theories under consideration, the dependence on C_j and C_T appears only in spaces with even dimensions, see Eqs. (2.177) and (2.186).

The second terms in Eq. (3.46) and in Eq. (3.47) denote anomalous operator contributions of the scalar fields $P_j(x)$ and $P_T(x)$, for which the following unified notation is useful:

$$P(x) = \{P_j(x), P_T(x)\} . \tag{3.50}$$

Both fields have the same dimension

$$d_{P_j} = d_{P_T} = D - 2 . \tag{3.51}$$

The conformally invariant Green functions of the fields (3.50) have the form

$$\langle P_j(x_1)P_j(x_2)\rangle \sim \langle P_T(x_1)P_T(x_2)\rangle \sim (x_{12}^2)^{-D+2} , \tag{3.52}$$

$$\langle \varphi(x_1)\varphi^+(x_2)P_j(x_3)\rangle \sim \langle \varphi(x_1)\varphi^+(x_2)P_T(x_3)\rangle = g_T^P \left(\frac{x_{12}^2}{x_{13}^2 x_{23}^2}\right)^{(D-2)/2} \langle \varphi(x_1)\varphi^+(x_2)\rangle , \tag{3.53}$$

where g_T^P is the coupling constant.

In two-dimensional space the fields P_j and P_T become constants

$$\begin{aligned} &P_j(x)|_{D=2} = g_j^P, \quad P_T(x)|_{D=2} = g_T^P , \\ &\langle P(x_1)P(x_2)\rangle|_{D=2} = \text{const.}, \quad \langle \varphi(x_1)\varphi^+(x_2)P(x_3)\rangle|_{D=2} \sim \langle \varphi(x_1)\varphi(x_2)\rangle . \end{aligned} \tag{3.54}$$

The two leading contributions in Eqs. (3.46) and (3.47) become c-numbers, their sum coinciding with the term proportional to the central charge of two-dimensional theory.

Note that for even $D \geq 4$ the fields $P_T(x)$ and $T_{\mu\nu}(x)$ are secondary fields generated by the c-number contribution C_T. Similarly, the fields $P_j(x)$ and $j_\mu(x)$ are secondary fields generated by C_j.

In conclusion we list the anomalous Ward identities based on the expansions (3.46)–(3.48). They are derived from the conditions of conformal invariance of the r.h.s. and have the following form [3,4,15] (see Ref. [35] for more details)

$$\begin{aligned}
&\partial_\mu^{x_1}\langle j_\mu(x_1) j_\nu(x_2)\varphi(x_3)\varphi^+(x_4)\rangle \\
&= -[\delta(x_{13})-\delta(x_{14})]\,\langle j_\nu(x_2)\varphi(x_3)\varphi^+(x_4)\rangle \\
&\quad + C_j\partial_\nu^{x_1}\Box^{(D-2)/2}\,\delta(x_{12})\langle\varphi(x_3)\varphi^+(x_4)\rangle + \partial_\nu^{x_1}\delta(x_{12})\langle P_j(x_2)\varphi(x_3)\varphi^+(x_4)\rangle\,,
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
&\partial_\mu^{x_1}\langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle \\
&= C_T\,\{\partial_\nu\partial_\rho\partial_\sigma - ((D-1)/2D)\,(\delta_{\nu\rho}\partial_\sigma + \delta_{\nu\sigma}\partial_\rho)\Box - (1/D^2)\,\delta_{\rho\sigma}\partial_\nu\Box\}\,\Box^{(D-2)/2}\,\delta(x_{12})\langle\varphi(x_1)\varphi(x_2)\rangle \\
&\quad - \{\delta(x_{13})\partial_\nu^{x_3} + \delta(x_{14})\partial_\nu^{x_4} - (d/D)\partial_\nu^{x_1}\,[\delta(x_{13})+\delta(x_{14})]\}\,\langle T_{\rho\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle \\
&\quad + \{\hat F_{\nu,\rho\sigma}(\partial^{x_1},\partial^{x_2})\,\langle P_T(x_2)\varphi(x_3)\varphi(x_4)\rangle\} \\
&\quad + \{[-\delta(x_{12})\partial_\nu^{x_2} + (1-(2/D)\,a)\partial_\nu^{x_1}\delta(x_{12})]\,\langle T_{\rho\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle \\
&\quad + \tfrac{1}{2}(1+a)\,[\partial_\rho^{x_1}\,\delta(x_{12})\,\langle T_{\nu\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle \\
&\quad + \partial_\sigma^{x_1}\delta(x_{12})\,\langle T_{\nu\rho}(x_2)\varphi(x_3)\varphi(x_4)\rangle - (2/D)\,\delta_{\rho\sigma}\partial_\lambda^{x_1}\,\delta(x_{12}) \\
&\quad \times\langle T_{\lambda\nu}(x_2)\varphi(x_3)\varphi(x_4)\rangle] + \tfrac{1}{2}(a-1)\,[\delta_{\nu\rho}\partial_\lambda^{x_1}\,\delta(x_{12}) \\
&\quad \times\langle T_{\lambda\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle + \delta_{\nu\sigma}\partial_\lambda^{x_1}\,\delta(x_{12})\,\langle T_{\lambda\rho}(x_2)\varphi(x_3)\varphi(x_4)\rangle \\
&\quad - (2/D)\,\delta_{\rho\sigma}\partial_\lambda^{x_1}\,\delta(x_{12})\,\langle T_{\nu\lambda}(x_2)\varphi(x_3)\varphi(x_4)\rangle]\}
\end{aligned} \tag{3.56}$$

where a as a free parameter and $\hat F_{\nu,\rho,\sigma}$ is the following differential operator:

$$\begin{aligned}
\hat F_{\nu,\rho\sigma}(\partial^{x_1},\partial^{x_2}) = \Bigg[&\frac{1}{2}(1+f)\,\partial_\nu^{x_1}\partial_\rho^{x_1}\partial_\sigma^{x_1}\,\delta(x_{12}) + \frac{1}{(D-2)}(2+f)\partial_\nu^{x_1}\partial_\sigma^{x_1}\,\delta(x_{12})\partial_\rho^{x_2} \\
&- \frac{1}{2(D-2)}[fD+(D+2)]\partial_\rho^{x_1}\partial_\sigma^{x_1}\,\delta(x_{12})\partial_\nu^{x_2} + \frac{1}{2}\,\delta_{\nu\rho}\partial_\sigma^{x_1}\Box_{x_1}\delta(x_{12}) \\
&- \frac{1}{(D-2)^2}\left(Df + \frac{D^2+2D-2}{D-1}\right)\delta_{\nu\rho}\partial_\sigma^{x_1}\delta(x_{12})\Box_{x_2} \\
&+ \frac{1}{(D-2)}\,\delta_{\nu\rho}\Box_{x_1}\delta(x_{12})\partial_\sigma^{x_2} - \frac{D}{2(D-2)}\,\delta_{\nu\rho}\partial_\sigma^{x_1}\partial_\tau^{x_1}\delta(x_{12})\partial_\tau^{x_2} \\
&- \frac{D}{2(D-1)(D-2)}\,\partial_\sigma^{x_1}\delta(x_{12})\partial_\nu^{x_2}\partial_\rho^{x_2} + \frac{1}{(D-1)(D-2)}\,\partial_\nu^{x_1}\delta(x_{12})\partial_\rho^{x_2}\partial_\tau^{x_2} \\
&- \frac{D}{2(D-1)(D-2)}\,\partial_\tau^{x_1}\delta(x_{13})\delta_{\nu\rho}\partial_\sigma^{x_2}\partial_\tau^{x_2}\Bigg] + (\rho\leftrightarrow\sigma) - \text{trace in } \rho,\sigma\,.
\end{aligned} \tag{3.57}$$

Here $(\rho\leftrightarrow\sigma)$ stands for the expression in square brackets with ρ,σ indices interchanged, f is a free parameter.

There are four groups of terms (each placed between curly brackets) of different origin in this identity. The second group is due to common contribution of the commutator $\delta(x^0-y^0)[T_{0\nu}(x),\varphi(y)]$. The third group is the $\hat{P}_T$ field contribution. The fourth group consists of anomalous terms due to anomalous gradient corrections to the commutator $[T,T]$, which contain the energy–momentum tensor components. On account of Eq. (3.57), four free parameters enter this Ward identity (for even D):

$$C_T, f, g_T^P, a\,. \tag{3.58}$$

One can show that conformally invariant Ward identities corresponding to the expansion (3.48) have the following form [15,35]:

$$\begin{aligned}
&\partial_\mu^{x_1}\langle T_{\mu\nu}(x_1)j_\sigma(x_2)\varphi(x_3)\varphi^+(x_4)\rangle\\
&\quad= -\left\{\delta(x_{13})\partial_\nu^{x_3}+\delta(x_{14})\partial_\nu^{x_4}-\frac{d}{D}\partial_\nu^{x_1}[\delta(x_{13})+\delta(x_{14})]\right\}\\
&\quad\times\langle j_\sigma(x_2)\varphi(x_3)\varphi^+(x_4)\rangle-\delta(x_{12})\partial_\nu^{x_2}\langle j_\sigma(x_2)\varphi(x_3)\varphi^+(x_4)\rangle\\
&\quad+\left(1-\frac{2}{D}b\right)\partial_\nu^{x_1}\delta(x_{12})\langle j_\sigma(x_2)\varphi(x_3)\varphi^+(x_4)\rangle\\
&\quad+b\partial_\sigma^{x_1}\delta(x_{12})\langle j_\nu(x_2)\varphi(x_3)\varphi^+(x_4)\rangle+(b-1)\,\delta_{\nu\sigma}\partial_\tau^{x_1}\,\delta(x_{12})\,\langle j_\tau(x_2)\varphi(x_3)\varphi^+(x_4)\rangle\,,
\end{aligned} \tag{3.59}$$

where b is the free parameter,

$$\begin{aligned}
&\partial_\sigma^{x_2}\langle T_{\mu\nu}(x_1)j_\sigma(x_2)\varphi(x_3)\varphi^+(x_4)\rangle\\
&\quad= -[\delta(x_{23})-\delta(x_{24})]\,\langle T_{\mu\nu}(x_1)\varphi(x_3)\varphi^+(x_4)\rangle\\
&\quad+b\Big\{\partial_\mu^{x_2}\delta(x_{12})\,\langle j_\nu(x_1)\varphi(x_3)\varphi^+(x_4)\rangle+\partial_\nu^{x_2}\delta(x_{12})\,\langle j_\mu(x_1)\varphi(x_3)\varphi^+(x_4)\rangle\\
&\quad-\frac{2}{D}\,\delta_{\mu\nu}\,\partial_\tau^{x_2}\delta(x_{12})\,\langle j_\tau(x_1)\varphi(x_3)\varphi^+(x_4)\rangle\Big\}\,.
\end{aligned} \tag{3.60}$$

The free parameter enters both identities in such a way as to produce the same results after taking the derivatives $\partial_\sigma^{x_2}$ in Eq. (3.59) and $\partial_\mu^{x_1}$ in Eq. (3.60).

3.4. Null states of dynamical sector

Let us show that any dynamical model may be defined by a certain self-consistent constraint on the states of dynamical sector. To do this, let us compose the superpositions of the fields (3.44) having equal scale dimensions, see Eq. (3.85). Denote these superpositions as Q_s. Let us choose the coefficients in the superpositions in a way that ensures the "normal" form of its commutators with j_0 and $T_{0\nu}$, i.e. the absence of anomalous terms (which are, for example, present in Eq. (3.18)). Then the field $Q_s(x)$ transforms as a primary field:

$$\delta(x^0 - y^0)\,[\,j_0(x), Q_s(y)] = -\,\delta^{(D)}(x-y)Q_s(y)\,,$$
$$\delta(x^0 - y^0)\,[T_{0\mu}(x), Q_s(y)] = \mathrm{i}\delta^{(D)}(x-y)\partial_\mu Q_s(y) + \cdots\,. \tag{3.61}$$

In the last commutator the dots stand for the gradient term $\sim \partial^x_\mu \delta^{(D)}(x-y)Q_s(y)$.

Such a form of the commutators is guaranteed by the following self-consistency conditions:

$$\langle Q^j_s\, P^{s_1\ldots s_k}_{s'} j_\mu\rangle = 0 \quad \text{or} \quad \langle Q^T_s\, P^{s_1\ldots s_k}_{s'} T_{\mu\nu}\rangle = 0 \tag{3.62}$$

for all $s' = 0, 1, \ldots, s-1$. These conditions ensure the cancellation of anomalous contributions into the commutators (3.61) and, simultaneously, lead to the system of algebraic equations on the coefficients of the superposition Q_s. This is discussed in detail in the next section using particular examples.

Note that on account of operator product expansions of the type (3.42) the self-consistency conditions (3.62) may be replaced by the following equations:

$$\langle Q^j_s \varphi^+ j_{\mu_1} \ldots j_{\mu_n}\rangle = 0, \quad n = 1, 2, \ldots, \tag{3.63}$$

or

$$\langle Q^T_s \varphi T_{\mu_1\nu_1} \ldots T_{\mu_n\nu_n}\rangle = 0, \quad n = 1, 2, \ldots, \tag{3.64}$$

which are equivalent to the set of conditions (3.62) for all values of s':

$$s' = 0, 1, \ldots$$

Suppose that conditions (3.62) are satisfied. Then the equation

$$Q_s(x) = 0 \tag{3.65}$$

gives a self-consistent condition on the states of dynamical sector. Any such equation defines a certain exactly solvable model. The Lagrangean models also belong to this family, see Refs. [2–4,15].

Consider the simplest model. It is defined by the requirement of vanishing of the field $P_\mu = P^{d+1}_\mu$ having the scale dimension $d_1 = d + 1$

$$P_\mu(x) = P_s(x)|_{s=1} = 0\,. \tag{3.66}$$

This equation means that the states $P_\mu(x)\,|\,0\rangle$ disappear in the dynamical sector

$$P_\mu(x)\,|\,0\rangle = 0\,. \tag{3.67}$$

The Euclidean Green functions

$$\langle \varphi(x_1) \ldots \varphi^+(x_{2n})\chi(x_{2n+1}) \ldots \chi(x_{2n+m})\rangle \tag{3.68}$$

will satisfy the following system of differential equations:

$$\langle P_\mu(x_1)\varphi(x_2) \ldots \varphi^+(x_{2n})\chi(x_{2n+1}) \ldots \chi(x_{2n+m})\rangle = 0\,, \tag{3.69}$$

or, owing to Eq. (3.32)

$$\hat{P}_\mu(x, \partial^x)\langle\varphi(x_1)\ldots\varphi^+(x_{2n})\chi(x_{2n+1})\ldots\chi(x_{2n+m})\rangle = 0\,. \tag{3.70}$$

The latter is a vector equation. Thus one has a system of differential equations for each Green function. Consider the equations

$$\hat{P}_\mu(x, \partial^x)\,\langle\varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle = 0\,, \tag{3.71}$$

$$\hat{P}_{j,\mu}(x, \partial^x)\,\langle\varphi(x_1)\varphi^+(x_2)j_\nu(x_3)\rangle = 0\,, \tag{3.72}$$

$$\hat{P}_{T,\mu\nu}(x, \partial^x)\,\langle\varphi(x_1)\varphi^+(x_2)T_{\rho\sigma}(x_3)\rangle = 0\,. \tag{3.73}$$

The $\hat{P}_{j,\mu}$ and $\hat{P}_{T,\mu\nu}$ operators depend on parameters of anomalous terms entering Ward identities. Since the coordinate dependence of three-point functions is known, we get the equations on the free parameters of the theory, i.e. the scale dimensions

$$d,\ \Delta \tag{3.74}$$

and the parameters entering anomalous Ward identities. (One can show that the additional constraints on the parameters appear during the solution.)

Depending on the choice of anomalous terms in Ward identities (3.55) and (3.56) the following three variants of a model are possible:[4]

$$P^j_\mu(x) + \beta P^T_\mu(x) = 0, \quad P^j_\mu(x) = 0, \quad P^T_\mu(x) = 0\,, \tag{3.75}$$

where β is a constant.

All these variants are dealt with in Refs. [4,5,15], see also Section 4. In the second and third cases one of the two equations (3.72) and (3.73) survives.

A more complicated model is given by the equation

$$Q_{\mu\nu}(x) = P_{\mu\nu}(x) + \alpha\bar{P}_{\mu\nu}(x) = 0\,, \tag{3.76}$$

where α is unknown parameter and

$$P_{\mu\nu}(x) = P_s(x)\,|_{s=2}, \quad \bar{P}_{\mu\nu}(x) = P^{s_1}_s(x)\,|_{s=2,s_1=1}\,.$$

Eq. (3.76) means vanishing of corresponding states in dynamical sector:

$$Q_{\mu\nu}(x)\,|\,0\rangle = 0\,. \tag{3.77}$$

The Green functions (3.68) satisfy differential equations

$$\hat{Q}_{\mu\nu}(x, \partial^x)\langle\varphi(x_1)\ldots\varphi^+(x_{2n})\chi(x_{2n+1})\ldots\chi(x_{2n+m})\rangle = 0\,, \tag{3.78}$$

where

$$\hat{Q}_{\mu\nu}(x, \partial^x) = \hat{P}_{\mu\nu}(x, \partial^x) + \alpha\hat{\bar{P}}_{\mu\nu}(x, \partial^x)\,, \tag{3.79}$$

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[4] In two-dimensional space the $P^T_\mu(x)$ field is absent. For $D \geq 3$ it appears under the definite choice of anomalous operator terms.

and $\hat{P}_{\mu\nu}(x, \partial^x)$ is the operator which defines the field $P_{\mu\nu}(x)$, see Eq. (3.32) or Eq. (3.38); $\hat{\bar{P}}_{\mu\nu}(x, \partial^x)$ is analogous operator, defining $\bar{P}_{\mu\nu}(x)$. The free parameters (3.74) as well as the parameters in anomalous Ward identities are, similar to an above model, calculated from the equations

$$\hat{Q}_{\mu\nu}(x, \partial^x) \langle \varphi(x_1)\varphi^+(x_2)\chi(x_3)\rangle = 0\,, \tag{3.80}$$

$$\hat{Q}_{j,\mu\nu}(x, \partial^x) \langle \varphi(x_1)\varphi^+(x_2)j_\lambda(x_3)\rangle = 0\,, \tag{3.81}$$

$$\hat{Q}_{T,\mu\nu}(x, \partial^x) \langle \varphi(x_1)\varphi^+(x_2)T_{\lambda\sigma}(x_3)\rangle = 0\,. \tag{3.82}$$

Moreover, the following self-consistency conditions are present in this model:

$$\hat{Q}_{P_1,\mu\nu}(x, \partial^x) \langle \varphi(x_1)P^+_\rho(x_2)j_\lambda(x_3)\rangle = 0, \quad \hat{Q}_{P_1,\mu\nu}(x, \partial^x) \langle \varphi(x_1)P^+_\rho(x_2)T_{\lambda\sigma}(x_3)\rangle = 0\,, \tag{3.83}$$

fixing the value of the α parameter. These conditions are non-trivial due to Eq. (3.18) and the anomalous Ward identities for Green functions (3.19). As before, the three variants of a model are possible:

$$Q^j_{\mu\nu}(x) + \beta Q^T_{\mu\nu}(x) = 0, \quad Q^j_{\mu\nu}(x) = 0, \quad Q^T_{\mu\nu}(x) = 0\,. \tag{3.84}$$

As an example, in Section 4 we discuss the solution of the model defined by equation $Q^j_{\mu\nu}(x) = 0$.

Finally, let us consider a general case

$$Q_s(x) = P_s(x) + \sum_{k=1}^{s-1} \sum_{s_1,\dots,s_k} \alpha_{s_1,\dots,s_k} P_s^{s_1,\dots,s_k}(x)\,, \tag{3.85}$$

where

$$s_1 \le s_2 \le \cdots \le s_k, \quad s_1 + s_2 + \cdots + s_k \le s-1\,.$$

The coefficients $\alpha_{s_1,\dots,s_k}$ are determined by consistency conditions (3.62) for each of the fields

$$P_{s'}^{s_1,\dots,s_k}(x) \quad \text{where } s_1 + \cdots + s_k \le s' < s\,.$$

No more principal differences with the previous model exist.

Some of the models are equivalent to Lagrangian models [15]. A possible conjecture is that the three-dimensional Ising model corresponds to one of the solutions of a model

$$Q^T_{\mu\nu} = 0\,.$$

The constraints on the states of dynamical sector, viewed as a means to define a model, were first studied by the authors in 1978. It was shown [2,14,36] that the solutions of trivial models (the Thirring model and gradient model for $D = 4$) are defined by these conditions. Later [33] this problem was examined in a slightly different context. However, the far better understanding of this scheme, especially its features related to the necessity of introduction of fields $P_s^{s_1,\dots,s_k}$ [3] into dynamical spectrum and to the role of self-consistency conditions, has come to us after the works [16,18]. The scheme described above has a striking resemblance to the structure of two-dimensional models. The states

$$Q_s(x)\,|\,0\rangle \tag{3.86}$$

are analogous to null vectors of two-dimensional models. As will be shown in the next section the states (3.86) for $D = 2$ literally coincide with null vectors, and all the two-dimensional models known to the present time may be solved by the method described in this work without any reference to Virasoro algebra, as if we were completely unaware of its existence. Having in mind the explicit analogy between two-dimensional models and those described here, it looks quite probable that the latter models should correspond to yet unknown realization of D-dimensional analogue of Virasoro algebra.

4. Examples of exactly solvable models in D-dimensional space

We consider a solution of several models discussed above under assumption that only a c-number analogue of the central charge exists, while the operator ones are absent:

$$P_j(x) = P_T(x) = 0 . \tag{4.1}$$

In the space of even dimension $D \geq 4$ it proves useful to work in terms of the potential $A_\mu(x) \equiv A_\mu^{\mathrm{long}}(x)$ rather than Abelian current j_μ, see Section 2.7. The self-consistency conditions (3.63) take the form

$$\langle Q_s^j \varphi^+ A_{\mu_1} \dots A_{\mu_k} \rangle = 0, \quad k = 1, 2, \dots . \tag{4.2}$$

4.1. A model of a scalar field

To illustrate the main ideas and calculation specifics we start with the simplest model in the space of even dimension $D \geq 4$ defined by equation (see also [5,15])

$$P_\mu^j(x) = 0 . \tag{4.3}$$

This model is a scalar version of the pure gauge model discussed in Ref. [36]. It is presented here due to methodical reasonings.

Due to Eq. (3.31), the Green functions of the field P_μ^j are calculated from the equation

$$\langle P_\mu^j(x_1)\varphi(x_2) \dots \varphi^+(x_{2n})\rangle$$

$$= -\Lambda_1 \operatorname*{res}_{l=d+1} \int \mathrm{d}y_1\, \mathrm{d}y_2\, \tilde{C}_\mu^l(x_1 y_1 y_2) \partial_\nu^{y_2} \langle j_\nu(y_2)\varphi(y_1)\varphi(x_2) \dots \varphi^+(x_{2n})\rangle , \tag{4.4}$$

where Λ_1 is a constant and

$$\tilde{C}_\mu^l(x_1 x_2 x_3) = \lambda_\mu^{x_1}(x_3 x_2)(x_{12}^2)^{-(l-d-1+D)/2}\,(x_{13}^2)^{-(l+d-1-D)/2}\,(x_{23}^2)^{(l+d-1-D)/2} . \tag{4.5}$$

Using the Ward identities (1.53) one can transform the r.h.s. of Eq. (4.4) to get

$$-\Lambda_1 \operatorname*{res}_{l=d+1} \int \mathrm{d}y_1 \left[\sum_{r=2}^{n} \tilde{C}_\mu^l(x_1 y_1 x_r) - \sum_{r=n+1}^{2n} \tilde{C}_\mu^l(x_1 y_1 x_r) \right] \langle \varphi(y_1)\varphi(x_2) \dots \varphi^+(x_{2n})\rangle . \tag{4.6}$$

Note that the term containing $\delta(y_1 - y_2)$ is omitted: it is multiplied by the power factor $[(y_1 - y_2)^2]^{(l+d-1-D)/2}$ and does not contribute to the integral. Let us thread the residue symbol $\mathrm{res}_{l=d+1}$ through the integral sign and make use of relation

$$\operatorname*{res}_{l=d+1} \tilde{C}^l_\mu(x_1 x_2 x_3) = \frac{1}{2}\frac{\pi^{D/2}}{\Gamma\left(\frac{D+4}{2}\right)}(D+2)\left[-\partial^{x_2}_\mu + 2d\frac{(x_{13})_\mu}{x^2_{13}}\right]\delta(x_{12}), \tag{4.7}$$

which follows from Eq. (2.34) for $k = 0$. As the result we obtain

$$\langle P^j_\mu(x_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle$$
$$= A_1(D+2)\gamma\left[\partial^{x_1}_\mu - 2d\sum_{r=2}^{n}\frac{(x_{1r})_\mu}{x^2_{1r}} + 2d\sum_{r=n+1}^{2n}\frac{(x_{1r})_\mu}{x^2_{1r}}\right]\langle\varphi(x_1)\ldots\varphi^+(x_{2n})\rangle, \tag{4.8}$$

where we introduced

$$\gamma = \frac{1}{2}\frac{\pi^{D/2}}{\Gamma\left(\frac{D+4}{2}\right)}. \tag{4.9}$$

Eq. (4.3) leads to differential equations of the first order for all the Green functions of the model. For $n = 2$ we get the equation

$$\left\{\partial^{x_1}_\mu + 2d\left[-\frac{(x_{12})_\mu}{x^2_{12}} + \frac{(x_{13})_\mu}{x^2_{13}} + \frac{(x_{14})_\mu}{x^2_{14}}\right]\right\}\langle\varphi(x_1)\varphi(x_2)\varphi^+(x_3)\varphi^+(x_4)\rangle = 0. \tag{4.10}$$

It has the solution

$$\langle\varphi(x_1)\varphi(x_2)\varphi^+(x_3)\varphi^+(x_4)\rangle = \langle\varphi(x_1)\varphi^+(x_3)\rangle\,\langle\varphi(x_2)\varphi^+(x_4)\rangle\left(\frac{x^2_{12}x^2_{34}}{x^2_{14}x^2_{23}}\right)^d, \tag{4.11}$$

where $\langle\varphi(x_1)\varphi^+(x_2)\rangle = (\frac{1}{2}x^2_{12})^{-d}$.

Consider the self-consistency conditions (4.2) for the model (4.3). One must find all the Green functions $\langle P^j_\mu\varphi^+ A_{\mu_1}\ldots A_{\mu_k}\rangle$, $k = 1, 2, \ldots$. Consider the simplest case $k = 2$ first. We have

$$\langle P^j_\mu(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)A_{\mu_2}(x_4)\rangle$$
$$= -A_1\operatorname*{res}_{l=d+1}\int dy_1\,dy_2\,\tilde{C}^l_\mu(x_1 y_1 y_2)\partial^{y_2}_\nu\langle j_\nu(y_2)\varphi(y_1)\varphi^+(x_2)A_{\mu_1}(x_3)A_{\mu_2}(x_4)\rangle. \tag{4.12}$$

Employ the Ward identity:

$$\partial^x_\nu\,\langle j_\nu(x)\varphi(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)A_{\mu_2}(x_4)\rangle$$
$$= [-\delta(x - x_1) + \delta(x - x_2)]\,\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)A_{\mu_2}(x_4)\rangle$$
$$+\,\partial^x_{\mu_1}\delta(x - x_3)\,\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_2}(x_4)\rangle + \partial^x_{\mu_2}\delta(x - x_4)\,\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle. \tag{4.13}$$

As before, the first term $\sim \delta(x-x_1)$ does not contribute to the integral (4.12),

$$\langle P^j_\mu(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)A_{\mu_2}(x_4)\rangle = \Lambda_1 \operatorname*{res}_{l=d+1} \int \mathrm{d}y\, \{\partial^{x_3}_{\mu_1} \tilde{C}^l_\mu(x_1yx_3)\, \langle\varphi(y)\varphi^+(x_2)A_{\mu_2}(x_4)\rangle$$
$$+\, \partial^{x_4}_{\mu_2} \tilde{C}^l_\mu(x_1yx_4)\, \langle\varphi(y)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle$$
$$-\, \tilde{C}^l_\mu(x_1yx_2)\, \langle\varphi(y)\varphi^+(x_2)A_{\mu_1}(x_3)A_{\mu_2}(x_4)\rangle\}\,. \tag{4.14}$$

Let us substitute the explicit expressions for the Green functions $\langle\varphi\varphi^+A_\mu\rangle$, $\langle\varphi\varphi^+A_{\mu_1}A_{\mu_2}\rangle$ from Eq. (2.183):

$$\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_1}(x_1)A_{\mu_2}(x_2)\rangle = \langle\varphi(x_1)\varphi^+(x_2)\rangle\,\langle A_{\mu_1}(x_1)A_{\mu_2}(x_2)\rangle$$
$$+\,(g_A)^2\lambda^{x_3}_{\mu_1}(x_1x_2)\lambda^{x_4}_{\mu_2}(x_1x_2)\,\langle\varphi(x_1)\varphi^+(x_2)\rangle\,. \tag{4.15}$$

Note that disconnected component of this Green function does not contribute to Eq. (4.14) due to Eq. (4.3) for $n=1$:

$$\left[\partial^{x_1}_\mu + 2d\,\frac{(x_{12})_\mu}{x^2_{12}}\right]\langle\varphi(x_1)\varphi^+(x_2)\rangle = 0\,. \tag{4.16}$$

Recalling also that due to Eq. (4.7)

$$\partial^{x_3}_{\mu_1} \operatorname*{res}_{l=d+1} \tilde{C}^l_\mu(x_1x_2x_3) = -\gamma(D+2)2d\,\frac{1}{x^2_{13}}\,g_{\mu\mu_1}(x_{13})\delta(x_{12})\,, \tag{4.17}$$

we find from Eq. (4.14)

$$\langle P^j_\mu(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)A_{\mu_2}(x_4)\rangle$$
$$= -\Lambda_1\gamma(D+2)\,(2d+g_A)\left[\frac{1}{x^2_{13}}\,g_{\mu\mu_1}(x_{13})\,\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_2}(x_4)\rangle\right.$$
$$\left. +\,\frac{1}{x^2_{14}}\,g_{\mu\mu_2}(x_{14})\,\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle\right]. \tag{4.18}$$

Making analogous calculations in the general case $k>2$ will lead to the following result:

$$\langle P^j_\mu(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots A_{\mu_k}(x_k)\rangle$$
$$= -\Lambda_1\gamma(D+2)\,(2d+g_A)\left[\sum_{r=1}^k \frac{1}{(y_1-x_r)^2}\,g_{\mu\mu_r}(y_1-x_r)\right.$$
$$\left.\times\,\langle\varphi(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots\hat{A}_{\mu_r}(x_r)\ldots A_{\mu_k}(x_k)\rangle\right], \tag{4.19}$$

where the symbol $\hat{A}_{\mu_r}$ denotes the omission of the field $A_{\mu_r}(x_r)$ in this expression. In the derivation of this equation we have used the explicit form of the Green functions $\langle\varphi\varphi^+A_{\mu_1}\ldots A_{\mu_k}\rangle$, see Eqs. (2.183) and (4.15).

According to the general program described in the end of the previous section, the dependence of dimension d on the parameter C_j is determined by the equation

$$\langle P^j_\mu(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle = 0\,, \tag{4.20}$$

while the vanishing of all the higher Green functions

$$\langle P^j_\mu(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\dots A_{\mu_k}(x_k)\rangle = 0, \quad k = 2, 3\dots \tag{4.21}$$

is interpreted as the infinite set of self-consistency conditions of the model. The Green function $\langle P_\mu\varphi^+A_{\mu_1}\rangle$ has the form

$$\langle P^j_\mu(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle = -\Lambda_1\gamma(D+2)\,(2d+g_A)\frac{1}{x^2_{13}}\,g_{\mu\mu_1}(x_{13})\,\langle\varphi(x_1)\varphi^+(x_2)\rangle\,.$$

Eq. (4.20) implies $(2d + g_A) = 0$, or, owing to Eq. (2.184)

$$d = (-1)^{D/2}\left[(4\pi)^{D/2}\Gamma\left(\frac{D}{2}\right)\right]^{-1}\frac{1}{C_j}\,. \tag{4.22}$$

Let us remark that the general expression (4.19) contains the factor $(2d + g_A)$ which is independent on the value of k. So if Eq. (4.22) is taken into account, self-consistency conditions for the model hold identically.

The model described here is the simplest D-dimensional analogue of two-dimensional exactly solvable models: the Thirring model and Wess–Zumino–Witten model. The first one was solved by the authors of the present article using the method discussed here as far back as in 1978 in the works [2,14], and the second, in the work [37], see also Refs. [3,5,15]. Both methodically and technically, the method of solution of these models is analogous to the solution of pure gauge model. Not surprisingly, the results of the work [18] are reproduced in discussion therein.

4.2. A model in the space of even dimension $D \geq 4$ defined by two generations of secondary fields

Consider the second of the models (3.84)

$$Q^j_{\mu\nu}(x) = P^j_{\mu\nu}(x) + \alpha\bar{P}^j_{\mu\nu}(x) = 0\,, \tag{4.23}$$

where α is unknown parameter, and the fields $P^j_{\mu\nu}$ and $\bar{P}^j_{\mu\nu}$ are those appearing in the operator product expansion of $j_\mu\varphi$ and $j_\mu P^j_\nu$:

$$\begin{aligned} j_\mu(x_1)\varphi(x_2) &= [\varphi] + [P^j_\nu] + [P^j_{\rho\sigma}] + \cdots\,,\\ j_\mu(x_1)P^j_\nu(x_2) &= [\varphi] + [P^j_\lambda] + [\bar{P}^j_{\rho\sigma}] + \cdots\,. \end{aligned} \tag{4.24}$$

Both tensor fields have the same scale dimension

$$l_2 = \bar{l}_2 = d + 2\,.$$

Three dimensionless parameters exist in this model:

$$d,\ C_j,\ \alpha\,. \tag{4.25}$$

According to Eq. (3.62), these parameters are related by a pair of algebraic equations which may be derived from the system

$$\langle Q^j_{\mu\nu}\varphi^+ j_\tau\rangle = \langle P^j_{\mu\nu}\varphi^+ j_\tau\rangle + \alpha\langle \bar{P}^j_{\mu\nu}\varphi^+ j_\tau\rangle = 0\,, \tag{4.26}$$

$$\langle Q^j_{\mu\nu}P^{+j}_\rho j_\tau\rangle = \langle P^j_{\mu\nu}P^{+j}_\rho j_\tau\rangle + \alpha\langle \bar{P}^j_{\mu\nu}P^{+j}_\rho j_\tau\rangle = 0\,. \tag{4.27}$$

It is also necessary to convince oneself that the self-consistency conditions hold.

Below the solution of the model (4.23) will be found. We will obtain a pair of algebraic equations for the parameters (4.25) and the closed set of differential equations for higher Green functions.

4.2.1. Self-consistency conditions

We must prove that

$$\langle Q^j_{\mu\nu}(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots A_{\mu_k}(x_k)\rangle = 0 \quad \text{where } k = 1, 2 \ldots . \tag{4.28}$$

Similar to the previous model, we will concurrently find the parameters (4.25).

Consider the Green functions of the field $P^j_{\mu\nu}$. According to Eq. (3.31) we have

$$\langle P^j_{\mu\nu}(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots A_{\mu_k}(x_k)\rangle$$
$$= -\Lambda_2 \operatorname*{res}_{l=d+2} \int dz_1\, dz_2\, \tilde{C}^l_{\mu\nu}(y_1 z_1 z_2)\partial^{z_2}_\rho\langle j_\rho(z_2)\varphi(z_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots A_{\mu_k}(x_k)\rangle\,, \tag{4.29}$$

where

$$\tilde{C}^l_{\mu\nu}(x_1x_2x_3) = \lambda^{x_1}_{\mu\nu}(x_3x_2)(x^2_{12})^{-(l-d-2+D)/2}\,(x^2_{23}/x^2_{13})^{(l+d-2-D)/2}\,. \tag{4.30}$$

The result is, see Ref. [5] for more details

$$\langle P_{\mu\nu}(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots A_{\mu_k}(x_k)\rangle$$
$$= \gamma\Lambda_2\left\{(g^2_A + 4(d+1)\,g_A)\left[\sum_{\substack{r,t=1\\ r\neq t}}^{k}\frac{1}{(y_1-x_r)^2}\,g_{\mu_r\mu}(y_1-x_r)\frac{1}{(y_1-x_t)^2}\,g_{\mu_t\nu}(y_1-x_t)\right.\right.$$
$$\left.\times\langle\varphi(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots\hat{A}_{\mu_r}\ldots\hat{A}_{\mu_t}\ldots A_{\mu_k}(x_k)\rangle - \text{trace in } \mu,\nu\right]$$
$$+ (4d(d+1) - 2g_A)\left(\sum_{r=1}^{k}\left[\frac{1}{(y_1-x_r)^2}\,g_{\mu_r\mu}(y_1-x_r)\lambda^{y_1}_\nu(y_2x_r)\right.\right.$$
$$\left.\left.\left.\times\langle\varphi(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\ldots\hat{A}_{\mu_r}\ldots A_{\mu_k}(x_k)\rangle + (\mu\leftrightarrow\nu) - \text{trace in } \mu,\nu\right]\right)\right\}, \tag{4.31}$$

where γ is the parameter (4.9), and hats on $\hat{A}_{\mu_r}$ and $\hat{A}_{\mu_t}$ imply dropping of the fields $A_{\mu_r}(x_r)$, $A_{\mu_t}(x_t)$. Note that for $k = 1$ the first term in Eq. (4.31) is absent:

$$\langle P^j_{\mu\nu}(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle = \gamma\Lambda_2(4d(d+1) - 2g_A)\,g_A$$
$$\times\left[\frac{1}{x^2_{13}}\,g_{\mu_1\mu}(x_{13})\,\lambda^{x_1}_\nu(x_2x_3) + (\mu\leftrightarrow\nu) - \text{trace in } \mu,\nu\right]\langle\varphi(x_1)\varphi^+(x_2)\rangle\,. \tag{4.32}$$

Consider the Green functions of the field $\bar{P}^j_{\mu\nu}$. Recall that this field enters the operator product expansion of $j_\mu P^j_\nu$. Hence the analogue of Eq. (3.31) reads

$$\langle \bar{P}^j_s(x_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle$$

$$= \bar{\Lambda}^j_s \operatorname*{res}_{\sigma=\sigma_s} \int \mathrm{d}y_1\,\mathrm{d}y_2\, \tilde{C}^l_{\mu_1\dots\mu_s,\mu,\rho}(x_1y_1y_2)\langle j_\mu(y_2)P^j_\rho(y_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle\,, \tag{4.33}$$

where $\bar{\Lambda}^j_s$ are some constants, and

$$\tilde{C}^l_{\mu_1\dots\mu_s,\mu,\rho}(x_1x_2x_3) = \langle P^l_{\mu_1\dots\mu_s}(x_1)\tilde{P}_\rho(x_2)A^{\mathrm{long}}_\mu(x_3)\rangle \tag{4.34}$$

is the longitudinal conformally invariant function. Here $\tilde{P}_\rho$ denotes the conformal partner of the field P_ρ. Its dimension equals to $D-d-1$. Note that the general conformally invariant expression for the function of the type $\langle P^l_{\mu_1\dots\mu_s}P_\rho A_\mu\rangle$ includes five independent structures. One can check that only two independent combinations of the latter are longitudinal. The function (4.34) is longitudinal due to equation of the type (2.77) for the Green functions $\langle j_\mu P_\rho\dots\rangle$. The general conformally invariant expression for the longitudinal function may be shown to have the form

$$\tilde{C}^l_{\mu_1\dots\mu_s,\mu,\rho}(x_1x_2x_3) = \partial^{x_3}_\mu\, \tilde{C}^l_{\mu_1\dots\mu_s,\rho}(x_1x_2x_3)\,,$$

where

$$\tilde{C}^l_{\mu_1\dots\mu_s,\rho}(x_1x_2x_3)$$

$$= \left\{\lambda^{x_2}_\rho(x_1x_3)\lambda^{x_1}_{\mu_1\dots\mu_s}(x_3x_2) + \beta\frac{1}{x^2_{12}}\left[\sum_{r=1}^{s} g_{\rho\mu_r}(x_{12})\lambda^{x_1}_{\mu_1\dots\hat{\mu}_r\dots\mu_s}(x_3x_2) - \text{traces in } \mu_1\dots\mu_s\right]\right\}$$

$$\times (x_{12})^{-(D+l-d-s-2)/2}\left(\frac{x^2_{23}}{x^2_{13}}\right)^{(l+d-D-s+2)/2}\,, \tag{4.35}$$

where β is arbitrary parameter.

Substituting Eq. (4.35) into Eq. (4.33) we get

$$\langle \bar{P}^j_s(x_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle$$

$$= -\Lambda^j_s \operatorname*{res}_{\sigma=\sigma_s} \int \mathrm{d}y_1\,\mathrm{d}y_2\, \tilde{C}^l_{\mu_1\dots\mu_s,\rho}(x_1y_1y_2)\partial^{y_2}_\mu\, \langle j_\mu(y_2)P^j_\rho(y_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle\,. \tag{4.36}$$

Thus the Green functions of the fields $\bar{P}^j_s$ which arise in the operator product expansions $P_\rho j_\mu$ may be calculated from the Ward identities as well.

The Green functions of the field $\bar{P}^j_{\mu\nu}$ containing the fields A_μ have a similar representation

$$\langle \bar{P}^j_{\mu\nu}(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\dots A_{\mu_k}(x_k)\rangle$$

$$= -\bar{\Lambda}_2 \operatorname*{res}_{l=d+2} \int \mathrm{d}z_1\,\mathrm{d}z_2\, \tilde{C}^l_{\mu\nu,\rho}(y_1z_1z_2)\partial^{z_2}_\lambda\, \langle j_\lambda(z_2)P^j_\rho(z_1)\varphi^+(y_2)A_{\mu_1}(x_1)\dots A_{\mu_k}(x_k)\rangle\,, \tag{4.37}$$

where

$$\tilde{C}^{l}_{\mu\nu,\rho}(x_1x_2x_3) = \left\{\lambda^{x_2}_{\rho}(x_1x_3)\lambda^{x_1}_{\mu\nu}(x_3x_2) + \beta\frac{1}{x_{12}^2}\left[g_{\rho\mu}(x_{12})\lambda^{x_1}_{\nu}(x_3x_2) + (\mu\leftrightarrow\nu) - \text{trace in } \mu,\nu\right]\right\} \times (x_{12}^2)^{-(D+l-d-4)/2}\left(\frac{x_{23}^2}{x_{13}^2}\right)^{(l+d-D)/2} . \quad (4.38)$$

The calculation of the r.h.s. of Eq. (4.37) is presented in Ref. [5]. The result is

$$\langle \bar{P}^j_{\mu\nu}(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\dots A_{\mu_k}(x_k)\rangle = 2\gamma^2(D+2)\Lambda_1\tilde{\Lambda}_2(2d+g_A) \times\left\{(g_A+2(d+2))\left[\sum_{\substack{r,t=1\\ r\neq t}}^{k}\frac{1}{(y_1-x_r)^2}g_{\mu\mu_r}(y_1-x_r)\frac{1}{(y_1-x_t)^2}g_{\nu\mu_t}(y_1-x_t) \times\langle\varphi(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\dots\hat{A}_{\mu_r}\dots\hat{A}_{\mu_t}\dots A_{\mu_k}(x_k)\rangle - \text{trace in }\mu,\nu\right] - \left(\sum_{r=1}^{k}\left[\frac{1}{(y_1-x_r)^2}g_{\mu\mu_r}(y_1-x_r)\lambda^{y_1}_{\nu}(y_2x_r)\langle\varphi(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\dots\hat{A}_{\mu_r}\dots A_{\mu_k}(x_k)\rangle + (\mu\leftrightarrow\nu) - \text{trace in }\mu,\nu\right]\right)\right\}, \quad (4.39)$$

where

$$\tilde{\Lambda}_2 = \bar{\Lambda}_2\left[1+\beta(D-2)\right] . \quad (4.40)$$

The symbols $\hat{A}_{\mu_r}$, $\hat{A}_{\mu_t}$ mean that the fields $A_{\mu_r}(x_r)$, $A_{\mu_t}(x_t)$ are omitted. For $k=1$ the first term in Eq. (4.39) is absent

$$\langle \bar{P}^j_{\mu\nu}(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle = -2\gamma^2(D+2)\Lambda_1\tilde{\Lambda}_2(2d+g_A) \times\left[\frac{1}{x_{13}^2}g_{\mu_1\nu}(x_{13})\,\lambda^{x_1}_{\mu}(x_2x_3) + (\mu\leftrightarrow\nu) - \text{trace in }\mu,\nu\right]\langle\varphi(x_1)\varphi^+(x_2)\rangle . \quad (4.41)$$

Consider the self-consistency conditions (4.28). The Green functions of the $Q^j_{\mu\nu}$ have the form

$$\langle Q^j_{\mu\nu}(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\dots A_{\mu_k}(x_k)\rangle = \gamma\Lambda_2\left\{N_1\left(\sum_{\substack{r,t=1\\ r\neq t}}^{k}\frac{1}{(y_1-x_r)^2}g_{\mu_r\mu}(y_1-x_r)\frac{1}{(y_1-x_t)^2}g_{\mu_t\nu}(y_1-x_t) \times\langle\varphi(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\dots\hat{A}_{\mu_r}\dots\hat{A}_{\mu_t}\dots A_{\mu_k}(x_k)\rangle - \text{trace in }\mu,\nu\right) + N_2\left(\sum_{r=1}^{k}\left[\frac{1}{(y_1-x_r)^2}g_{\mu\mu_r}(y_1-x_r)\lambda^{y_1}_{\nu}(y_2x_r)\langle\varphi(y_1)\varphi^+(y_2)A_{\mu_1}(x_1)\dots\hat{A}_{\mu_r}\dots A_{\mu_k}(x_k)\rangle + (\mu\leftrightarrow\nu) - \text{trace in }\mu,\nu\right]\right)\right\}, \quad (4.42)$$

where

$$N_1 = g_A^2 + 4(d+1)g_A + \frac{2\pi^{D/2}}{\Gamma\left(\frac{D+2}{2}\right)} \frac{\Lambda_1 \tilde{\Lambda}_2}{\Lambda_2} (2d + g_A)\,[g_A + 2(d+2)]\,\alpha\,, \tag{4.43}$$

$$N_2 = 4d(d+1) - 2g_A - \frac{2\pi^{D/2}}{\Gamma\left(\frac{D+2}{2}\right)} \frac{\Lambda_1 \tilde{\Lambda}_2}{\Lambda_2} (2d + g_A)\,\alpha\,. \tag{4.44}$$

All the Green functions (4.42) vanish if one sets

$$N_1 = N_2 = 0\,.$$

This leads to the following equations:

$$g_A^2 + 4(d+1)g_A + \alpha N\,[g_A + 2(d+2)] = 0\,, \tag{4.45}$$

$$2g_A - 4d(d+1) + \alpha N = 0\,. \tag{4.46}$$

where

$$N = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D+2}{2}\right)} \frac{\Lambda_1 \tilde{\Lambda}_2}{\Lambda_2} (2d + g_A)\,. \tag{4.47}$$

Let us remind that the coupling constant g_A is expressed through the parameter C_j, see Eq. (2.184). Eliminating the factor αN from the system (4.45), (4,46) we obtain the equation which expresses dimension d in terms of parameter C_j:

$$g_A^2 - 4(d^2 + d - 1)g_A - 8d(d+1)(d+2) = 0\,. \tag{4.48}$$

One easily see that this equation has a solution satisfying physical requirements

$$d > D/2 - 1, \quad C_j > 0 \tag{4.49}$$

for any (even) space dimension $D \geq 4$.

4.2.2. Differential equations for Green functions of fundamental fields

Consider the equation

$$\langle Q^j_{\mu\nu}(x_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle = 0\,. \tag{4.50}$$

It represents a source to differential equations for Green functions $\langle\varphi_1 \ldots \varphi^+_{2n}\rangle$. The Green functions of the fields $P^j_{\mu\nu}$ and $\bar{P}^j_{\mu\nu}$ are calculated from Eqs. (3.31) and (4.36) for $s = 2$. Applying the Ward identities to these equations, we get

$$\langle P^j_{\mu\nu}(x_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle = \Lambda_2 \operatorname*{res}_{l=d+2} \int \mathrm{d}y_1 \left[\sum_{r=2}^{n} \tilde{C}^l_{\mu\nu}(x_1 y_1 x_r)\langle\varphi(y_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle \right.$$
$$\left. - \sum_{r=n+1}^{2n} \tilde{C}^l_{\mu\nu}(x_1 y_1 x_r)\langle\varphi(y_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle \right], \tag{4.51}$$

$$\langle \bar{P}^j_{\mu\nu}(x_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle = \bar{\Lambda}_2 \operatorname*{res}_{l=d+2} \int dy_1 \Bigg[\sum_{r=2}^{n} \tilde{C}^l_{\mu\nu,\rho}(x_1y_1x_r)\langle P^j_\rho(y_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle - \sum_{r=n+1}^{2n} \tilde{C}^l_{\mu\nu,\rho}(x_1y_1x_r)\langle P^j_\rho(y_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle \Bigg]. \quad (4.52)$$

As the result we obtain, see Ref. [5] for more details

$$\langle P^j_{\mu\nu}(x_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle = \hat{P}^j_{\mu\nu}(x,\partial^{x_1})\,\langle\varphi(x_1)\dots\varphi^+(x_{2n})\rangle\,, \quad (4.53)$$

where

$$\hat{P}^j_{\mu\nu}(x,\partial^x) = \gamma\Lambda_2 \Bigg\{ \partial^{x_1}_\mu\partial^{x_1}_\nu + 2(d+1)\Bigg[-\sum_{r=2}^{n} \frac{(x_{1r})_\mu}{x^2_{1r}}\partial^{x_1}_\nu + \sum_{r=n+1}^{2n} \frac{(x_{1r})_\mu}{x^2_{1r}}\partial^{x_1}_\nu + (\mu\leftrightarrow\nu)\Bigg] + 4d(d+1)\Bigg[-\sum_{r=2}^{n} \frac{(x_{1r})_\mu(x_{1r})_\nu}{(x^2_{1r})^2} + \sum_{r=n+1}^{2n} \frac{(x_{1r})_\mu(x_{1r})_\nu}{(x^2_{1r})^2}\Bigg] - \text{trace}\Bigg\}, \quad (4.54)$$

$$\langle \bar{P}^j_{\mu\nu}(x_1)\varphi(x_2)\dots\varphi^+(x_{2n})\rangle = \hat{\bar{P}}^j_{\mu\nu}(x,\partial^{x_1})\,\langle\varphi(x_1)\dots\varphi^+(x_{2n})\rangle\,, \quad (4.55)$$

where

$$\hat{\bar{P}}^j_{\mu\nu}(x,\partial^{x_1}) = \{\hat{P}^j_{1\mu}(x,\partial^{x_1})\hat{P}^j_\nu(x,\partial^{x_1}) + (\mu\leftrightarrow\nu) - \text{trace}\}\,. \quad (4.56)$$

Here $\hat{P}^j_{1\mu}$ and $\hat{P}^j_\nu$ are the differential operators of the first order

$$\hat{P}^j_{1\mu} = -\gamma\tilde{\Lambda}_2 \Bigg\{ \partial^{x_1}_\mu + 2(d+2)\Bigg[-\sum_{r=2}^{n} \frac{(x_{1r})_\mu}{x^2_{1r}} + \sum_{r=n+1}^{2n} \frac{(x_{1r})_\mu}{x^2_{1r}}\Bigg]\Bigg\}, \quad (4.57)$$

$$\hat{P}^j_\nu = \gamma\Lambda_1(D+2)\Bigg\{ \partial^{x_1}_\nu + 2d\Bigg[-\sum_{r=2}^{n} \frac{(x_{1r})_\nu}{x^2_{1r}} + \sum_{r=n+1}^{2n} \frac{(x_{1r})_\nu}{x^2_{1r}}\Bigg]\Bigg\}. \quad (4.58)$$

Note that the operator $\hat{P}^j_\nu$ is given by Eq. (4.8).

Introduce the differential operator

$$\hat{Q}^j_{\mu\nu}(x,\partial^{x_1}) = \hat{P}^j_{\mu\nu}(x,\partial^{x_1}) + \alpha\hat{\bar{P}}^j_{\mu\nu}(x,\partial^{x_1})\,. \quad (4.59)$$

where $\hat{P}^j_{\mu\nu}$ and $\hat{\bar{P}}^j_{\mu\nu}$ are defined by Eqs. (4.54) and (4.56). From Eq. (4.50) one finds

$$\hat{Q}^j_{\mu\nu}(x,\partial^{x_1})\,\langle\varphi(x_1)\dots\varphi^+(x_{2n})\rangle = 0\,. \quad (4.60)$$

Thus we obtain a closed set of differential equations for any Green function of the model. Note that $\hat{Q}^j_{\mu\nu}$ is a tensor operator. One can show that each Eq. (4.60) is equivalent to a set of several equations of the second order in variables $\xi_\alpha = x^2_{ik}x^2_{mn}/x^2_{im}x^2_{kn}$. It can be shown that the tensor equation (4.60) written in these variables is equivalent to a system of the three differential equations of the second order. The derivation of these equations and their solution was published in Ref. [38].

4.3. Primary and secondary fields

The commutators of fields with the zero components of current or energy–momentum tensor determine the transformation properties of the fields. The gradient terms in commutators might turn out to be significant provided that a higher symmetry like the D-dimensional analogue of the Virasoro algebra would be found. Above we have introduced the concepts of primary and secondary fields for $D \geq 3$. The fundamental field is primary by definition. The commutator of the field $\varphi(x)$ with j_0 has the standard form

$$\delta(x^0 - y^0)\,[\,j_0(x), \varphi(y)] = -\,\delta^{(D)}(x-y)\varphi(y)\,. \tag{4.61}$$

Similarly, the Ward identities in this case also have the standard form.

Note that the primary fields have the following property. Let $\Phi_s^l(x)$ be any primary field of dimension l and tensor rank s. It is known that invariant three-point functions $\langle \Phi_s^l \varphi^+ j_\mu \rangle$ of these fields are either zero or are transverse. The latter follows from the Ward identity

$$\partial_\mu^{x_3} \langle \Phi_s^l(x_1)\varphi^+(x_2) j_\mu(x_3)\rangle = -\,[\delta(x_{13}) - \delta(x_{23})]\,\langle \Phi_s^l(x_1)\varphi^+(x_2)\rangle = 0\,,$$

due to orthogonality of conformal fields $\langle \Phi_s^l(x_1)\varphi^+(x_2)\rangle = 0$, if $s \neq 0, l \neq d$. The transversal invariant functions may not appear in the theory if the condition (2.77) holds. Thus one has

$$\langle \Phi_s^l(x_1)\varphi^+(x_2) j_\mu(x_3)\rangle = 0 \quad \text{if} \quad s \neq 0,\, l \neq d \tag{4.62}$$

for any primary field Φ_s^l. The Green functions

$$\langle P_s^j \varphi^+ j_\mu \rangle, \quad \langle P_s^T \varphi^+ T_{\mu\nu}\rangle$$

are nonzero owing to the presence of anomalous terms in the Ward identities. The anomalous Ward identities are pertinent to any generation of secondary fields.

The fields Q_s^j and Q_s^T are constructed as superpositions of secondary fields satisfying the usual commutation relations (3.61). Due to the discussion above, see Eq. (4.62), the latter is guaranteed by the self-consistency conditions (3.62) which are equivalent to Eqs. (3.63) and (3.64). It is essential that the fields with such properties arise only for specific dependence of dimension on the central charge (i.e. the parameter C_j or C_T). The situation resembles the one in two-dimensional conformal models, and the dependence mentioned above is analogous to the Kac formula.

To illustrate what has been said let us find the anomalous Ward identities for the case of fields P_μ^j, $P_{\mu\nu}^j$, $\bar{P}_{\mu\nu}^j$ and demonstrate that the dependence of dimension on the parameter C_j is the consequence of the requirement for Q_μ^j and $Q_{\mu\nu}^j$ to be primary fields.

Consider the Ward identities in the case of the field P_μ^j. The most general form of anomalous Ward identities reads:

$$\begin{aligned}\partial_\lambda^x \langle j_\lambda(x) P_\mu^j(x_1)\varphi(x_2)\ldots\varphi^+(x_{2n})\rangle &= -\,\delta(x-x_1)\,\langle P_\mu^j(x_1)\cdots\varphi^+(x_{2n})\rangle \\ &\quad + a\partial_\mu^x\delta(x-x_1)\,\langle\varphi(x_1)\cdots\varphi^+(x_{2n})\rangle + \cdots\,,\end{aligned} \tag{4.63}$$

where the dots stand for all the "usual" contributions. To find the constant a we consider the Ward identity for the Green function $\langle j_\lambda P_\mu^j \varphi^+ A_{\mu_1}\rangle$. Using Eqs. (2.181) and (4.19) we get:

$$\begin{aligned}&\partial_{\mu_2}^{x_4} \langle P_\mu^j(x_1)\varphi^+(x_2) A_{\mu_1}(x_3) j_{\mu_2}(x_4)\rangle \\ &= C_j\,\Box_{x_4}^{(D-2)/2}\,\partial_{\mu_2}^{x_4}\langle P_\mu^j(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)A_{\mu_2}(x_4)\rangle \\ &= -\,\Lambda_1\gamma(D+2)(2d+g_A)\left[\frac{1}{x_{13}^2}\,g_{\mu\mu_1}(x_{13})C_j\,\Box_{x_4}^{(D-2)/2}\,\partial_{\mu_2}^{x_4}\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_2}(x_4)\rangle\right. \\ &\quad \left. + C_j\,\Box_{x_4}^{(D-2)/2}\,\partial_{\mu_2}^{x_4}\frac{1}{x_{14}^2}\,g_{\mu\mu_2}(x_{14})\,\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle\right].\end{aligned} \tag{4.64}$$

Finally we obtain, see Ref. [5] for more details

$$\partial^x_\lambda \langle j_\lambda(x) P^j_\mu(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle = -[\delta(x-x_1)-\delta(x-x_2)]\langle P^j_\mu(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle + a\partial^x_\mu\delta(x-x_1)\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle\,, \tag{4.65}$$

where a is the same constant as in Eq. (4.63). The derivation makes use of the fact that the Green function $\langle P_\mu\varphi^+A_{\mu_1}\rangle$ is defined by Eq. (4.19) for $k=1$. The constant a turns out to be

$$a = -\Lambda_1(2d+g_A)C_j(-1)^{(D+2)/2}(4\pi)^{D/2}\,\Gamma\left(\frac{D+2}{2}\right). \tag{4.66}$$

Setting $a=0$ we obtain the previous result (4.22): $g_A=-2d$. So this result is a consequence of the requirement for $Q^j_\mu = P^j_\mu$ to be a primary field.

Consider the anomalous Ward identities for the fields $P^j_{\mu\nu}$ and $\bar{P}^j_{\mu\nu}$. Each of them involves a pair of independent parameters, one of the two being related to anomalous contribution of the field P^j_μ, and the other, with a contribution of the field φ. Evaluating the quantity

$$C_j\,\square^{(D-2)/2}_{x_4}\,\partial^{x_4}_{\mu_2}\langle P^j_{\mu\nu}(x_1)\varphi(x_2)A_{\mu_1}(x_3)A_{\mu_2}(x_4)\rangle$$

and making use of the results of Section 4.2, we get, see Ref. [5] for more details

$$\begin{aligned}\partial^x_\lambda &\langle j_\lambda(x)P^j_{\mu\nu}(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle\\ &= -[\delta(x_1-x)-\delta(x_2-x)]\langle P^j_{\mu\nu}(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle\\ &\quad + b_1[\partial^x_\nu\delta(x-x_1)\langle P^j_\mu(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle + (\mu\leftrightarrow\nu) - \text{trace}]\\ &\quad + \left\{b_2\partial^x_\mu\partial^x_\nu\delta(x_1-x) + b_3\left[\frac{(x_{12})_\mu}{x_{12}^2}\partial^x_\nu\delta(x_1-x)+(\mu\leftrightarrow\nu)\right] - \text{trace}\right\}\\ &\quad \times\langle\varphi(x_1)\varphi^+(x_2)A_{\mu_1}(x_3)\rangle\,. \end{aligned}\tag{4.67}$$

One obtains the following expressions for the constants:

$$b_1 = -\tfrac{1}{2}b_0(g_A^2+4(d+1)g_A)[\Lambda_1(D+2)(2d+g_A)]^{-1}\,,$$
$$b_2 = -\tfrac{1}{2}b_3 = \tfrac{1}{4}b_0[4d(d+1)-2g_A]\,,$$

where

$$b_0 = (-1)^{D/2+1}(4\pi)^{D/2}\,\Gamma\left(\frac{D}{2}\right)C_j\gamma\Lambda_2\,.$$

The coefficients in anomalous terms of the Ward identity for the Green function $\langle j_\lambda\bar{P}^j_{\mu\nu}\varphi^+A_{\mu_1}\rangle$ are calculated analogously. This Ward identity coincides with Eq. (4.67) after the change $P^j_{\mu\nu}\to\bar{P}^j_{\mu\nu}$; $b_i\to\bar{b}_i$, $i=1,2,3$ in the latter identity. The coefficient $\bar{b}_i$ turn out to be

$$\bar{b}_1 = -\tfrac{1}{2}\bar{b}_0(2(d+2)+g_A)[\Lambda_1(D+2)(2d+g_A)]^{-1},\quad \bar{b}_2 = -\tfrac{1}{2}\bar{b}_3 = \tfrac{1}{4}\bar{b}_0\,,$$

where

$$\bar{b}_0 = Nb_0\,,$$

and the coefficient N is given by the formula (4.47).

Consider the field $Q^j_{\mu\nu} = P^j_{\mu\nu} + \alpha\bar{P}^j_{\mu\nu}$. Let us demand this field to be primary. It means that the Ward identity for the Green function

$$\langle j_\lambda Q^j_{\mu\nu}\varphi^+ A_\mu\rangle$$

should comprise "usual" terms only. This requirement leads to the equations

$$b_1 + \alpha\bar{b}_1 = 0,\quad b_2 + \alpha\bar{b}_2 = 0\,,$$

which are easily seen to coincide with Eqs. (4.45) and (4.46). Thus Eq. (4.48) which relates the parameters

$$d,\, C_j$$

is the consequence of the fact that the field $Q^j_{\mu\nu}$ is a primary field. In Ref. [6] we show, see also below Eqs. (4.89) and (4.94), that for $D = 2$ such an approach leads to the well-known [16,19] results. In particular, the Kac formula [39] arises as a consequence of the second equation from Eq. (3.62). This feature may prove useful in the derivation of the analogue of the Kac formula in D-dimensional space.

4.4. A model of two scalar fields in D-dimensional space

Let us consider the model defined by the first equation from Eq. (3.75):

$$P^j_\mu(x) + \beta P^T_\mu(x) = 0\,. \tag{4.68}$$

Analysing partial wave expansions of the Green functions $\langle T_{\mu\nu}\varphi T_{\rho\sigma}\varphi^+\rangle$ and $\langle T_{\mu\nu}\varphi\varphi^+\chi\rangle$ one can show that after the setting

$$a = -\frac{d(D-2)^2 - D(D^2 - 2D + D)}{(D-2)\,[d(D-4) - D(D-2)]}$$

in Eq. (3.56) there exists a pair of mutually orthogonal fields $P^T_{1\mu}$ and $P^T_{2\mu}$,

$$T_{\mu\nu}(x_1)\varphi(x_2) = [\varphi] + [P^T_{1\rho}] + [P^T_{2\rho}] + \cdots\,, \tag{4.69}$$

with one of them, say $P^T_{2\mu}$, having a negative norm. Due to that the contributions of the fields $P^T_{1\mu}$ and $P^T_{2\mu}$ into each of the Green functions mentioned above, compensate each other. On the other hand, the contribution of the field $P^T_\mu \equiv P^T_{1\mu}$ into the partial wave expansion of the function $\langle T_{\mu\nu}\varphi j_\rho\varphi^+\rangle$ is still uncompensated, if the parameter b in the Ward identities (3.59) and (3.60) which determine the latter function, equals to zero. One can show that

$$\langle P^j_\mu(x_1)P^T_\nu(x_2)\rangle \neq 0 \quad \text{only if } b = 0\,, \tag{4.70}$$

but $\langle P^j_\mu P^T_{2\nu}\rangle = 0$ for all b. The calculations needed to prove these statements and to find the solution of the model (4.68), are analogous to those in the previous sections (though are more cumbersome), and will be published separately, see also Refs. [3,15]. Here we just present a final result.

Consider the equations

$$\langle P^j_\mu \varphi^+ j_\nu\rangle + \beta\langle P^T_\mu \varphi^+ j_\nu\rangle = 0, \quad \langle P^j_\mu \varphi^+ T_{\nu\rho}\rangle + \beta\langle P^T_\mu \varphi^+ T_{\nu\rho}\rangle = 0\,, \tag{4.71}$$

$$\langle P^j_\mu \varphi^+ \chi\rangle + \beta\langle P^T_\mu \varphi^+ \chi\rangle = 0\,, \tag{4.72}$$

$$\langle P^j_\mu(x_1)\varphi^+(x_2)\varphi(x_3)\varphi^+(x_4)\rangle + \beta\langle P^T_\mu(x_1)\varphi^+(x_2)\varphi(x_3)\varphi^+(x_4)\rangle = 0\,. \tag{4.73}$$

One can show that for $C_j = 0$ the first three equations lead to the following algebraic relations:

$$\Delta = \frac{D(2d-D)}{D-1}, \quad \beta = 4D\,\frac{(d-D/2+1)}{D^2-d(d+1)}, \tag{4.74}$$

$$2d^2(D^2-2D-4) - dD(4D^2-5D-6) + D^3(2D-1) = 0\,.$$

The solutions read:

$$d \simeq 2.7, \quad \Delta = 3.6 \quad \text{for } D = 3,$$

$$d \simeq 3.65, \quad \Delta = 4.4 \quad \text{for } D = 4\,,$$

$$d \simeq 6.24, \quad \Delta = 6.24 \quad \text{for } D = 6\,.$$

Eq. (4.73) gives

$$\hat P^j_\mu(x,\partial_{x_1})\,\langle\varphi(x_1)\varphi^+(x_2)\varphi(x_3)\varphi^+(x_4)\rangle + \hat P^T_\mu(x,\partial_{x_1})\,\langle\varphi(x_1)\varphi^+(x_2)\varphi(x_3)\varphi^+(x_4)\rangle = 0\,, \tag{4.75}$$

where $\hat P^j_\mu$ is the operator (3.33), while $\hat P^T_\mu$ is defined by Eq. (3.37) for $s = 1$. It is convenient to represent the general conformally invariant expression for the four-point Green functions in the form

$$G(x_1x_2x_3x_4) = (x^2_{12}x^2_{34})^{-d}\,\Phi(\xi,\eta)\,, \tag{4.76}$$

where

$$\xi = \ln\frac{x^2_{12}x^2_{34}}{x^2_{13}x^2_{24}}, \quad \eta = \ln\frac{x^2_{12}x^2_{34}}{x^2_{14}x^2_{23}}$$

are the conformally invariant variables. Written in these variables, Eq. (4.75) may be shown to have the form:

$$\{\lambda^{x_1}_\mu(x_2x_3)\hat G_1(\xi,\eta,\partial_\xi,\partial_\eta) + \lambda^{x_1}_\mu(x_2x_4)\,\hat G_2(\xi,\eta,\partial_\xi,\partial_\eta)\}\,\Phi(\xi,\eta) = 0\,, \tag{4.77}$$

where

$$\hat G_1(\xi,\eta\,\partial_\xi,\partial_\eta) = \Big\{(e^{-\xi}-e^{-\eta}-1)\partial^2_\xi + 2(e^{-\xi}-e^{-\eta}+1)\partial_\xi\partial_\eta$$

$$+ (e^{-\xi}+e^{\eta-\xi}-2e^{-\eta}+2)\partial^2_\eta + 2d\left(e^{-\eta}-e^{-\xi}+\frac{D-3}{2D}\right)\partial_\xi$$

$$+ 2d(e^{-\eta} - e^{-\xi} - 1)\partial_\eta + d\left(e^{-\xi} - e^{-\eta} + \frac{1}{D}\right)$$

$$+ \frac{1}{4\beta}\left(d - \frac{D}{2} + 1\right)^{-1}(d + \partial_\xi)\Bigg\}, \tag{4.78}$$

$$\hat{G}_2(\xi, \eta\, \partial_\xi, \partial_\eta) = \Big\{(e^{-\eta} + e^{\xi-\eta} - 2e^{-\xi} + 2)\partial_\xi^2 + 2(e^{-\eta} - e^{-\xi} + 1)\partial_\xi\partial_\eta + (e^{-\eta} - e^{-\xi} - 1)\partial_\eta^2$$

$$+ 2d(e^{-\xi} - e^{-\eta} - 1)\partial_\xi + 2d\left(e^{-\xi} - e^{-\eta} + \frac{D-3}{2D}\right)\partial_\eta + d\left(e^{-\eta} - e^{-\xi} + \frac{1}{D}\right)$$

$$+ \frac{1}{4\beta}\left(d - \frac{D}{2} + 1\right)^{-1}(-d + \partial_\eta)\Bigg\}. \tag{4.79}$$

Thus the Green function (4.76) is determined by the pair of differential equations:

$$\hat{G}_1(\xi, \eta, \partial_\xi, \partial_\eta)\Phi(\xi, \eta) = 0, \quad \hat{G}_2(\xi, \eta, \partial_\xi, \partial_\eta)\Phi(\xi, \eta) = 0\,. \tag{4.80}$$

4.5. Two-dimensional conformal models

Let φ be a neutral scalar field of dimension d. Let us pass to the complex components $T = T_\pm$ and $P_s = P_{s\pm}$, see Eqs. (2.165) and (2.168). As already explained in Section 2.6, when $D = 2$, all the Green functions of the energy–momentum tensor satisfy Eq. (2.149) identically, and consequently are completely determined by the Ward identities, see Eq. (2.171). Write down the operator product expansion $T\varphi$ as

$$T(x_1)\varphi(x_2) = [\varphi] + \sum_{s=2}^{\infty} [P_s]\,. \tag{4.81}$$

The Green functions of the fields P_s are uniquely determined by Eq. (3.37) for $D = 2$. Taking the integrals in the r.h.s., see Refs. [41,6,15] for more details, we get (for $s \geq 2$):

$$\langle P_s(x)\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle \sim \frac{1}{2}(s-1)(s+1)(d+s-2)\,(\partial_x)^s\,\langle\varphi(x)\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle$$

$$- \sum_{k=3}^{s+1} \frac{\Gamma(s+2)\Gamma(d+s)}{\Gamma(k+1)\Gamma(s-k+2)\Gamma(d+s-k+1)}$$

$$\times \left\{\sum_{r=1}^{m} (x - x_r)^{-(k-2)}\left[\partial_{x_r} + \frac{1}{2}(k-2)\frac{d_r}{x - x_r}\right](\partial_x)^{s-k+1}\right\}$$

$$\times \langle\varphi(x)\varphi_1(x_1)\ldots\varphi_m(x_m)\rangle\,. \tag{4.82}$$

Here we have used the complex variables (2.162), $x = x^\pm$ and $\partial_x = \partial_\pm$ for the component $P_s = P_{s\pm}$, $\varphi(x) = \varphi(x^+, x^-)$, $P_s = P_s(x^+, x^-)$,

$$\langle P_{s\pm}(x)\varphi(x)T_\pm(x)\rangle = g_s^T(d, C)\left(\frac{x_{12}^\pm}{x_{13}^\pm x_{23}^\pm}\right)^2\left(\frac{x_{23}^\pm}{x_{12}^\pm x_{13}^\pm}\right)^s\langle\varphi(x_1)\varphi(x_2)\rangle\,, \tag{4.83}$$

where $\langle\varphi(x_1)\varphi(x_2)\rangle = (x_{12}^2)^{-d}$, $g_s^T(d, C)$ is the coupling constant:

$$g_s^T(d, C) \sim \left\{\frac{C}{12} + \frac{d(d-1)(d-2)}{(d+s)(d+s-1)s(s-1)}\left[\frac{d+s-1}{d+s-2} - \frac{1}{s+1}\right]\right.$$
$$\left. + (-1)^{s+1}\,\Gamma(s-1)\frac{\Gamma(d+1)}{\Gamma(d+s+1)}\left[1 - \frac{1}{4}(s+1)d(d+s-2)\right]\right\}, \tag{4.84}$$

C is the central charge.

The same way as it was done in Section 3, one can introduce a complete family of secondary fields

$$P_s,\ P_s^{s_1},\ P_s^{s_1,s_2},\ \dots, \tag{4.85}$$

which begets a basis of the dynamical sector. This sector comprises the states of the type

$$T(x_1)\dots T(x_k)\varphi(x)|0\rangle \quad \text{where } k = 0, 1, 2, \dots . \tag{4.86}$$

Any exactly solvable conformal model is defined by the equation, see Eq. (3.85)

$$Q_{s\pm}(x) = 0 \quad \text{where } Q_{s\pm}(x) = P_{s\pm}(x) + \sum_{k=1}^{s-1}\sum_{s_1\dots s_k}\alpha_{s_1\dots s_k}P_{s\pm}^{s_1\dots s_k}(x), \tag{4.87}$$

where $2 \le s_1 \le s_2 \le \cdots \le s_k$, $s_1 + s_2 + \cdots + s_k \le s-2$.

The simplest models are defined by the equations

$$Q_2(x) \equiv P_2(x) = 0 \quad \text{or} \quad Q_3(x) \equiv P_3(x) = 0 . \tag{4.88}$$

Setting the Green function (4.83) to zero for $s = 2$ or $s = 3$, and using Eq. (4.84), we find

$$C = \frac{d(5-4d)}{d+1} \quad \text{or} \quad C = -\frac{3d^2 - 14d + 8}{2(d+2)} . \tag{4.89}$$

For the first model one has [41,34], see Eq. (4.82)

$$\left[\frac{3}{2(d+1)}\partial_{x_1}^2 - \frac{1}{x_{12}}\partial_{x_2} - \frac{1}{x_{13}}\partial_{x_3} - \frac{1}{2}\left(\frac{d}{x_{12}^2} + \frac{\Delta}{x_{13}^2}\right)\right]\langle\varphi(x_1)\varphi(x_2)\chi(x_3)\rangle = 0 , \tag{4.90}$$

$$\left[\frac{3}{2(d+1)}\partial_{x_1}^2 - \sum_{k=2}^{4}\frac{1}{x_{1k}}\partial_{x_k} - \frac{d}{2}\frac{1}{x_{12}^2} - \frac{\Delta}{2}\left(\frac{1}{x_{13}^2} + \frac{1}{x_{14}^2}\right)\right]\langle\varphi(x_1)\varphi(x_2)\chi(x_3)\chi(x_4)\rangle = 0 . \tag{4.91}$$

Substituting the first equation into the conformally invariant expression for the function $\langle\varphi\varphi\chi\rangle$, we get [41,34]

$$d = \tfrac{3}{8}\Delta - \tfrac{1}{4} . \tag{4.92}$$

The more complex model is defined by the equation

$$Q_4(x) = P_4(x) + \beta\bar{P}_4(x) = 0 , \tag{4.93}$$

where the field $\bar{P}_4$ arises in the operator product expansion $T(x_1)P_2(x_2)$. One can prove that the equation $\langle Q_4 \varphi T \rangle = 0$ has two solutions

$$C = 1 - 4d, \quad C = \frac{(d-2)(33-4d)}{5(d+3)}, \tag{4.94}$$

see Refs. [6,15] for details.

The results (4.89)–(4.94) coincide with known the results derived in Refs. [16,19] on the basis of infinite dimensional conformal symmetry. To compare, one must factorize all the Green functions (i.e., to make transition to the fields which solely depend either on x^+ or on x^-), and also introduce the new quantum numbers $\delta_s = (l+s)/2$, $\bar{\delta}_s = (l-s)/2$ in place of (l, s). For one-dimensional fields $\varphi(x^+)$, $P_s(x^+)$ the latter means

$$\delta = d/2, \quad \delta_s = d/2 + s, \tag{4.95}$$

see Refs. [6,15] for more details. One easily checks that the values of the central charge (4.89) and (4.94) coincide with those following from the Kac formula [39,40]. Eqs. (4.90) and (4.91) coincide with the equations for the Ising and Potts models. Let us stress that in our consideration its derivation is solely based on the six-parametric symmetry (1.2). The solution to the Ising model in this formulation is studied in detail in Refs. [41,34], see also Ref. [15], while the solution of the Wess–Zumino–Witten model is studied in Refs. [37,15].

One can demonstrate [5,15] that the whole list of results known in two-dimensional conformal theories is reproduced in the framework of the approach developed herein. It is readily seen that the infinite-dimensional symmetry is implicitly present in this formulation. Indeed, the Ward identities are completely determined by the symmetry (1.2). On the other hand, knowledge of Ward identities for $D = 2$ amounts to a definition of the commutators $[T_\pm(x_1), T_\pm(x_2)]$ which the Virasoro algebra (or its central extension, to be exact) follows from. Thus the two approaches do coincide in principle, differing only by technical details. One can expect that the family of models defined in Section 3 is also related to a certain D-dimensional analogue of the Virasoro algebra, see Refs. [42,43] for example.

When $D = 2$, the dynamical sector coincides with the representation space of the Virasoro algebra, and the states $Q_s(x)|0\rangle$ coincide with the null vectors. Indeed, consider a family of secondary fields [16]

$$\varphi^{(k_1, \dots, k_m)}(x) = L_{k_1}(x) L_{k_2}(x) \dots L_{k_m}(x) \varphi(x), \tag{4.96}$$

where L_k are the generators of the Virasoro algebra. The two families (4.85) and (4.96) are easily seen to be isomorphic. Both of them arise as the result of operator product expansions (4.86). For example, consider the expansion

$$T(x_1)\varphi(x_2) = \sum_{k=0}^{\infty} (x_{12})^{-2+k} \varphi^{(-k)}(x_2) = \sum_{k=0}^{\infty} (x_{12})^{-2+k} L_{-k}(x)\varphi(x). \tag{4.97}$$

Passing to one-dimensional fields P_s, compare the above with the expansion (4.81). The latter is realized by the combinations of secondary fields (4.96) covariant under the transformations of the group SL(2, R):

$$P_s = [L_{-s} + \alpha_1 L_{-1} L_{-s+1} + \alpha_2 (L_{-1})^2 L_{-s+2} + \cdots + \alpha_{s-1} (L_{-1})^s] \varphi,$$

which satisfy the conditions

$$L_{-1}P_s = \partial P_s, \quad L_0 P_s = \left(\frac{d}{2} + s\right) P_s, \quad L_1 P_s = 0 . \tag{4.98}$$

If one makes use of the commutation relations of the Virasoro algebra and the identity $L_1(x)\varphi(x) = 0$, the coefficients $\alpha_a, \dots, \alpha_s$ are determined from the last condition. For example, one has

$$P_2(x) = \left[L_{-2}(x) - \frac{3}{2(2\delta+1)}(L_{-1}(x))^2\right]\varphi(x) = \varphi^{(-2)}(x) - \frac{3}{2(2\delta+1)}\partial_x^2\varphi(x) ,$$
$$P_3(x) = \varphi^{(-3)}(x) - \frac{2}{\delta+2}\partial_x\varphi^{(-2)}(x) + \frac{1}{(\delta+1)(\delta+2)}\partial_x^3\varphi(x) , \tag{4.99}$$

et cetera. Here $\delta = d/2$.

Each of the fields $Q_s(x)$ may also be expressed either through the fields (4.85), or the fields (4.96). The Green functions containing any of the fields (4.85) satisfy the anomalous Ward identities (3.20) for $D = 2$. Using the relations of the type (4.99), the anomalous terms may also be expressed through the anomalous terms in the case of the fields (4.96). By definition, each field Q_s is constructed as a combination of secondary fields (4.85) which represents a primary field, see Eq. (3.61). It may be expressed through the secondary fields (4.96) as well. The condition of cancellation of anomalous terms for the case of Green functions of the field Q_s leads to identical results independently on the choice of the type of secondary fields which the field Q_s is expressed through. The cancellation of anomalous terms in the Ward identities for the Green functions $\langle TQ_s \dots \rangle$ is guaranteed by Eq. (3.62) which determine the dependence of dimension of the field φ on the central charge (and the coefficients of the superposition (3.85) as well). The latter is demonstrated in Section 4.3 on an example of the Green functions $\langle j_\mu Q_s^j \dots \rangle$. It is evident from the above that the Kac formula in our approach results as the consequence of the second group of Eq. (3.62) for $D = 2$. The latter is demonstrated to a greater extent in Refs. [6,15].

5. Conformal invariance in gauge theories

5.1. Inclusion of the gauge interactions

This section has two goals. First we are going to discuss how the gauge interactions could be taken into account. Our second aim is to present a new viewpoint on the irreducibility conditions (2.77) and (2.149) for the current and energy–momentum tensor, which define the models discussed above. Formally, these conditions are the ones allowing to derive a unique solution to the conformal Ward identities in $D \geq 3$. According to Section 2, a general solution of the Ward identities may be uniquely represented as a sum of the two conformally invariant terms, see Eqs. (2.52) and (2.102). The second term is transversal and is caused by gauge interactions, while the first one contains the information on equal-time commutators of the j_0 and $T_{0\mu}$ components with the matter fields, see Ref. [15] for more details. The fields j_μ and $T_{\mu\nu}$, being determined by the

Ward identities, transform by direct sums of irreducible representations[5]

$$\tilde{Q}_j \oplus Q_j^{tr}, \quad \tilde{Q}_T \oplus Q_T^{tr}. \tag{5.1}$$

The conformal partners A_μ, $h_{\mu\nu}$ are, correspondingly, transformed by the direct sums

$$Q_A^{long} \oplus \tilde{Q}_A, \quad Q_h^{long} \oplus \tilde{Q}_h. \tag{5.2}$$

(Recall that the initial representations Q_j, Q_T and Q_A, Q_h defined by the transformation laws (2.3), (2.4) and (2.9), (2.10), are undecomposable).

The importance of these results is due to the following reason. The renormalized Schwinger–Dyson equations contain the integrals over internal lines of gauge fields. Let $\Delta^A_{\mu\nu}(x_{12}) = \langle A_\mu(x_1) A_\nu(x_2) \rangle$ be the propagator of the gauge field. Consider the integrals

$$\int dx\, dy\, G_\mu(x \ldots)\, (\Delta^A_{\mu\nu})^{-1}(x-y) G_\nu(y \ldots) = \int dx\, dy\, \Gamma_\mu(x \ldots) \Delta^A_{\mu\nu}(x-y) \Gamma_\nu(y \ldots),$$

where $G_\mu(x \ldots) = \langle A_\mu(x) \ldots \rangle$ are the Green functions, and $\Gamma_\mu(x \ldots)$ are the corresponding vertices. Analogous integrals appear in the approach developed herein (which is based on the solution [1,2] of Schwinger–Dyson equations). As shown in Sections 5.4 and 5.5, the calculation of these integrals in the case of conformal field theory is reduced to a calculation of contractions of Euclidean fields

$$\int dx\, dy\, A_\mu(x) \Delta^j_{\mu\nu}(x-y) A_\nu(y), \quad \int dx\, dy\, j_\mu(x) \Delta^A_{\mu\nu}(x-y) j_\nu(y),$$

where $\Delta^j_{\mu\nu}$ is the propagator of the current. The claim that the fields A_μ, j_μ transform by the direct sums of representations (5.1) and (5.2) manifests in a quite specific structure of the contractions. Below we shall show that due to Eqs. (5.1) and (5.2) each of these contractions could be represented in the following form:

$$\int dx\, A_\mu^{long}(x) \tilde{j}_\mu(x) + \int dx\, \tilde{A}_\mu(x) j_\mu^{tr}(x),$$

where A_μ^{long}, $\tilde{A}_\mu$ and $\tilde{j}_\mu$, j_μ^{tr} are the conformal fields transforming by irreducible representations Q_A^{long}, $\tilde{Q}_A$ and $\tilde{Q}_j$, Q_j^{tr}, respectively. Note that this expression does not include the formally invariant "cross-term"

$$\int dx\, \tilde{A}_\mu(x) \tilde{j}_\mu(x).$$

This means that the transversal part of the current $\tilde{j}_\mu$ does not contribute to the gauge interaction. Similarly, the longitudinal part of the field $\tilde{A}_\mu$ decouples from the gauge sector. In other words, the

[5] Let us remind that the conformal partial wave expansion of the current Green functions contains two terms, each being identified unambiguously. The first term is an expansion into invariant three-point functions $C^l_{1\mu,\mu_1\ldots\mu_l}$, see Eq. (2.68), which correspond to the direct irreducible representation $\tilde{Q}_j$. The second term is decomposed into invariant transversal functions $C^{l,\,tr}_{\mu,\mu_1\ldots\mu_l}$, see Eq. (2.65), which correspond to the irreducible representation Q_j^{tr}. An analogous situation arises in the case of the energy–momentum tensor. More comprehensive comments may be found in Ref. [15], see also Section 6.

"transversal current" $\tilde{j}^{\rm tr}_\mu = (\delta_{\mu\nu} - \partial_\mu\partial_\nu/\Box)\tilde{j}_\nu$ and the "longitudinal field" $\tilde{A}^{\rm long}_\mu = (\partial_\mu\partial_\nu/\Box)\tilde{A}_\nu$ do not contribute to physical phenomena.[6] From the mathematical point of view it implies the orthogonality of subspaces generated by the conformal currents $\tilde{j}_\mu$ and $j^{\rm tr}_\mu$ (or the fields $A^{\rm long}_\mu$, $\tilde{A}_\mu$). The orthogonality conditions (2.56) and (2.58) derived in Section 2 are the reflections of this feature. All the above admits a straightforward generalization to the case of the fields $h_{\mu\nu}(x)$ and $T_{\mu\nu}(x)$, see Section 5.5.

Due to importance of these results, in the latter sections we reproduce them explicitly from the requirement of conformal invariance for the generating functional of the gauge theory. Basing on this requirement we shall show that the fields A_μ and $h_{\mu\nu}$ transform by direct sums of representations (5.2), while the invariant contractions $\int A_\mu j_\mu$ and $\int h_{\mu\nu} T_{\mu\nu}$ have the structure described above. It provides one with a new standpoint concerning the irreducibility conditions (2.77) and (2.149), which select out the models of direct (non-gauge) interaction of the matter fields.

A more general family of models may be obtained by the introduction of gauge interactions into the models discussed above. To achieve this, it is necessary to give up the irreducibility conditions for the current and the energy–momentum tensor which have been accepted in Section 2. After that the dynamical sector acquires the states of the kind

$$j^{\rm tr}_\mu(x_1)\varphi(x_2)|0\rangle, \quad T^{\rm tr}_{\mu\nu}(x_1)\varphi(x_2)|0\rangle, \; \dots ,$$

where $j^{\rm tr}_\mu$ and $T^{\rm tr}_{\mu\nu}$ are the transversal conformal fields (2.5) corresponding to representations of the type Q_0. Let us pass to reducible fields transforming by the representations of the type (2.51):

$$\tilde{j}_\mu \to \tilde{j}_\mu + j^{\rm tr}_\mu, \quad \tilde{T}_{\mu\nu} \to \tilde{T}_{\mu\nu} + T^{\rm tr}_{\mu\nu} . \tag{5.3}$$

The transversal components $j^{\rm tr}_\mu$ and $T^{\rm tr}_{\mu\nu}$ are generated by gauge interactions. The electromagnetic interaction leads to the appearance of transversal parts $\langle j^{\rm tr}_\mu \cdots \rangle$ in all the Green functions, Eqs. (2.43) and (2.62) in particular. Analogously, the gravitational interaction begets a non-zero transversal part $G^{\rm tr}_{\mu\nu\rho\sigma}$ in Eq. (2.136) and non-vanishing kernels (2.146) in the case of higher Green functions $\langle T_{\mu\nu}\varphi \dots \varphi\rangle$. We stress that these transversal parts are to be "found" from the electromagnetic and gravitational interactions, meaning that one should evaluate the asymptotic operator product expansions $j^{\rm tr}_\mu(x_1)\varphi(x_2)$ and $T^{\rm tr}_\mu(x_1)\varphi(x_2)$. The latter is sufficient for the setting up all the states of the dynamical sector.

In the same manner as in Section 3, the dynamical sector is defined as the set of states of the type

$$j_{\mu_1}(x_1)\dots j_{\mu_k}(x_k)\varphi(x)\rangle|0\rangle, \quad T_{\mu_1\nu_1}(x_1)\dots T_{\mu_r\nu_r}(x_r)\varphi(x)|0\rangle , \qquad k, r = 0,1, \dots ,$$

where the current and the energy–momentum tensor are defined as in Eq. (5.3). After that one considers a family of the models (3.65). The primary fields Q_s still represent combinations of the secondary fields (3.44). However the latter are calculated taking into account electromagnetic and gravitational interactions. In particular, the simplest model (3.66) is generalized as

$$P^j_\mu(x) + R^j_\mu(x) = 0 ,$$

provided that non-vanishing field R^j_μ exists in the expansion (2.54).

---

[6] Recall that the conformal fields $\tilde{j}_\mu$ and $\tilde{A}_\mu$ are representatives of the equivalence classes, see Eqs. (2.38) and (2.40). The fields $\tilde{j}_\mu - \tilde{j}^{\rm tr}_\mu$ and $\tilde{A}_\mu - \tilde{A}^{\rm long}_\mu$ are different representatives of the same classes and thus are physically equivalent to the fields $\tilde{A}_\mu$ and $\tilde{j}_\mu$.

5.2. Conformal transformations of the gauge fields

As should be clear from Section 2, the formal transition of usual conformal transformation laws to the case of gauge theories poses a number of obstacles. The problem consists in the following. When considering the gauge field A_μ as a conformal vector with the transformation law (2.9), the requirement of conformal invariants leads to the purely longitudinal expression (2.28) of the propagator $\langle A_\mu A_\nu \rangle$. Note that this expression arises as long as the dimension of the field A_μ is canonical ($l_A = 1$). The canonical dimension in $D = 4$ conformal QED results from linearity (in the field A_μ) of the Maxwell equations. In (non-linear) non-Abelian theories one could expect that anomalous corrections to the dimension should appear. As shown in Ref. [44] (see also Ref. [15]), the latter is not the case in $D = 4$. This is the tensor field $F_{\mu\nu}$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)] \tag{5.4}$$

that acquires anomalous dimension, while the dimension of the field $A_\mu(x)$ remains canonical. Thus the difficulty mentioned above persists in non-Abelian theories.

The group-theoretical source of this difficulty is rooted in undecomposability of the representation Q_A given by the transformation law (2.9). The invariant longitudinal propagator (2.28) is related to the irreducible representation Q_A^{long} which acts in the space of gauge degrees of freedom. In non-Abelian case the latter are described by the field

$$A_\mu(x) = g(x)\partial_\mu g^{-1}(x)\,. \tag{5.5}$$

Below we show that in the Lagrangean approach, the invariance with respect to transformation (2.9) is possible only on the space of fields (5.5), i.e. under the condition $F_{\mu\nu} = 0$.

When $F_{\mu\nu} \neq 0$, the transformation law (2.9) needs to be modified. Let us remind that non-trivial (i.e., having a transversal part) field $\tilde{A}_\mu(x)$ is a certain representative of the equivalence class which the irreducible representation $\tilde{Q}_A$ is defined on. The transformation (2.9) relates different equivalence classes. The fields entering the same class are connected by gauge transformations. Hence one easily concludes that the more consistent transformation law for the field $\tilde{A}_\mu$ must be a combination of the transformation (2.9) and a gauge transformation of definite sort. The combined transformation law is discussed in the next three subsections.

In what follows we utilize the infinitesimal form of special conformal transformations

$$\delta\Phi(x) = \varepsilon_\lambda K_\lambda^x \Phi(x)\,,$$

where ε_λ are the parameters while K_λ are the generators of conformal transformations. For the scalar and vector fields of dimension l one has (see, example Refs. [2,22,15] and references therein):

$$K_\lambda^x \Phi^l(x) = (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau - 2x_\lambda l)\Phi^l(x)\,,$$

$$K_\lambda^x \Phi_\mu^l(x) = (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau - 2x_\lambda l)\Phi_\mu^l(x) + 2x_\mu \Phi_\lambda(x) - 2\delta_{\lambda\mu} x_\tau A_\tau(x)\,.$$

The invariance condition for the propagator of the field A_μ of dimension $l = 1$ has the form

$$(K_\lambda^{x_1} + K_\lambda^{x_2})\langle A_\mu(x_1) A_\nu(x_2)\rangle = 0\,, \tag{5.6}$$

where

$$K_\lambda A_\mu(x) = (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau - 2x_\lambda)A_\mu(x) + 2x_\mu A_\lambda(x) - 2\delta_{\lambda\mu}x_\tau A_\tau(x) . \tag{5.7}$$

The solution of Eq. (5.6) is the longitudinal function (2.28).

One easily checks that Eq. (5.4) and the action

$$S_0 = \frac{1}{4}\int dx\ \mathrm{Sp}\ F_{\mu\nu}^2$$

are invariant under the transformations

$$\delta A_\mu(x) = \varepsilon_\lambda K_\lambda A_\mu(x) . \tag{5.8}$$

Consider the gauge term. There exists a unique choice of Lorentz- and scale invariant gauge term, namely

$$S_{\mathrm{gauge}} \sim \int dx\ \mathrm{Sp}\ (\partial_\mu A_\mu)^2 . \tag{5.9}$$

It is not difficult to see that in general this term breaks the invariance under Eq. (5.7)

$$\delta S_{\mathrm{gauge}} \sim \varepsilon_\lambda \int dx\ A_\lambda^a \partial_\mu A_\mu^a \neq 0 . \tag{5.10}$$

However, on the pure gauge space (5.5) its invariance revives. Considering $g(x)$ as a scalar field of zero dimension

$$\delta g(x) = \varepsilon_\lambda K_\lambda g(x) = \varepsilon_\lambda (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau) g(x) ,$$

we have

$$\delta S_{\mathrm{gauge}} \sim \int dx\ \mathrm{Sp}\ [\partial_\mu(\partial_\mu g g^{-1})\partial_\lambda g g^{-1}]$$

$$= -\varepsilon_\lambda \int dx\ \mathrm{Sp}\ [\Box g^{-1}\partial_\lambda g + \partial_\mu g^{-1}\partial_\lambda \partial_\mu g] = 0 .$$

The result (5.10) is clearly understandable from the viewpoint of the above discussion: as far as the proper choice of a representative in the equivalence class of the field $\tilde{A}_\mu$ had not been made, the latter should contain an uncontrollable longitudinal part breaking down the gauge choice in the form (5.9).

5.3. *Invariance of the generating functional of a gauge field in a non-Abelian case*

Let us consider the generating functional of a non-Abelian theory

$$Z(J) = \frac{1}{N}\int dA_\mu\, dC\, d\bar{C} \exp\left\{\int dx \left[-\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2 - \bar{C}\partial \nabla C + A_\mu^a J_\mu^a \right]\right\} , \tag{5.11}$$

where $C, \bar{C}$ are ghost fields, ∇_μ is a covariant derivative

$$\nabla_\mu = \partial_\mu + [A_\mu, \dots], \quad \partial\nabla = \partial_\mu \nabla_\mu .$$

Invariance of the functional the under linear the conformal transformations (5.8) is violated by the gauge and ghost terms.

So, the term $\int dx\,(F^a_{\mu\nu})^2$ is invariant under the direct product of conformal and gauge groups, while the term

$$dA_\mu\, dC\, d\bar{C} \exp\left\{\int dx \left[\frac{1}{2\alpha}(\partial_\mu A^a_\mu)^2 - \bar{C}\partial\nabla C\right]\right\} \tag{5.12}$$

breaks either of the symmetries.

The idea of the approach proposed is as follows [44–46]. We shall show that *a complete change of the term* (5.12) *under special conformal transformations* (5.7) *and* (5.8) *can be compensated by a certain gauge transformation whose parameter depends in a special way on the field* A_μ. *This will thus prove the existence of combined transformations under which the term* (5.12) *remains invariant.* They consist of special conformal and compensating gauge transformations. Then we shall show that *these combined transformations form a non-linear representation of the conformal group.*

Let us consider the variation of the first term in Eq. (5.12) under special conformal transformations:

$$\int dx\,(\partial_\mu A^a_\mu)^2 \to \int dx\,(\partial_\mu A^a_\mu + 4\varepsilon_\lambda A^a_\lambda)^2 . \tag{5.13}$$

We shall find the variation of the ghost term. The ghost fields C and $\bar{C}$ are transformed as conformal scalars with scale dimensions d_C and $d_{\bar{C}}$, and from scale invariance it follows that $d_C + d_{\bar{C}} = 2$. It will be shown that the property of compensation of conformal and gauge transformations occurs under the condition

$$d_C = 0, \quad d_{\bar{C}} = 2 . \tag{5.14}$$

Special conformal transformations corresponding to these values are

$$\delta C(x) = \varepsilon_\lambda K_\lambda C(x), \quad \delta\bar{C}(x) = \varepsilon_\lambda K_\lambda \bar{C}(x) , \tag{5.15}$$

where

$$K_\lambda C(x) = (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau)C(x) ,$$

$$K_\lambda \bar{C}(x) = (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau - 4x_\lambda)\bar{C}(x) .$$

It can be readily verified that the quantity $\nabla_\mu C(x)$ is transformed as $A_\mu(x)$, while $\partial\nabla C(x)$ is transformed as $\partial_\mu A_\mu(x)$. As a result we have

$$\delta \int dx\, \bar{C}(x)\partial\nabla C(x) = \varepsilon_\lambda \int dx\, \{4\bar{C}\nabla_\lambda C + (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau - 8x_\lambda)\bar{C}\partial\nabla C\} .$$

The second term in braces can be omitted since it is a total derivative. Combining this result with Eq. (5.13), we find

$$\mathrm{d}A_\mu\,\mathrm{d}C\,\mathrm{d}\bar{C}\exp\left\{\int \mathrm{d}x\left[\frac{1}{2\alpha}(\partial_\mu A^a_\mu)^2 - \bar{C}\partial\nabla C\right]\right\}$$
$$\to \mathrm{d}A_\mu\,\mathrm{d}C\,\mathrm{d}\bar{C}\exp\left\{\int \mathrm{d}x\left[\frac{1}{2\alpha}(\partial_\mu A^a_\mu + 4\varepsilon_\mu A^a_\mu)^2 - \bar{C}(\partial\nabla + 4\varepsilon_\mu \nabla_\mu)C\right]\right\}. \quad (5.16)$$

In the right-hand side there exist additional terms

$$\frac{4}{\alpha}\varepsilon_\lambda\partial_\mu A^a_\mu A^a_\lambda - 4\varepsilon_\lambda\bar{C}\nabla_\lambda C$$

expressing symmetry breaking. We can easily make sure that to compensate these terms it is necessary to make the following BRST transformation:

$$\delta A_\mu(x) = -\nabla_\mu C(x)\varepsilon,\ \delta C^a(x) = -\frac{1}{2}t^{abc}C^b(x)C^c(x)\varepsilon,\ \delta\bar{C}^a(x) = -\frac{1}{\alpha}\partial_\mu A^a_\mu(x)\varepsilon\,, \quad (5.17)$$

where ε is a parameter of BRST transformation. To compensate the additional terms, the parameter ε should be chosen in the form

$$\varepsilon = 4\varepsilon_\lambda\int \mathrm{d}y\,\bar{C}^a(y)A^a_\lambda(y)\,.$$

Specific for these transformations is a non-linear dependence of the parameter on the fields. Such BRST transformations were studied in Refs. [47–50]. It is of importance that these transformations affect the measure of integration in the functional integral (5.11). This should necessarily be taken into account in a verification of invariance of the term (5.12) under combined transformations. So, we substitute in Eq. (5.12) the fields transformed according to Eqs. (5.7), (5.8), (5.15), (5.16) and (5.17) and examine the first-order terms in ε_λ. For the constant parameter ε the term (5.12) is invariant under BRST transformations, and therefore it is necessary to trace only their contribution from the measure variation. It can be easily verified that this contribution compensates the variation (5.16). We have thus provided that the generating functional is invariant under the following combined transformations [44,45]

$$\delta A_\mu(x) = \varepsilon_\lambda\tilde{K}_\lambda A_\mu(x),\ \delta C(x) = \varepsilon_\lambda\tilde{K}_\lambda C(x),\ \delta\bar{C}(x) = \varepsilon_\lambda\tilde{K}_\lambda\bar{C}(x)\,, \quad (5.18)$$

$$\tilde{K}_\lambda A_\mu(x) = K_\lambda A_\mu(x) - 4\nabla_\mu C(x)\int \mathrm{d}y\,\bar{C}^b(y)A^b_\lambda(y)\,, \quad (5.19)$$

$$\tilde{K}_\lambda C^a(x) = K_\lambda C^a(x) - 2t^{abd}C^b(x)C^d(x)\int \mathrm{d}y\,\bar{C}^f(y)A^f_\lambda(y)\,, \quad (5.20)$$

$$\tilde{K}_\lambda\bar{C}^a(x) = K_\lambda\bar{C}^a(x) - \frac{4}{\alpha}\partial_\mu A^a_\mu(x)\int \mathrm{d}y\,A^b_\lambda(y)\bar{C}^b(y)\,. \quad (5.21)$$

In the next section we check that the new operators $\tilde{K}_\lambda$ are actually generators of special conformal transformations.

These results can be represented in an alternative form where the ghost fields are integrated. To this end, we integrate Eq. (5.11) over the ghost fields to obtain[7]

$$Z(J)|_{J=0} = \frac{1}{N}\int \mathrm{d}A_\mu \det|\partial\nabla| \exp\left\{\int \mathrm{d}x\left[-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A^a_\mu)^2\right]\right\}. \tag{5.22}$$

Combined transformations of a gauge field, which consist of conformal transformations and compensating gauge transformations have the form [45,46]:

$$\delta A_\mu(x) = \varepsilon_\lambda \tilde{K}_\lambda A_\mu(x), \tag{5.23}$$

where

$$\tilde{K}_\lambda A_\mu(x) = K_\lambda A_\mu(x) - 4\nabla_\mu \frac{1}{\partial\nabla} A_\lambda(x). \tag{5.24}$$

We note that the generator $\tilde{K}_\lambda$ is obtained from Eq. (5.19) by a formal substitution

$$C(x)\bar{C}(y) \to \frac{1}{\partial\nabla}\delta(x-y).$$

Invariance of the generating functional (5.22) under the transformations (5.23) is readily proved by a direct verification. Consider the factor

$$\det|\partial\nabla| \exp\left\{\int \mathrm{d}x \frac{1}{2\alpha}(\partial_\mu A^a_\mu)^2\right\}.$$

Let us make a change $A_\mu \to A_\mu + \varepsilon_\lambda K_\lambda A_\mu$. It may be shown that

$$\det|\partial\nabla| \to \det|\partial\nabla + \varepsilon_\lambda \partial_\mu K_\lambda A_\mu| \sim \det|\partial\nabla + 4\varepsilon_\lambda A_\lambda|.$$

The factor independent of A_μ is omitted. As a result, we find

$$\det|\partial\nabla| \exp\left[\frac{1}{2\alpha}\int \mathrm{d}x\,(\partial_\mu A^a_\mu)^2\right] \to \det|\partial\nabla + 4\varepsilon_\lambda A_\lambda| \exp\left[\frac{1}{2\alpha}\int \mathrm{d}x(\partial_\mu A^a_\mu + 4\varepsilon_\mu A^a_\mu)^2\right].$$

We shall now make a compensating gauge transformation. If we choose it in the form of the second term in Eq. (5.24)

$$\delta' A_\mu = -4\varepsilon_\lambda \nabla_\mu \frac{1}{\partial\nabla} A_\lambda \tag{5.25}$$

then we again come to the initial expression for the generating functional $Z(J)$. For this purpose it is convenient to represent $\det|\partial\nabla + 4\varepsilon_\lambda A_\lambda|$ as the value of the functional $\Delta(A)$ determined

[7] For the sake of simplicity, we carry out all further calculations as if the operator $\partial\nabla$ had no zeros. In the presence of zeroes these expansions become much more complicated. But we do not consider this case here.

by the relation

$$\Delta(A)\int \mathrm{d}\omega\,\delta(\partial_\mu A_\mu^\omega(x) + 4\varepsilon_\mu A_\mu^\omega(x) - g(x)) = 1\,,$$

where $A_\mu^\omega(x) = A_\mu - \nabla_\mu\omega$, on the surface $\partial_\mu A_\mu(x) + 4\varepsilon_\mu A_\mu(x) = g(x)$, and then use the standard technique.

Let us present a new law of conformal transformations of the tensor field $F_{\mu\nu}$. Supplementing the previous transformation law with the gauge transformation (5.25), we find

$$\delta F_{\mu\nu}(x) = \varepsilon_\lambda \tilde{K}_\lambda F_{\mu\nu}(x) = \varepsilon_\lambda K_\lambda F_{\mu\nu}(x) + 4\varepsilon_\lambda\left[\frac{1}{\partial\nabla}A_\lambda(x), F_{\mu\nu}(x)\right],$$

where

$$K_\lambda F_{\mu\nu}(x) = (x^2\partial_\lambda - 2x_\lambda x_\tau\partial_\tau - 2d_F)F_{\mu\nu}(x) + 2x_\rho[\Sigma_{\lambda\rho}, F_{\mu\nu}(x)]\,.$$

Here d_F is an anomalous dimension of the field $F_{\mu\nu}$. As distinguished from d_A, there occur no restrictions on its value.

Let us consider the invariance conditions of the Green functions under the combined transformations

$$\begin{aligned} 0 = \delta\langle A_\mu A_\nu\rangle &= \delta\langle F_{\mu\nu}A_\rho\rangle = \delta\langle F_{\mu\nu}F_{\rho\tau}\rangle = \delta\langle C\bar{C}\rangle \\ &= \delta\langle A_\mu A_\nu A_\rho\rangle = \delta\langle F_{\mu\nu}A_\rho A_\tau\rangle = \cdots \end{aligned}$$

and so forth. Formally they can be obtained in a usual way in an analysis of the generating functional with allowance for its invariance. Substitution of the field variations (5.18)–(5.21), (5.23) and (5.24) gives the relations that link different Green functions. As distinguished from the usual conformal Ward identities, these relations are non-linear in the field due to a non-linear character of the transformations.

Consider the first of these identities:

$$\begin{aligned} \delta\langle A_\mu A_\nu\rangle &= \langle\delta A_\mu(x_1)A_\nu(x_2)\rangle + \langle A_\mu(x_1)\delta A_\nu(x_2)\rangle \\ &= \varepsilon_\lambda\Bigg[\langle K_\lambda A_\mu(x_1)A_\nu(x_2)\rangle + \langle A_\mu(x_1)K_\lambda A_\nu(x_2)\rangle - (6-2d_A) \\ &\quad\times\left(\langle\nabla_\mu\frac{1}{\partial\nabla}A_\lambda(x_1)A_\nu(x_2)\rangle + \langle A_\mu(x_1)\nabla_\nu\frac{1}{\partial\nabla}A_\lambda(x_2)\rangle\right)\Bigg] = 0\,. \end{aligned} \tag{5.26}$$

Here we have used an extension of the transformation (5.23), (5.24) to the case of anomalous dimension d_A of a gauge field. The condition $d_A = 1$ is a consequence[8] of the group law [15]. As will be shown right now, formal invariance of the Green functions holds for any d_A values.

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[8] This can be proved directly from the equations, see Refs. [44,15].

We shall analyse Eq. (5.26) in the transverse gauge and show [46] that its solution is a transverse power-law function

$$D^{\mathrm{tr}}_{\mu\nu}(x_{12}) \sim \left(\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}\right)\frac{1}{(x_{12}^2)^{d_A}}\,. \tag{5.27}$$

Let us examine the two last terms in Eq. (5.26). We take the notation

$$D_{\mu\lambda\nu}(x_{12}) = \left\langle \nabla_\mu \frac{1}{\partial \nabla} A_\lambda(x_1) A_\nu(x_2) \right\rangle .$$

In the transverse gauge we have

$$\partial_\mu D_{\mu\lambda\nu}(x) = D_{\lambda\nu}(x), \quad \partial_\nu D_{\mu\lambda\nu}(x) = 0\,.$$

The most general scale-invariant solution of these equations is

$$D_{\mu\lambda\nu}(x_{12}) = \frac{\partial^{x_1}_\mu}{\Box_{x_1}} D_{\lambda\nu}(x_{12}) + \beta\, \frac{\partial^{x_1}_\lambda}{\Box_{x_1}} D_{\mu\nu}(x_{12})\,,$$

where β is an unknown constant. When substituted in Eq. (5.26), the terms $\sim \beta$ are cancelled, and we come to the linear integro-differential equation

$$\langle K_\lambda A_\mu(x_1) A_\nu(x_2)\rangle + \langle A_\mu(x_1) K_\lambda A_\nu(x_2)\rangle - (6 - 2d_A)\left[\frac{\partial^{x_1}_\mu}{\Box_{x_1}} D_{\lambda\nu}(x_{12}) + \frac{\partial^{x_2}_\nu}{\Box_{x_2}} D_{\mu\lambda}(x_{12})\right] = 0\,.$$

Its solution is the function (5.27).

We shall analyse the Green function

$$D^{FA}_{\mu\nu,\tau}(x_{12}) = \langle F_{\mu\nu}(x_1) A_\tau(x_2)\rangle\,.$$

We shall show that the general conformally invariant expression for this function is

$$D^{FA}_{\mu\nu,\tau}(x_{12}) = C(\delta_{\mu\tau}\partial_\nu - \delta_{\nu\tau}\partial_\mu)\frac{1}{(x_{12}^2)^{d_F/2}}\,, \tag{5.28}$$

where d_F is an anomalous dimension of the field $F_{\mu\nu}$. Note that due to a non-linear character of conformal transformations of the fields $F_{\mu\nu}$, A_τ this function is non-zero for $d_F \neq 2$. The corresponding Ward identity for it is of the form

$$(K^{x_1}_\lambda + K^{x_2}_\lambda) D^{FA}_{\mu\nu,\tau}(x_{12}) - 4\left\langle F_{\mu\nu}(x_1) \nabla_\tau \frac{1}{\partial \nabla} A_\lambda(x_2)\right\rangle + 4\left\langle \left[\frac{1}{\partial \nabla} A_\lambda(x_1), F_{\mu\nu}(x_1)\right] A_\tau(x_2)\right\rangle = 0\,,$$

where the action of the operator $K^{x_1}_\lambda$ is defined above. In the Abelian case this equation becomes linear

$$(K^{x_1}_\lambda + K^{x_2}_\lambda) D^{FA}_{\mu\nu,\tau}(x_{12}) - 4\,\frac{\partial^{x_2}_\tau}{\Box_{x_2}} D^{FA}_{\mu\nu,\lambda}(x_{12}) = 0\,.$$

One can readily make sure that the expression (5.28) satisfies this equation only under the condition

$$d_F = 2\,.$$

In a non-Abelian case for $d_F \neq 2$ we have from Eq. (5.28)

$$(K_\lambda^{x_1} + K_\lambda^{x_2}) D^{FA}_{\mu\nu,\tau}(x_{12}) = -2(d_F-4)\frac{\partial_\tau^{x_2}}{\Box_{x_2}} D^{FA}_{\mu\nu,\lambda}(x_{12}) + (d_F-2)(d_F-4)\frac{\partial_\lambda^{x_1}}{\Box_{x_1}} D^{FA}_{\mu\nu,\tau}(x_{12}) + 2(d_F-2)C(\delta_{\mu\tau}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\tau})(x_{12}^2)^{-d_F/2}\,.$$

Substitution of this expression into the Ward identity yields some restrictions on the form of non-linear terms, for more details see Ref. [44]. Thus, anomalous dimensions $d_F \neq 2$ do not contradict the Ward identity. One can similarly examine the Green function $\langle F_{\mu\nu} F_{\rho\tau}\rangle$ and show that the most general expression for it, compatible with the Ward identity, is

$$D^{FF}_{\mu\nu,\rho\tau}(x_{12}) = \langle F_{\mu\nu}F_{\rho\tau}\rangle = f_1[g_{\mu\rho}(x_{12})g_{\nu\tau}(x_{12}) - g_{\mu\tau}(x_{12})g_{\nu\rho}(x_{12})](x_{12}^2)^{-d_F} + f_2(d_F-2)(\delta_{\mu\rho}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\rho})(x_{12}^2)^{-d_F}\,,$$

where $f_{1,2}$ are arbitrary constants.

5.4. Conformal QED in D = 4

Let us analyse the group-theoretical structure of the transformations (5.23) and (5.24) in Abelian case, where they take the form:

$$\delta A_\mu(x) = \varepsilon_\lambda K_\lambda A_\mu(x) - 4\varepsilon_\lambda \frac{\partial_\mu}{\Box} A_\lambda(x)\,. \tag{5.29}$$

Consider the transformations of the form

$$\delta A_\mu(x) = \varepsilon_\lambda \tilde{K}_\lambda A_\mu(x) = \varepsilon_\lambda \left[K_\lambda A_\mu(x) - 4\partial_\mu \frac{1}{\Box} A_\lambda^{\rm tr}(x) \right], \tag{5.30}$$

where $A_\lambda^{\rm tr}(x) = (\delta_{\lambda\tau} - \partial_\lambda\partial_\tau/\Box)A_\tau(x)$. They differ from Eq. (5.29) by the gauge transformation

$$\delta_g A_\mu(x) = \varepsilon_\lambda \partial_\mu \frac{\partial_\lambda\partial_\tau}{\Box^2} A_\tau(x)\,,$$

which leaves the gauge term (5.9) invariant. Thus for the modified conformal transformations one may choose either Eq. (5.29) or Eq. (5.30). For the discussion in this section it is convenient to utilize Eq. (5.30).

Introduce the projection operators

$$P^{\rm tr} A_\mu(x) = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) A_\nu(x), \quad P^{\rm long} A_\mu(x) = \frac{\partial_\mu\partial_\nu}{\Box} A_\nu(x)\,. \tag{5.31}$$

The generator $\tilde{K}_\lambda$ entering the transformations (5.30) may be represented in the form

$$\tilde{K}_\lambda A_\mu(x) = P^{\rm tr} K_\lambda P^{\rm tr} A_\mu(x) + K_\lambda P^{\rm long} A_\mu(x)\,. \tag{5.32}$$

To prove this, let us make use of the relations

$$\partial_\mu K_\lambda A_\mu(x) = (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau - 4x_\lambda)\partial_\mu A_\mu(x) + 4A_\lambda(x)\,,$$

$$\frac{1}{\Box}[(x^2\partial_\lambda - 2x_\lambda x_\tau\partial_\tau - 4x_\lambda)\partial_\mu A_\mu(x)] = (x^2\partial_\lambda - 2x_\lambda x_\tau \partial_\tau)\frac{1}{\Box}\partial_\mu A_\mu(x) - 4\frac{\partial_\lambda}{\Box^2}\partial_\mu A_\mu(x),$$

$$P^{\mathrm{long}} K_\lambda P^{\mathrm{long}} = K_\lambda P^{\mathrm{long}}\,,$$

and put the r.h.s. of Eq. (5.30) into the form

$$\tilde{K}_\lambda A_\mu(x) = K_\lambda A_\mu(x) - 4\frac{1}{\Box}\,\partial_\mu A_\lambda^{\mathrm{tr}}(x) = P^{\mathrm{tr}} K_\lambda P^{\mathrm{tr}} A_\mu(x) + K_\lambda P^{\mathrm{long}} A_\mu(x)\,. \tag{5.33}$$

So the modified special conformal transformations in QED have the form [27–29]:

$$\delta A_\mu(x) = \varepsilon_\lambda [P^{\mathrm{tr}} K_\lambda P^{\mathrm{tr}} A_\mu(x) + K_\lambda P^{\mathrm{long}} A_\mu(x)]\,. \tag{5.34}$$

The modified global conformal transformations have the similar structure:

$$\tilde{U}_g A_\mu(x) = P^{\mathrm{tr}} U_g P^{\mathrm{tr}} A_\mu(x) + U_g P^{\mathrm{long}} A_\mu(x)\,, \tag{5.35}$$

where U_g are the operators in the Hilbert space which generate a vector representation of the conformal group (for $d = 1$). In particular, the modified transformations of conformal inversion have the form [27–29]:

$$A_\mu(x) \overset{R}{\to} A'_\mu(x) = \left(\delta_{\mu\nu} - \frac{\partial_\mu^x \partial_\nu^x}{\Box_x}\right)\frac{1}{x^2} g_{\nu\rho}(x)\left(\delta_{\rho\tau} - \frac{\partial_\rho^{Rx}\partial_\tau^{Rx}}{\Box_{Rx}}\right) A_\tau(Rx)$$
$$+ \frac{1}{x^2} g_{\mu\rho}(x) \frac{\partial_\rho^{Rx}\partial_\tau^{Rx}}{\Box_{Rx}} A_\tau(Rx)\,. \tag{5.36}$$

One easily checks that the operators $\tilde{U}_g$ satisfy the group law

$$\tilde{U}_{g1}\tilde{U}_{g2} = \tilde{U}_{g1g2} \quad \text{if } U_{g1}U_{g2} = U_{g1g2}\,.$$

To demonstrate this one employs the relations

$$P^{\mathrm{tr}} U_g P^{\mathrm{long}} A_\mu(x) = 0, \quad P^{\mathrm{long}} U_g P^{\mathrm{long}} A_\mu(x) = U_g P^{\mathrm{long}} A_\mu(x)\,. \tag{5.37}$$

The transformation law (5.35) defines a reducible representation of the conformal group:

$$\tilde{Q}_A \oplus Q_A^{\mathrm{long}}\,. \tag{5.38}$$

Indeed, the representation Q_A^{long} is realized on the space of longitudinal fields A_μ^{long} and corresponds to second terms in each of Eqs. (5.34), (5.35) and (5.36). The first terms correspond to the representation $\tilde{Q}_A$. Recall that the latter acts on the space of equivalence classes $\{\tilde{A}_\mu\}$, each class comprising all the fields with a given transversal part $A_\mu^{\mathrm{tr}} = P^{\mathrm{tr}}\tilde{A}_\mu$

$$\{\tilde{A}\} : \tilde{A}_\mu = A_\mu^{\mathrm{tr}}(x) + \partial_\mu \varphi_0(x)\,,$$

where A^{tr}_μ is fixed, and $\varphi_0(x)$ is any scalar field. The transformations U_g map different equivalence classes into one another. To fix a definite realization of the transformations $\tilde{U}_g$, it is necessary to select a certain representative in each equivalence class. In the Eqs. (5.34), (5.35) and (5.36), the role of such a representative is played by the transversal part of the field $\tilde{A}_\mu$:

$$A_\mu(x) = A^{tr}_\mu(x) + A^{long}_\mu(x) \quad \text{where } A^{tr}_\mu = P^{tr}\tilde{A}_\mu(x)\,. \tag{5.39}$$

As the result, the irreducible representation $\tilde{Q}_A$ is herein realized (unlike the realization in Section 2) on the space of transversal functions. Any transformation in this realization may be represented as a sequence of the three transformations:

1. Gauge transformation inside within equivalence class: $\tilde{A}_\mu \to A^{tr}_\mu = P^{tr}\tilde{A}_\mu = P^{tr}A_\mu$;
2. Conformal transformation to a new equivalence class: $\{\tilde{A}\} \to \{\tilde{A}'\}$ (or $A^{tr}_\mu \to \tilde{A}'_\mu$);
3. Gauge transformation within a new equivalence class: $\tilde{A}'_\mu \to P^{tr}\tilde{A}'_\mu = P^{tr}A'_\mu$.

Finally, the r.h.s. of the transformation (5.35) (and, similarly, Eqs. (5.34) and (5.36)) may be represented in terms of the fields $\tilde{A}_\mu$, A^{long}_μ dealt with in Section 2:

$$A'_\mu = P^{tr}A'_\mu + A_\mu^{\prime long} = P^{tr}U_g P^{tr}\tilde{A}_\mu + U_g A^{long}_\mu\,.$$

Consider the conditions of invariance with respect to the new transformations. An infinitesimal form of these conditions is derived from Eq. (5.30). One gets for the propagator of the field A_μ:

$$(K^{x_1}_\lambda + K^{x_2}_\lambda)\langle A_\mu(x_1)A_\nu(x_2)\rangle - 4\frac{\partial^{x_1}_\mu}{\Box_{x_1}}\langle A^{tr}_\lambda(x_1)A_\nu(x_2)\rangle - 4\frac{\partial^{x_2}_\nu}{\Box_{x_2}}\langle A_\mu(x_1)A^{tr}_\lambda(x_2)\rangle = 0\,. \tag{5.40}$$

The general solution of these equations reads (up to a normalization) [27–29]:

$$\Delta^A_{\mu\nu}(x_{12}) = \langle A_\mu(x_1)A_\nu(x_2)\rangle \sim \frac{1}{4\pi^2}\left[\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\frac{1}{x^2_{12}} + \frac{\alpha}{4}\partial_\mu\partial_\nu \ln x^2_{12}\right], \tag{5.41}$$

where α is the gauge parameter,

$$\partial^{x_1}_\mu \Box\langle A_\mu(x_1)A_\nu(x_2)\rangle = \alpha\partial_\nu\delta(x_{12})\,. \tag{5.42}$$

The result (5.41) may also be derived from the condition of R-invariance

$$\langle A_\mu(x_1)A_\nu(x_2)\rangle = \langle A'_\mu(x_1)A'_\nu(x_2)\rangle\,,$$

where A'_μ is given by the expression (5.36).

The photon and the current propagators appear on internal lines in the Schwinger–Dyson equations (exact ones or those in the skeleton approximation[9]),as well as in the equations of the formalism presented in this paper. So let us examine the invariant contractions

$$\int dx\, dy\, j_\mu(x)\Delta^A_{\mu\nu}(x-y)j_\nu(y),\quad \int dx\, dy\, A_\mu(x)\Delta^j_{\mu\nu}(x-y)A_\nu(y),\quad \int dx\, j_\mu(x)A_\mu(x)\,, \tag{5.43}$$

[9] The skeleton approximation of conformal theories is described in the reviews [2,26,15], see also references therein.

where $\Delta^j_{\mu\nu}$ is the propagator of the current, see below. Unlike the discourse of Sections 2–4, the calculations of these contractions should now involve the electromagnetic interaction. Let us remind that there exists a pair of types of conformal currents: $j^{\rm tr}_\mu(x)$ and $\tilde{j}_\mu(x)$, transforming by irreducible representations $Q^{\rm tr}_j$ and $\tilde{Q}_j$, see Section 2. The field $\tilde{j}_\mu$ (and also the field $\tilde{A}\mu$) is nothing but a certainly fixed representative of the equivalence class $\{\tilde{j}_\mu\}$. The physics resulting does not depend on the choice of representative. Above we have already passed from the field $\tilde{A}_\mu$ to the field $A^{\rm tr}_\mu$. The results of calculations should coincide no matter which of these fields had been used. In the case of the current the situation is analogous. To examine the contractions (5.43) it proves more useful to utilize the representatives $A^{\rm tr}_\mu$ and $j^{\rm long}_\mu = P^{\rm long}\tilde{j}_\mu$ instead of the fields $\tilde{A}_\mu$ and $\tilde{j}_\mu$. Recall that the field $\tilde{j}_\mu$ is the representative of the equivalence class $\{\tilde{j}_\mu\}$ which includes all the fields $\tilde{j}_\mu$ with a fixed longitudinal part. Formally, the transition

$$\tilde{j}_\mu \to j^{\rm long}_\mu = P^{\rm long}\tilde{j}_\mu \tag{5.44}$$

leads to the change of the current's transformation law, see below. On account of Eq. (5.44) the total current should be represented as

$$j_\mu(x) = j^{\rm tr}_\mu(x) + j^{\rm long}_\mu(x) = j^{\rm tr}_\mu(x) + P^{\rm long}\tilde{j}_\mu(x)\,. \tag{5.45}$$

This differs from Eq. (2.52): the transversal part of the field $\tilde{j}_\mu$ is now omitted. According to Section 2, this part does not contribute into electromagnetic interaction. The current (5.45) corresponds to a new realization of the reducible representation

$$Q^{\rm tr}_j \oplus \tilde{Q}_j\,. \tag{5.46}$$

The transformation law of the current in the new realization reads [27–29]:

$$\delta j_\mu(x) = \varepsilon_\lambda[K_\lambda P^{\rm tr} j_\mu(x) + P^{\rm long}K_\lambda P^{\rm long} j_\mu(x)]\,, \tag{5.47}$$

or in the case of global transformations

$$j'_\mu = [U_g P^{\rm tr} + P^{\rm long}U_g P^{\rm long}]\, j_\mu(x)\,. \tag{5.48}$$

The first and the second terms describe the transformations of transversal and longitudinal parts of the current, respectively. Note that the second term manifests itself (alike the case of $\tilde{A}_\mu$ field, see above) as a combination of the three transformations: a transformation within equivalence class towards a longitudinal representative, conformal transformation to a new equivalence class, and a transformation towards a longitudinal representative within a new class. The Green function $\langle j_\mu j_\nu\rangle$, which is invariant under these transformations, may be shown [27–29] to have the following form:

$$\Delta^j_{\mu\nu}(x_{12}) = \langle j_\mu(x_1) j_\nu(x_2)\rangle = f_j(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\delta(x_{12}) + C_j\partial_\mu\partial_\nu\delta(x_{12})\,, \tag{5.49}$$

where f_j is some constant and C_j is the central charge introduced above.

Let us remind that the representations (5.38) and (5.46) are equivalent in virtue of Eq. (2.26), while the propagators $\Delta^A_{\mu\nu}$ and $\Delta^j_{\mu\nu}$ are the kernels of intertwining operators. Fix the free parameters entering Eqs. (5.41) and (5.49) by the constraint

$$\int dx_3\, \Delta^A_{\mu\rho}(x_{13})\Delta^j_{\rho\nu}(x_{32}) = \delta_{\mu\nu}\delta(x_{12})\,. \tag{5.50}$$

Then the equivalence conditions for the representations in the realizations (5.39) and (5.45) are written as

$$A_\mu(x) = \int dy\, \Delta^A_{\mu\nu}(x-y) j_\nu(y),\; j_\mu(x) = \int dy\, \Delta^j_{\mu\nu}(x-y) A_\nu(y)\,. \tag{5.51}$$

The second condition leads [28,29] to the Maxwell equations.[10]

Let us come back to the contractions (5.43). From Eqs. (5.50) and (5.51) one gets

$$\int dx\, dy\, j_\mu(x)\Delta^A_{\mu\nu}(x-y)j_\nu(y) = \int dx\, dy\, A_\mu(x)\Delta^j_{\mu\nu}(x-y)A_\nu(y) = \int dx\, j_\mu(x)A_\mu(x)\,. \tag{5.52}$$

The invariant propagators $\Delta^A_{\mu\nu}$ and $\Delta^j_{\mu\nu}$ coincide with the kernels of invariant scalar products on the direct sums of subspaces (see Section 2)

$$M^{tr}_j \oplus \tilde{M}_j \quad \text{and} \quad \tilde{M}_A \oplus M^{long}_A\,, \tag{5.53}$$

respectively. Each contraction in Eq. (5.52) may be represented as a sum of the two conformally invariant terms correspondent to two terms in the sums (5.53), both being independent on the choice of realization of the spaces $\tilde{M}_j$ and $\tilde{M}_A$ (i.e., on the fixing of representatives in equivalence classes). For example, consider the second contraction. Decompose it into a sum of transversal and longitudinal contributions

$$\int dx\, dy\, A^{tr}_\mu(x)\Delta^{j,\,tr}_{\mu\nu}(x-y)A^{tr}_\nu(y) + \int dx\, dy\, A^{long}_\mu(x)\Delta^{j,\,long}_{\mu\nu}(x-y)A^{long}_\nu(y)\,. \tag{5.54}$$

The transversal part $\Delta^{j,\,tr}_{\mu\nu}$ coincides with invariant propagator $D^{tr}_{\mu\nu} = \langle j^{tr}_\mu j^{tr}_\nu \rangle$ introduced in Section 2, see Eq. (2.33). Considering the first term, it is useful to pass to conformal fields $\tilde{A}_\mu$ belonging to the same equivalence class as A^{tr}_μ:

$$\int dx\, dy\, A^{tr}_\mu(x)\Delta^{j,\,tr}_{\mu\nu}(x-y)A^{tr}_\nu(y) = \int dx\, dy\, \tilde{A}_\mu(x)D^{tr}_{\mu\nu}(x-y)\tilde{A}_\nu(y)\,.$$

The second term includes conformal fields $A^{long}_\mu \in M^{long}_A$. Hence one can add a singular longitudinal part to the kernel $\Delta^{j,\,long}_{\mu\nu}$ and make transition to a singular invariant propagator $D^j_{\mu\nu}$ (see Eq. (2.32) and, for more details, Section 2.7):

$$\int dx\, dy\, A^{long}_\mu(x)\Delta^{j,\,long}_{\mu\nu}(x-y)A^{long}_\nu(y) = \int dx\, dy\, A^{long}_\mu(x)D^j_{\mu\nu}(x-y)A^{long}_\nu(y)\,.$$

Finally the expression (5.54) may be rewritten as

$$\int dx\, dy\, A_\mu(x)\Delta^j_{\mu\nu}(x-y)A_\nu(y)$$
$$= \int dx\, dy\, \tilde{A}_\mu(x)D^{tr}_{\mu\nu}(x-y)\tilde{A}_\nu(y) + \int dx\, dy\, A^{long}_\mu(x)D^j_{\mu\nu}(x-y)A^{long}_\nu(y)\,. \tag{5.55}$$

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[10] Introduce the tensor of the field $F_{\mu\nu}$ which transforms by irreducible representation Q_F of the conformal group. According to Refs. [21,26], we have $Q_F \sim \tilde{Q}_A$, $Q_F \sim Q^{tr}_j$. Calculating the intertwining operators for these equivalence conditions one can show that $F_{\mu\nu}(x) = \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x)$, $\partial_\nu F_{\mu\nu}(x) = j^{tr}_\mu(x)$. The second condition in Eq. (5.51) is equivalent to Maxwell equations in α-gauge.

It is essential that both terms in this expression are conformally invariant in usual sense (i.e., as was understood in Sections 1 and 2). Let us also stress that the "cross-term"

$$\int \mathrm{d}x\, \mathrm{d}y\, \tilde{A}_\mu(x) D^j_{\mu\nu}(x-y) \tilde{A}_\nu(y) \tag{5.56}$$

is absent. This term is formally invariant, but is divergent. The third contraction may be represented analogously:

$$\int \mathrm{d}x\, j_\mu(x) A_\mu(x) = \int \mathrm{d}x\, j^{\mathrm{tr}}_\mu(x) A^{\mathrm{tr}}_\mu(x) + \int \mathrm{d}x\, j^{\mathrm{long}}_\mu(x) A^{\mathrm{long}}_\mu(x)$$
$$= \int \mathrm{d}x\, j^{\mathrm{tr}}_\mu(x) \tilde{A}_\mu(x) + \int \mathrm{d}x\, j_\mu(x) A^{\mathrm{long}}_\mu(x) . \tag{5.57}$$

Here the r.h.s. is also expressed through the conformal fields $\tilde{A}_\mu, \tilde{j}_\mu$ introduced in Section 2. Moreover, the cross-term

$$\int \mathrm{d}x\, \tilde{j}_\mu(x) \tilde{A}_\mu(x) \tag{5.58}$$

is also absent. The absence of terms (5.56) and (5.58) is equivalent to the orthogonality condition, Eqs. (2.56) and (2.58). This condition means that the transversal part of the current $\tilde{j}_\mu$ is expelled from the interaction with the field A^{tr}_μ. The final result is that through a direct analysis of electromagnetic interaction we have managed to get the former orthogonality condition, Eqs. (2.56) and (2.58).

Let us remind that Euclidean fields are understood as if they were placed inside the averaging symbols. All the above relations for the fields A_μ, j_μ, including Eqs. (5.54), (5.55), (5.56), (5.57) and (5.58), should be treated as the relations between the Green functions. In Section 2 we have discussed two types of conformally invariant Green functions

$$G^{\mathrm{tr}}_\mu(x, \ldots) = \langle j^{\mathrm{tr}}_\mu(x) \varphi \ldots \varphi^+ \rangle, \;\; \tilde{G}^j_\mu(x, \ldots) = \langle \tilde{j}_\mu(x) \varphi \ldots \varphi^+ \rangle$$

together with the two types of conformally invariant Green functions of the potential

$$\tilde{G}^A_\mu(x, \ldots) = \langle \tilde{A}_\mu(x) \varphi \ldots \varphi^+ \rangle, \;\; G^{\mathrm{long}}_\mu(x, \ldots) = \langle A^{\mathrm{long}}_\mu(x) \varphi \ldots \varphi^+ \rangle ,$$

where the currents $j^{\mathrm{tr}}_\mu, \tilde{j}_\mu$ and the fields $\tilde{A}_\mu$, A^{long}_μ have the usual transformation laws (2.3) and (2.9). Relations (5.55) and (5.57) viewed in terms of these Green functions mean that the integrals over the internal photon line have the form:

$$\int \mathrm{d}x\, \mathrm{d}y\, \langle \varphi \ldots \varphi^+ \tilde{A}_\mu(x) \rangle D^{\mathrm{tr}}_{\mu\nu}(x-y) \langle \tilde{A}_\nu(y) \varphi \ldots \varphi^+ \rangle$$
$$+ \int \mathrm{d}x\, \mathrm{d}y\, \langle \varphi \ldots \varphi^+ A^{\mathrm{long}}_\mu(x) \rangle D^j_{\mu\nu}(x-y) \langle A^{\mathrm{long}}_\nu(y) \varphi \ldots \varphi^+ \rangle$$
$$= \int \mathrm{d}x\, \langle \varphi \ldots \varphi^+ j^{\mathrm{tr}}_\mu(x) \rangle \, \langle \tilde{A}_\mu(x) \varphi \ldots \varphi^+ \rangle + \int \mathrm{d}x\, \langle \varphi \ldots \varphi^+ \tilde{j}_\mu(x) \rangle \, \langle A^{\mathrm{long}}_\mu(x) \varphi \ldots \varphi^+ \rangle . \tag{5.59}$$

The divergent cross-terms

$$\int \mathrm{d}x\,\mathrm{d}y\,\langle\varphi\ldots\varphi^+\tilde{A}_\mu(x)\rangle D^j_{\mu\nu}(x-y)\langle\tilde{A}_\nu(y)\varphi\ldots\varphi^+\rangle\,,$$

$$\int \mathrm{d}x\,\langle\varphi\ldots\varphi^+\tilde{j}_\mu(x)\rangle\,\langle\tilde{A}_\mu(x)\varphi\ldots\varphi^+\rangle$$

are absent. Recall that in order to evaluate the second terms in both sides of the equality (5.59) one should introduce a regularization, see Section 2.7.

As an example, let us consider the case of spinor QED. The three-point functions invariant under Eqs. (2.3) and (2.9) have the form (see Refs. [22,15] and references therein):

$$G^{\mathrm{tr}}_\mu(x_1x_2|x_3)=\langle\psi(x_1)\bar{\psi}(x_2)j^{\mathrm{tr}}_\mu(x_3)\rangle\sim\frac{\hat{x}_{13}}{(x_{13}^2)^2}\gamma_\mu\frac{\hat{x}_{32}}{(x_{23}^2)^2}\frac{1}{(x_{12}^2)^{d-3/2}}-\frac{\hat{x}_{12}}{(x_{12}^2)^{d-1/2}}\frac{1}{x_{13}^2x_{23}^2}\lambda^{x_3}_\mu(x_1x_2)\,,$$

$$\tilde{G}^j_\mu(x_1x_2|x_3)=\langle\psi(x_1)\bar{\psi}(x_2)\tilde{j}_\mu(x_3)\rangle\sim\frac{\hat{x}_{12}}{(x_{12}^2)^{d-1/2}}\frac{1}{x_{13}^2x_{23}^2}\lambda^{x_3}_\mu(x_1x_2)\,. \qquad (5.60)$$

The analogous Green functions for the fields $\tilde{A}_\mu$, A^{long}_μ read:

$$\tilde{G}^A_\mu(x_1x_2|x_3)=\langle\psi(x_1)\bar{\psi}(x_2)\tilde{A}_\mu(x_3)\rangle\sim\frac{1}{(x_{12}^2)^{d-1/2}}\frac{\hat{x}_{13}}{x_{13}^2}\gamma_\mu\frac{\hat{x}_{32}}{x_{23}^2}\,,$$

$$G^{\mathrm{long}}_\mu(x_1x_2|x_3)=\langle\psi(x_1)\bar{\psi}(x_2)A^{\mathrm{long}}_\mu(x_3)\rangle\sim\partial^{x_3}_\mu\left[\ln\frac{x_{23}^2}{x_{13}^2}\frac{\hat{x}_{12}}{(x_{12}^2)^{d+1/2}}\right]. \qquad (5.61)$$

These functions satisfy the following invariance conditions:

$$(K^{x_1}_\lambda+K^{x_2}_\lambda+K^{x_3}_\lambda)\,\langle\psi(x_1)\bar{\psi}(x_2)\tilde{A}_\mu(x_3)\rangle=(K^{x_1}_\lambda+K^{x_2}_\lambda+K^{x_3}_\lambda)\,\langle\psi(x_1)\bar{\psi}(x_2)A^{\mathrm{long}}_\mu(x_3)\rangle=0\,. \qquad (5.62)$$

The infinitesimal transformations of the spinor field, i.e., the form of $K_\lambda\psi(x)$ may be found in reviews [22,2,15], see also references therein. Introduce the invariant under Eqs. (5.47) and (5.33) Green functions of currents and potentials. The Green functions $G^{\mathrm{tr}}_\mu(x_1x_2|x_3)$ and $G^{\mathrm{long}}_\mu(x_1x_2|x_3)$ remain unchanged, while the functions $\tilde{G}^j_\mu$ and $\tilde{G}^A_\mu$, as may be shown [27–29], are replaced by the following expressions:

$$\tilde{G}^{j,\,\mathrm{long}}(x_1x_2x_3)=\langle\psi(x_1)\bar{\psi}(x_2)j^{\mathrm{long}}_\mu(x_3)\rangle\sim\frac{\partial^{x_3}_\mu\partial^{x_3}_\nu}{\Box_{x_3}}\left[\frac{\hat{x}_{12}}{(x_{12}^2)^{d-1/2}}\frac{1}{x_{13}^2x_{23}^2}\lambda^{x_3}_\nu(x_1x_2)\right], \qquad (5.63)$$

$$\tilde{G}^{A,\,\mathrm{tr}}(x_1x_2x_3)=\langle\psi(x_1)\bar{\psi}(x_2)A^{\mathrm{tr}}_\mu(x_3)\rangle\sim\left(\delta_{\mu\nu}-\frac{\partial^{x_3}_\mu\partial^{x_3}_\nu}{\Box_{x_3}}\right)\left[\frac{1}{(x_{12}^2)^{d-1/2}}\frac{\hat{x}_{13}}{x_{13}^2}\gamma_\nu\frac{\hat{x}_{32}}{x_{23}^2}\right]. \qquad (5.64)$$

Unlike Eq. (5.62), the function (5.64) satisfies the following condition of infinitesimal invariance:

$$(K^{x_1}_\lambda+K^{x_2}_\lambda+K^{x_3}_\lambda)\,\langle\psi(x_1)\bar{\psi}(x_2)A_\mu(x_3)\rangle-4\partial^{x_3}_\mu\frac{1}{\Box_{x_3}}\left(\delta_{\lambda\rho}-\frac{\partial^{x_3}_\lambda\partial^{x_3}_\tau}{\Box_{x_3}}\right)\langle\psi(x_1)\bar{\psi}(x_2)A_\rho(x_3)\rangle=0\,. \qquad (5.65)$$

As before, the integral over internal photon line may be written through the functions (5.63) and (5.64), and then expressed through the functions (5.60) and (5.61):

$$\int \mathrm{d}x\, \mathrm{d}y\, \langle \psi(x_1)\bar{\psi}(x_2)A_\mu(x)\rangle \Delta^j_{\mu\nu}(x-y)\langle A_\nu(y)\psi(x_3)\bar{\psi}(x_4)\rangle$$

$$= \int \langle \psi\bar{\psi}\tilde{A}_\mu\rangle D^{\mathrm{tr}}_{\mu\nu}\langle \tilde{A}_\nu\psi\bar{\psi}\rangle + \int \langle \psi\bar{\psi}A^{\mathrm{long}}_\mu\rangle D^j_{\mu\nu}\langle A^{\mathrm{long}}_\nu\psi\bar{\psi}\rangle$$

$$= \int \langle \psi\bar{\psi}j^{\mathrm{tr}}_\mu\rangle \langle \tilde{A}_\mu\psi\bar{\psi}\rangle + \int \langle \psi\bar{\psi}\tilde{j}_\mu\rangle \langle A^{\mathrm{long}}_\mu\psi\bar{\psi}\rangle . \tag{5.66}$$

These results, as well as the conformal QED in skeleton approximation based on them, were obtained by authors in Refs. [27–29], see also Ref. [15]. The conformal bootstrap in spinor QED was discussed in Ref. [27,15]. Let us remark that the different version of conformal QED was examined independently by several authors in Refs. [51–54].

In conclusion we note that Eq. (5.65) only describe the linear part of exact non-linear invariance conditions discussed in the previous subsection. Indeed, the modified transformations of the spinor field include a gauge transformation:

$$\delta\psi(x) + \varepsilon_\lambda K^x_\lambda\psi(x) - \mathrm{i}e4\varepsilon_\lambda\left(\frac{1}{\Box}A^{\mathrm{tr}}_\lambda(x)\right)\psi(x),$$

where e is the electric charge. As the result, one has

$$(K^{x_1}_\lambda + K^{x_2}_\lambda + K^{x_3}_\lambda)\langle \psi(x_1)\bar{\psi}(x_2)A_\mu(x_3)\rangle - 4\partial^{x_3}_\mu\frac{1}{\Box_{x_3}}\langle \psi(x_1)\bar{\psi}(x_2)A_\lambda(x_3)$$

$$- \mathrm{i}e\left\langle\left(\frac{1}{\Box}A_\lambda(x_1)\right)\psi(x_1)\bar{\psi}(x_2)A_\mu(x_3)\right\rangle + \mathrm{i}e\left\langle \psi(x_1)\left(\frac{1}{\Box}A_\lambda(x_2)\right)\bar{\psi}(x_2)A_\mu(x_3)\right\rangle = 0 . \tag{5.67}$$

Eq. (5.65) may be viewed as an approximation of these equations when $e \ll 1$. It is evident that Eq. (5.67) are linearized when passing to gauge invariant combination

$$\Psi(x_1x_2) = \bar{\psi}(x_1)\exp\left[\mathrm{i}e\int_\Gamma \mathrm{d}x_\mu\, A_\mu(x)\right]\varphi(x_2) .$$

The analysis of linearized equations and the evaluation of averages

$$\langle \Psi(x_1x_2)\rangle,\quad \langle \Psi(x_1x_2)A_\mu(x_3)\rangle$$

was conducted in Refs. [55,56], see also Ref. [15]. The formulation of conformally invariant gauge theories in terms of such string averages looks more natural and deserves further investigations.

5.5. *Linear conformal gravity in D = 4*

The metric field $h_{\mu\nu}$ is twinned to a pair of irreducible representations: $\tilde{Q}_h$ and Q^{long}_h. Section 2 dealt with the two types of fields $\tilde{h}_{\mu\nu}(x)$ and $h^{\mathrm{long}}_{\mu\nu}(x)$ which transformed by these representations. As in the case of QED, the representation $\tilde{Q}_h$ acts in the space of equivalence classes. The modified conformal transformations of the field $\tilde{h}_{\mu\nu}$ may be derived through an addition of gauge transformations inside the equivalence class $\{\tilde{h}_{\mu\nu}\}$. Applying the arguments presented in the previous subsection literally, one introduces the field

$$h_{\mu\nu}(x) = P^{\rm tr}\tilde{h}_{\mu\nu}(x) + h^{\rm long}_{\mu\nu}(x) \tag{5.68}$$

possessing the following transformation law [31]:

$$\delta h_{\mu\nu}(x) = \varepsilon_\lambda[P^{\rm tr}K_\lambda P^{\rm tr} + K_\lambda P^{\rm long}]h_{\mu\nu}(x)\,, \tag{5.69}$$

where

$$P^{\rm tr}h_{\mu\nu}(x) = P^{\rm tr}_{\mu\nu,\rho\sigma}(\partial^x)h_{\rho\sigma}(x),\; P^{\rm long}h_{\mu\nu}(x) = P^{\rm long}_{\mu\nu,\rho\sigma}(\partial^x)h_{\rho\sigma}(x)\,,$$

while the projection operators $P^{\rm tr}_{\mu\nu,\rho\sigma}$ and $P^{\rm long}_{\mu\nu,\rho\sigma}$ are defined in Eqs. (2.122), (2.123), (2.124), (2.125) and (2.126). As in the previous section, this transformation law corresponds to a direct sum of irreducible representations

$$\tilde{Q}_h \oplus Q^{\rm long}_h\,. \tag{5.70}$$

By analogy, introduce the irreducible energy–momentum tensor

$$T_{\mu\nu}(x) + T^{\rm tr}_{\mu\nu}(x) + P^{\rm long}\tilde{T}_{\mu\nu}(x)\,, \tag{5.71}$$

which transforms by the direct sum of representations

$$Q^{\rm tr}_T \oplus \tilde{Q}_T\,. \tag{5.72}$$

Its transformation law reads [31]

$$\delta T_{\mu\nu}(x) = \varepsilon_\lambda[K_\lambda P^{\rm tr} + P^{\rm long}K_\lambda P^{\rm long}]T_{\mu\nu}(x)\,. \tag{5.73}$$

One easily deduces that the propagators, which are invariant under Eqs. (5.69) and (5.73), have the form

$$\Delta^h_{\mu\nu,\rho\sigma}(x_{12}) = \langle h_{\mu\nu}(x_1)h_{\rho\sigma}(x_2)\rangle \sim P^{\rm tr}_{\mu\nu,\rho\sigma}(\partial^x)\ln x^2_{12} + D^{\rm long}_{\mu\nu,\rho\sigma}(x_{12})\,, \tag{5.74}$$

where the expression for $D^{\rm long}_{\mu\nu,\rho\sigma}$ is given by Eq. (2.110):

$$\Delta^T_{\mu\nu,\rho\sigma}(x_{12}) = \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle = D^{T,\,\rm tr}_{\mu\nu,\rho\sigma}(x_{12}) + D^{T,\,\rm long}_{\mu\nu,\rho\sigma}(x_{12})\,. \tag{5.75}$$

where $D^{T,\,\rm tr}_{\mu\nu,\rho\sigma}$ is given by Eq. (2.121) for $D = 4$, and the longitudinal part equals to [31]

$$D^{T,\,\rm long}_{\mu\nu,\rho\sigma}(x_{12}) = \partial_\mu H_{\nu\rho\sigma}(x_{12}) + \partial_\nu H_{\mu\rho\sigma}(x_{12}) - \tfrac{1}{2}\,\delta_{\mu\nu}\partial_\lambda H_{\lambda\rho\sigma}(x_{12})\,, \tag{5.76}$$

$$H_{\mu\rho\sigma}(x) \sim [22\partial_\mu\partial_\rho\partial_\sigma - 9(\delta_{\mu\rho}\partial_\sigma\Box + \delta_{\mu\sigma}\partial_\rho\Box) - \delta_{\rho\sigma}\partial_\mu\Box]\delta(x_{12})\,. \tag{5.77}$$

This function may be directly evaluated from the invariance conditions. Though in practice it proves more convenient to make use of the equivalence property for the representations (5.70) and (5.72), see Ref. [31] for details.

It is useful to normalize the propagators by the condition

$$\int dx_3\, \Delta^T_{\mu\nu,\tau\lambda}(x_{13})\Delta^h_{\tau\lambda,\rho\sigma}(x_{23}) = \tfrac{1}{2}(\delta_{\mu\rho}\delta_{\nu\tau} + \delta_{\mu\sigma}\delta_{\nu\rho} - \tfrac{1}{2}\delta_{\mu\nu}\delta_{\rho\sigma})\delta(x_{12})\,. \quad (5.78)$$

The equivalence conditions are expressed by the relations

$$T_{\mu\nu}(x) = \int dy\, \Delta^T_{\mu\nu,\rho\sigma}(x-y)h_{\rho\sigma}(y),\ h_{\mu\nu}(x) = \int dy\, \Delta^h_{\mu\nu,\rho\sigma}(x-y)T_{\rho\sigma}(y)\,, \quad (5.79)$$

The first of these conditions coincides with the equations of linear conformal gravity in a conformally invariant gauge (its explicit form is presented in Ref. [31]). Note that there exists another equivalence condition, which has not been mentioned before. Introduce the Weyl tensor (in linear approximation)

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \tfrac{1}{2}(\delta_{\mu\rho}R_{\nu\sigma} + \delta_{\nu\sigma}R_{\mu\rho} - \delta_{\mu\sigma}R_{\nu\rho} - \delta_{\nu\rho}R_{\mu\sigma}) + \tfrac{1}{6}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})R\,, \quad (5.80)$$

where $R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\mu\partial_\sigma h_{\nu\rho} + \partial_\nu\partial_\rho h_{\mu\sigma} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho})$ is the linear part of the Riemann curvature. The Weyl tensor transforms by irreducible representation of the conformal group. Let us denote it as Q_C. According to Refs. [21,26], there exist the following equivalence relations:

$$Q_C \sim Q_h^{tr},\quad Q_C \sim Q_T^{tr}\,.$$

Calculating the intertwining operators for these representations one can check that the first condition results in the constraint relating the fields $C_{\mu\nu\rho\sigma}, h_{\mu\nu}$ and coinciding with Eq. (5.80), while the second leads to the equation

$$\partial_\rho\partial_\sigma C_{\mu\nu\rho\sigma} \sim T_{\mu\nu}\,,$$

which coincides with Eq. (2.131a). Thus the equations of conformal gravity in linear approximation follow from the equivalence of corresponding representations of the conformal group.

Consider the invariant contractions

$$\int dx\, dy\, h_{\mu\nu}(x)\Delta^T_{\mu\nu,\rho\sigma}(x-y)h_{\rho\sigma}(y) = \int dx\, T_{\mu\nu}(x)h_{\mu\nu}(x)\,. \quad (5.81)$$

The first one may be brought into the form

$$\int dx\, dy\, h^{tr}_{\mu\nu}(x)\Delta^{T,tr}_{\mu\nu,\rho\sigma}(x-y)h^{tr}_{\rho\sigma}(y) + \int dx\, dy\, h^{long}_{\mu\nu}(x)\Delta^{T,\,long}_{\mu\nu,\rho\sigma}(x-y)h^{long}_{\rho\sigma}(y)$$

$$= \int dx\, dy\, \tilde{h}_{\mu\nu}(x)D^{tr}_{\mu\nu,\rho\sigma}(x-y)\tilde{h}_{\rho\sigma}(y) + \int dx\, dy\, h^{long}_{\mu\nu}(x)D^T_{\mu\nu,\rho\sigma}(x-y)h^{long}_{\rho\sigma}(y)\,, \quad (5.82)$$

where $D^{tr}_{\mu\nu,\rho\sigma}$ and $D^T_{\mu\nu,\rho\sigma}$ are conformally invariant (in usual sense) propagators (2.121) and (2.115), $\tilde{h}_{\mu\nu}(x)$ and $h^{long}_{\mu\nu}(x)$ are conformal fields transforming by the standard law (2.10). It is important that the (singular) "cross-term"

$$\int dx dy\, \tilde{h}_{\mu\nu}(x)D^T_{\mu\nu,\rho\sigma}(x-y)\tilde{h}_{\rho\sigma}(y) \quad (5.83)$$

is absent. Analogously, the second contraction is also written as the sum of terms which are invariant under the usual transformation laws (2.4) and (2.10):

$$\int \mathrm{d}x\, T_{\mu\nu}(x) h_{\mu\nu}(x) = \int \mathrm{d}x\, T^{\mathrm{tr}}_{\mu\nu}(x) \tilde{h}_{\mu\nu}(x) + \int \tilde{T}_{\mu\nu}(x) h^{\mathrm{long}}_{\mu\nu}(x)\,. \tag{5.84}$$

Here the cross-term

$$\int \mathrm{d}x\, \tilde{T}_{\mu\nu}(x) \tilde{h}_{\mu\nu}(x) \tag{5.85}$$

is also absent. It is not hard to understand that the absence of terms (5.83) and (5.85) in invariant contractions is equivalent to the orthogonality condition (2.105). Here it means that the transversal part of the field $\tilde{h}_{\mu\nu}$ does not interact with the transversal part of the tensor $\tilde{T}_{\mu\nu}$ which was related to a direct (non-gravitational) interaction of the matter fields in Section 2.

6. Concluding remarks

The approach developed here is based on a number of general principles of quantum field theory. we have inspected axiomatic hypotheses which select out definite classes of conformal models, in the same way as it is done in two-dimensional theory. The conformally invariant Ward identities for the energy–momentum tensor and the current play a principal role in our approach. Effectively, the latter contain the information on the quantization rules, and, to some extent, are equivalent to the definition of Hamiltonian. The conformal symmetry leads to a highly specific structure of the Hilbert space, essentially of its sector begotten by the current and the energy–momentum tensor. This Section presents a detailed review and a more comprehensive commentary concerning the properties of these fields, which are indispensable in conformal theory and altogether imperative in our approach. We investigate, in particular, the propagators and Green functions of the fields j_μ, $T_{\mu\nu}$ and also of their irreducible components $\tilde{j}_\mu$, j^{tr}_μ, $\tilde{T}_{\mu\nu}$, $\tilde{T}^{\mathrm{tr}}_{\mu\nu}$ in different realizations of the $\tilde{Q}$-type representations; see Ref. [65] for more detailed considerations.

6.1. Conformal models of non-gauge fields

Consider the Euclidean fields $j_\mu(x)$ and $T_{\mu\nu}(x)$. The Green functions $\langle j_\mu \ldots \rangle$ and $\langle T_{\mu\nu} \ldots \rangle$ are the Euclidean analogues of T-ordered vacuum expectation values in Minkowski space. Here we treat the Euclidean fields $j_\mu(x)$ and $T_{\mu\nu}(x)$ as the symbolic notation for the complete sets of Green functions $\langle j_\mu \ldots \rangle$ and $\langle T_{\mu\nu} \ldots \rangle$. Correspondingly, the derivatives of the Euclidean fields $\partial_\mu j_\mu(x)$ and $\partial_\mu T_{\mu\nu}(x)$ denote the derivatives of Green functions $\partial_\mu \langle j_\mu \ldots \rangle$ and $\partial_\mu \langle T_{\mu\nu} \ldots \rangle$. Calculating these derivatives, one encounters the two types of terms of different nature. Consider those terms on an example of the conserved current in Minkowski space. One gets for the propagator of the current:

$$\partial_\mu \langle 0| T\,\{ j_\mu(x) j_\nu(0)\} |0\rangle = \delta(x^0) \langle 0| [j_0(x), j_\nu(0)] |0\rangle + \langle 0| T\,\{ \partial_\mu j_\mu(x) j_\nu(0)\} |0\rangle\,.$$

The second term vanishes due to the conservation law

$$\partial_\mu j^{\mathrm{Mink}}_\mu(x) = 0\,.$$

To ensure the covariance of the T-ordered average, one should add quasilocal terms to the first term of the expression. The form of the total contribution of these terms and the commutator is imposed by conformal invariance, and reads

$$\partial_\mu \langle 0|T\,\{j_\mu(x)j_\nu(0)\}|0\rangle = C_j \partial_\nu \Box^{(D-2)/2}\delta(x)\,.$$

In Euclidean conformal theory, one associates the above pair of contributions with the irreducible components of the Euclidean current $j_\mu(x)$, so that

$$\partial_\mu j_\mu(x) = \partial_\mu \tilde{j}_\mu(x), \quad \partial_\mu j^{\rm tr}_\mu(x) = 0\,.$$

In particular, the total propagator of the current for even $D \geq 4$ may be represented as (see Section 5 and below):

$$\langle j_\mu(x_1)j_\nu(x_2)\rangle = \langle \tilde{j}_\mu(x_1)\tilde{j}_\nu(x_2)\rangle + \langle j^{\rm tr}_\mu(x_1)j^{\rm tr}_\nu(x_2)\rangle\,,$$

$$\partial_\mu\langle j_\mu(x_1)j_\nu(x_2)\rangle = \partial_\mu\langle \tilde{j}_\mu(x_1)\tilde{j}_\nu(x_2)\rangle = C_j\partial_\nu\Box^{(D-2)/2}\delta(x_{12})\,,$$

$$\partial_\mu\langle j^{\rm tr}_\mu(x_1)j^{\rm tr}_\nu(x_2)\rangle = 0\,.$$

An analogous expansion for the higher Green functions was dealt with in Sections 2 and 4.

As will be shown below, the irreducible components $\tilde{j}_\mu$ and $j^{\rm tr}_\mu$ have different physical meaning, and hence the different group-theoretic structure.

According to Section 5, see also the discussion below, only the current $j^{tr}_\mu(x)$, but not $\tilde{j}_\mu(x)$, induces a non-trivial contribution to the electromagnetic interaction. The Green functions of the current $\tilde{j}_\mu$ satisfy non-trivial Ward identities and contain the information on the (postulated) communication relations of the total current:

$$[j_0(x), j_k(0)]_{x^0=0},\ [j_0(x),\varphi(0)]_{x^0=0}, \ldots$$

As shown in Section 2, all the Green functions $\langle \tilde{j}_\mu \ldots\rangle$ are uniquely determined by the condition of conformal invariance and by the Ward identities.

Note that the fact that the fields P_s arise in the operator expansion

$$\tilde{j}_\mu(x)\varphi(0) = \sum_s [P_s]\,,$$

is a necessary consequence of the Ward identities (see Section 2), independent of the choice of dynamical model. Therefore the existence of the fields P_s, generated by the current $\tilde{j}_\mu$, follows from the two statements taken as postulates: the requirement of conformal symmetry and the definition of the equal-time commutator $[j_0(x), \varphi(0)]$. The dynamical requirement is the choice of null vectors. The latter depends on how the commutator $[j_0(x), j_k(0)]_{x^0=0}$ is defined. In Euclidean version of the theory, this commutator is determined by the type of the operator product expansion $\tilde{j}_\mu(x)\tilde{j}_\nu(0)$.

The expansion $\tilde{j}_\mu(x)\tilde{j}_\nu(0) = [C_j] + [P_j] + \cdots$ was considered in Section 3, while in Section 4 we have examined the models with $C_j \neq 0$, $P_j(x) = 0$.

One should remark that the current $\tilde{j}_\mu(x)$ arises as a representative of an equivalence class $\{\tilde{j}_\mu\} \subset \tilde{M}_j = M_j/M^{\rm tr}_j$. The conformal transformations of the current $\tilde{j}_\mu$ depend on the type of representatives chosen in each class, see Sections 2 and 5. Thus the transversal parts of the Green functions $\langle \tilde{j} \ldots\rangle$ may be redefined by performing a different choice of representatives. Particularly, in the non-local realization of conformal transformations (for $D = 4$), considered in Section 5, these Green functions are longitudinal. This realization is useful for conformal QED. However, it is essential that the local realization of conformal transformations of the current $\tilde{j}_\mu$ considered in Section 2, is needed for the analysis of the operator product expansions $\tilde{j}_\nu\varphi$ and $\tilde{j}_\mu\tilde{j}_\nu$. Below we discuss both realizations to a greater extent, and describe the relations between them. In a local realization, the Green functions $\langle\tilde{j}_\mu\ldots\rangle$ have quite definite transversal parts, which do not contribute to the electromagnetic interaction since an irreducible component $\tilde{j}_\mu$ of the total current only appears in contractions of the type $\int \mathrm{d}x\, \tilde{j}_\mu(x)A_\mu^{\mathrm{long}}(x)$, see below. As shown in Section 5, the interaction with the irreducible field $\tilde{A}_\mu$ is caused by the component j_μ^{tr} of the total current, and has the from $\int \mathrm{d}x\, j_\mu^{\mathrm{tr}}(x)\tilde{A}_\mu(x)$.

All what has been said is equally valid for the case of the energy–momentum tensor in conformal theory. The two irreducible components $\tilde{T}_{\mu\nu}$ and $T_{\mu\nu}^{\mathrm{tr}}$ have the same meaning as explained in the case of the current above. The component $T_{\mu\nu}^{\mathrm{tr}}$ leads to a non-trivial contribution to the gravitational interaction (with the fields $\tilde{h}_{\mu\nu}$). The component $\tilde{T}_{\mu\nu}$ appears only in the contractions $\int \tilde{T}_{\mu\nu}(x)h_{\mu\nu}^{\mathrm{long}}(x)$ and describes the postulated commutation relations of the total tensor $T_{\mu\nu}(x)$ with the fields, and with itself. Though the conservation laws

$$\partial_\mu T_{\mu\nu}^{\mathrm{Mink}}(x) = 0$$

are satisfied in Minkowski space, for the Euclidean propagator (and for the T-averages in Minkowski space) we have

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \langle\tilde{T}_{\mu\nu}(x)\tilde{T}_{\rho\sigma}(0)\rangle + \langle T_{\mu\nu}^{\mathrm{tr}}(x)T_{\rho\sigma}^{\mathrm{tr}}(0)\rangle\,,$$

where $\partial_\nu\langle T_{\mu\nu}^{\mathrm{tr}}(x)T_{\rho\sigma}^{\mathrm{tr}}(0)\rangle = 0$. In the case of $C_T \neq 0$, we get

$$\partial_\nu\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \partial_\nu\langle\tilde{T}_{\mu\nu}(x)\tilde{T}_{\rho\sigma}(0)\rangle \neq 0\,.$$

All the Green functions of the irreducible component $\tilde{T}_{\mu\nu}(x)$ are determined from the requirement of conformal symmetry and from the Ward identities, as it is shown in Section 2.

Let us consider the structure of the Hilbert space in more detail.

We have shown in the previous sections that the Hilbert space of conformal field theory contains two orthogonal sectors

$$\tilde{H}\oplus H_0\,, \tag{6.1}$$

which are generated by irreducible fields

$$\tilde{j}_\mu,\ \tilde{T}_{\mu\nu}\quad\text{and}\quad j^{\mathrm{tr}},\ T_{\mu\nu}^{\mathrm{tr}}\,. \tag{6.2}$$

The subspace $\tilde{H}$ is related to irreducible representations of the type $\tilde{Q}$, while H_0, to representations of the type Q_0, see Eq. (2.15). The orthogonality of the subspaces $\tilde{H}$ and H_0 means the vanishing of the Green functions

$$\langle j_\mu^{\mathrm{tr}}\tilde{j}_\nu\rangle = \langle\varphi j_\mu^{\mathrm{tr}}\tilde{j}_\nu\varphi^+\rangle = 0,\ \langle T_{\mu\nu}^{\mathrm{tr}}\tilde{T}_{\rho\sigma}\rangle = \langle\varphi T_{\mu\nu}^{\mathrm{tr}}\tilde{T}_{\rho\sigma}\varphi^+\rangle = 0\,. \tag{6.3}$$

Due to the conditions of equivalence for the representations (2.27), (2.131), the subspace H_0 includes electromagnetic and gravitational degrees of freedom (while the subspace $\tilde{H}$ includes gauge degrees of freedom only). The appearance of non-zero conformal fields j_μ^{tr} and $T_{\mu\nu}^{\mathrm{tr}}$ necessarily leads to the appearance of electromagnetic and gravitational fields $\tilde{A}_\mu$ and $\tilde{h}_{\mu\nu}$. The Green functions of current and energy–momentum tensor, which are calculated from Ward identities, generally include the total fields

$$j_\mu(x) = \tilde{j}_\mu(x) + j^{\rm tr}_\mu(x), \quad T_{\mu\nu}(x) = \tilde{T}_{\mu\nu}(x) + T^{\rm tr}_{\mu\nu}(x). \tag{6.4}$$

Thus the general conformally invariant solution of the Ward identities contains contributions of electromagnetic and gravitational interactions. To derive the models of non-gauge interactions considered in Sections 3 and 4, one must impose special conditions on the Green functions of current and energy–momentum tensor, see Eqs. (2.77) and (2.149). These conditions restrict the total space (6.1) to the subspace $\tilde{H}$, setting transversal fields to zero: $j^{\rm tr}_\mu = T^{\rm tr}_{\mu\nu} = 0$ on $\tilde{H}$.

We arrive at the following picture. The conformal symmetry, which arises as a non-perturbative effect, may take place in a special class of models (not necessarily Lagrangean). Starting with the structure of Hilbert space described above, it is natural to assume that gauge interactions are present initially (for $D > 2$) in conformal models based on Ward identities; both the components (6.2) contributing to the total expressions for current and energy–momentum tensor. A "true" conformal theory must contain the fields (6.4), and, consequently the fields, $\tilde{A}_\mu$, $\tilde{h}_{\mu\nu}$ due to Eqs. (2.27), (2.113), and (2.131) (see also Eqs. (6.33) and (6.42)). If we choose to consider the "approximate" models without gauge interactions, the solution is to be looked for in the restricted class of Green functions $\langle j_\mu \dots \rangle, \langle T_{\mu\nu} \dots \rangle$ corresponding to irreducible representations $\tilde{Q}_j$, $\tilde{Q}_T$. This class of Green functions is singled out by conditions (2.77) and (2.149). A theory with such set of conditions is non-trivial if the operator product expansions $j_\mu(x) j_\nu(0)$ and $T_{\mu\nu}(x) T_{\rho\sigma}(0)$, where $J_\mu(x) = \tilde{j}_\mu(x), T_{\mu\nu}(x) = \tilde{T}_{\mu\nu}(x)$ are irreducible fields, include anomalous terms $[C]$ and $[P(x)]$, see Eqs. (3.46) and (3.47). The conformal Green functions

$$\langle \tilde{j}_\mu(x)\varphi \dots \varphi^+ \rangle, \ \langle \tilde{T}_{\mu\nu}(x)\varphi \dots \varphi^+ \rangle$$

are uniquely determined from the Ward identities (Section 2). As shown in Sections 3 and 4, the latter feature allows one to derive a D-dimensional analogue of the family of exactly solvable two-dimensional models.

In this Section we examine the propagators and higher Green functions of the total fields (6.4). Each of the Green functions will be represented as a sum of two terms, the first one corresponding to the subspace $\tilde{H}$, and the second, to the subspace H_0. These terms have different group-theoretical structures and different partial wave expansions. Note that the results will be technically different for the spaces of even and odd dimensions. Here we restrict ourselves to the case of even D.

6.2. The propagators of the current and the energy–momentum tensor for even $D \geq 4$.

As shown in Section 5, the conformal transformation of the fields (6.4) have the following form:

$$j_\mu(x) \xrightarrow{R} V^j_R\, j_\mu(x) = U^j_R P^{\rm tr} j_\mu(x) + P^{\rm long} U^j_R P^{\rm long} j_\mu(x), \tag{6.5}$$

where $P^{\rm tr} = \delta_{\mu\nu} - \partial_\mu\partial_\nu/\Box$, $P^{\rm long} = \partial_\mu\partial_\nu/\Box$,

$$T_{\mu\nu} \xrightarrow{R} V^T_R T_{\mu\nu}(x) = U^T_R P^{\rm tr} T_{\mu\nu} + P^{\rm long} U^T_R P^{\rm long} T_{\mu\nu}(x), \tag{6.6}$$

where the projection operators P^{tr} and P^{long} are given by the expressions (2.122) and (2.125). The action of the operators U^j_R and U^T_R is given by the r.h.s. of the transformations (2.3) and (2.4). The conformally invariant propagators are determined by the conditions

$$\langle j_\mu(x_1) j_\nu(x_2)\rangle = \langle V^j_R j_\mu(x_1) V^j_R j_\nu(x_2)\rangle, \langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2)\rangle = \langle V^T_R T_{\mu\nu}(x_1) V^T_R T_{\rho\sigma}(x_2)\rangle .$$

For even $D \geq 4$ these equation have exceptional solutions which may not be obtained by the limiting transition from anomalous dimensions. One can check that the solutions have the form:

$$\Delta^j_{\mu\nu}(x_{12}) = \langle j_\mu(x_1) j_\nu(x_2)\rangle = f_j\,(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\Box^{(D-4)/2}\delta(x_{12}) + C_j\partial_\mu\partial_\nu\Box^{(D-4)/2}\delta(x_{12}) , \tag{6.7}$$

$$\Delta^T_{\mu\nu,\rho\sigma}(x_{12}) = \langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2)\rangle = f_T H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x)\Box^{(D-4)/2}\delta(x_{12}) + C_T H^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x)\Box^{(D-4)/2}\delta(x_{12}) , \tag{6.8}$$

where f_j and f_T are some constants, $H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}$ is given by expression (2.166), and $H^{\mathrm{long}}_{\mu\nu,\rho\sigma}$ has the form

$$H^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x) = \partial_\mu H_{\nu\rho\sigma}(\partial^x) + \partial_\nu H_{\mu\rho\sigma}(\partial^x) - \frac{2}{D}\,\delta_{\mu\nu}\partial_\lambda H_{\lambda\rho\sigma}(\partial^x) ,$$

where

$$H_{\mu\rho\sigma}(\partial^x) = \frac{2D^2 - 3D + 2}{2D(D-1)}\,\partial_\mu\partial_\rho\partial_\sigma - \frac{D-1}{2D}(\delta_{\mu\rho}\partial_\sigma + \delta_{\mu\sigma}\partial_\rho)\Box - \frac{1}{2D(D-1)}\delta_{\rho\sigma}\partial_\mu\Box .$$

Propagators (6.7) and (6.8) satisfy the Ward identities (2.177) and (2.186).

In Section 2, the Ward identities were obtained using the kernels invariant under (2.3) and (2.4):

$$D^{j\varepsilon}_{\mu\nu}(x_{12}) = \tilde{C}_j \frac{1}{(x^2_{12})^{D-1+\varepsilon}} g_{\mu\nu}(x_{12})\bigg|_{\varepsilon\ll 1} = \frac{\tilde{C}_j}{2(D-1+\varepsilon)(D-2+\varepsilon)}\left\{(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\frac{1}{(x^2_{12})^{D-2+\varepsilon}} - \frac{\varepsilon}{D-2+2\varepsilon}\delta_{\mu\nu}\Box\frac{1}{(x^2_{12})^{D-2+\varepsilon}}\right\}\bigg|_{\varepsilon\ll 1} \tag{6.9}$$

$$= \frac{\tilde{C}_j}{2(D-1+\varepsilon)(D-2+\varepsilon)}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\frac{1}{(x^2_{12})^{D-2+\varepsilon}} + C_j\partial_\mu\partial_\nu\Box^{(D-4)/2}\delta(x_{12}) + \mathrm{O}(\varepsilon) , \tag{6.10}$$

$$D^{T\varepsilon}_{\mu\nu,\rho\sigma}(x_{12}) = \tilde{C}_T\frac{1}{(x^2_{12})^{D+\varepsilon}}\left[g_{\mu\rho}(x_{12})g_{\nu\sigma}(x_{12}) + g_{\mu\sigma}(x_{12})g_{\nu\rho}(x_{12}) - \frac{2}{D}\,\delta_{\mu\nu}\delta_{\rho\sigma}\right]\bigg|_{\varepsilon\ll 1} \tag{6.11}$$

$$= \frac{\tilde{C}'_T}{\varepsilon}H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x)\frac{1}{(x^2_{12})^{D-2+\varepsilon}} + \tilde{C}''_T H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x)\Box^{(D-4)/2}\delta(x_{12}) + C_T H^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x)\Box^{(D-4)/2}\delta(x_{12}) + \mathrm{O}(\varepsilon) , \tag{6.12}$$

where $\tilde{C}'_T$ and $\tilde{C}''_T$ are some constants. The longitudinal terms in Eqs. (6.10) and (6.12) are derived with the help of relations (2.34) for $k = (D-4)/2$. Notice that their form coincides with longitudinal terms of the propagators (6.7) and (6.8). The kernels (6.9) and (6.11) define in the limit $\varepsilon \to 0$ the propagators of irreducible fields $\tilde{j}_\mu$ and $\tilde{T}_{\mu\nu}$; they were used in calculations in Sections 3 and 4. The transversal parts of these kernels, singular at $\varepsilon = 0$, do not contribute to conformally invariant graphs, see Eqs. (5.55), (5.59) and (5.82), and do not change the results. Technically, the latter manifests in the fact that the representations of the type $\tilde{Q}$ are defined on equivalence classes (see Sections 2 and 5), not on the fields. The kernels (6.9) and (6.11) for $\varepsilon = 0$ are related to longitudinal terms in Eqs. (6.10) and (6.12) by transformations inside the equivalence class, corresponding to the transition to the other realization of representations $\tilde{Q}_j$ and $\tilde{Q}_T$. The more detailed discussion is presented in the next subsection.

Another approach to a definition of propagators was recently considered in Refs. [59–61]. These works introduce a special regularization of the non-integrable function $(x_{12}^2)^{-D+2}$ in Eqs. (6.10) and (6.12) at $\varepsilon = 0$, which breaks the conformal symmetry. Moreover, the longitudinal terms[11] are dropped in Eqs. (6.10) and (6.12). After that, it is shown that such breakdown of the conformal symmetry may be related to conformal anomalies (the physical motivation and the methods of derivation of conformal anomalies are discussed in Refs. [62,63]). Note that the possibility of the breakdown of the symmetry also remains in our approach. Though, our principal aim is the analysis of models with exact conformal symmetry.

6.3. The propagators of irreducible components of the current and the energy–momentum tensor

Consider the irreducible currents j_μ^{tr} and $\tilde{j}_\mu$. The pair of invariant propagators (for even $D \geq 4$)

$$\langle j_\mu^{\mathrm{tr}}(x_1) j_\nu^{\mathrm{tr}}(x_2)\rangle \quad \langle \tilde{j}_\mu(x_1)\tilde{j}_\nu(x_2)\rangle$$

may be related to the kernels of invariant scalar products on the spaces $\tilde{M}_A$ and M_A^{long}, see Eq. (2.23). These kernels

$$D_{\mu\nu}^{j,\,\mathrm{tr}}(x_{12}) \quad D_{\mu\nu}^{j\varepsilon}(x_{12}) \tag{6.13}$$

are invariant under the transformation (2.3) and were examined above for the case $D \geq 4$. The first one is transversal and may be identified with the propagator of the current j_μ^{tr}

$$D_{\mu\nu}^{j,\,\mathrm{tr}}(x_{12}) = \langle j_\mu^{\mathrm{tr}}(x_1) j_\nu^{\mathrm{tr}}(x_2)\rangle \sim (\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\Box^{(D-4)/2}\delta(x_{12})\,. \tag{6.14}$$

The second kernel is singular at $\varepsilon = 0$, see Eq. (6.10), and gives a certain realization of the propagator $\langle \tilde{j}_\mu \tilde{j}_\nu\rangle$, see below.

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[11] The right-hand sides of the Ward identities (2.177) and (2.186) are non-zero owing to contributions of equal-time commutators of the current and energy–momentum tensor components between themselves, see Eqs. (3.46) and (3.47). Let us remind that the vacuum expectation values of T-ordered products of the fields (in Minkowski space) are defined up to quasilocal terms, allowing one to make a transition to transversal propagators. However, such a redefinition breaks the conformal symmetry, which fixes longitudinal parts uniquely. In a well-known example of two-dimensional theory, the conformal propagator is given by the expression (6.11) for $D = 2$, $\varepsilon \to 0$ and may be written as $\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle|_{D=2} \sim C_T[(\partial_\mu\partial_\nu - \frac{1}{2}\delta_{\mu\nu}\Box)(\partial_\rho\partial_\sigma - \frac{1}{2}\delta_{\rho\sigma}\Box) - \frac{1}{8}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\rho\sigma})\Box^2]1/\Box\,\delta(x)$. This propagator satisfies the Ward identity (2.186) for $D = 2$. Passing to the complex variables $z = x_1 + \mathrm{i}x_2$, $T_{zz} = T_{11} + T_{22}$ in this formula, we get the well-known results $\langle T_{zz}(z) T_{zz}(0)\rangle \sim C_T(\partial_z)^4(\partial_z\partial_{\bar{z}})^{-1}\delta^{(2)}(z,\bar{z}) \sim C_T/z^4, \partial_{\bar{z}}\langle T_{zz}(z) T_{zz}(0)\rangle \sim C_T(\partial_z)^3\delta^{(2)}(z,\bar{z})$, $\langle T_{z\bar{z}}(z,\bar{z}) T_{z,\bar{z}}(0,0)\rangle = 0$. This propagator differs by quasi-local terms from the non-invariant transversal expression $(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)(\partial_\rho\partial_\sigma - \delta_{\rho\sigma}\Box)\ln x^2$, which has a non-zero trace.

As already mentioned in Sections 5 and 2, the Green functions of the current $\tilde{j}_\mu$ depend on the choice of realization of the representation $\tilde{Q}_j$. Choosing different representatives in the equivalence class $\{\tilde{j}_\mu\}$, one can obtain different realizations of the Green functions $\langle \tilde{j}_\mu \ldots \rangle$. In the Section 5.4 we reflect upon the special realization (5.44) of the representation $\tilde{Q}_j$: there, the current $\tilde{j}_\mu$ is longitudinal, and its propagator reads

$$D^{j,\,\mathrm{long}}_{\mu\nu}(x_{12}) = \langle \tilde{j}^{\mathrm{long}}_\mu(x_1)\tilde{j}^{\mathrm{long}}_\nu(x_2)\rangle = C_j\partial_\mu\partial_\nu\Box^{(D-4)/2}\delta(x_{12})\,. \tag{6.15}$$

Conformal transformations are non-local in this realization, see Eq. (6.5), and differ from the transformations of the current j^{tr}_μ. The propagator of the total current j_μ, on account of Eq. (6.3), equals to the sum of the terms (6.14) and (6.15).

In Sections 3 and 4 we studied another realization of the representation $\tilde{Q}_j$. Its conformal transformations are local, coincide with transformations of the current j^{tr}_μ, and have the form (2.3). The propagator $\langle \tilde{j}_\mu \tilde{j}_\nu\rangle$ in this case also demand regularization and coincides with the Kernel $D^j_{\mu\nu}$

$$D^{j^\varepsilon}_{\mu\nu}(x_{12}) = \langle \tilde{j}_\mu(x_1)\tilde{j}_\nu(x_2)\rangle_\varepsilon = C_j\partial_\mu\partial_\nu\Box^{(D-4)/2}\delta(x_{12}) + A(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\frac{1}{(x^2_{12})^{D-2+\varepsilon}} + \mathrm{O}(\varepsilon)\,. \tag{6.16}$$

The physical significance has, however, the equivalence class $\{\tilde{j}_\mu\} \supset \tilde{j}_\mu, j^{\mathrm{long}}_\mu$ rather than the current $\tilde{j}_\mu$ by itself. The framework of the conformal theory described in the preceding sections is constructed in a manner which prevents the transformations inside an equivalence class from having an influence on the results. In particular, the transformation $\tilde{j}_\mu \to \tilde{j}^{\mathrm{long}}_\mu$ does not change the values of invariant contractions (5.52). The latter may be written either in the form (5.54), or in the form (5.55). The divergent at $\varepsilon = 0$ transversal component of the kernel $D^j_{\mu\nu}$ in Eq. (5.55) does not contribute, since the conformal Green functions $\langle A^{\mathrm{long}} \ldots \rangle$ are longitudinal in the leading order in ε (the conformally invariant regularization is used for calculation of such integrals, see Section 6.4).

All that was said above is equally valid for the field A_μ, which transforms by the representation $\tilde{Q}_A \oplus Q^{\mathrm{long}}_A$. The irreducible fields $\tilde{A}_\mu$ and A^{long}_μ correspond to the kernels

$$D^{A^\varepsilon}_{\mu\nu}(x_{12}) \text{ and } D^{A,\,\mathrm{long}}_{\mu\nu}(x_{12})\,,$$

which are invariant under the transformation (2.9). The kernel $D^{A^\varepsilon}_{\mu\nu}$ is singular at $\varepsilon = 0$

$$D^{A^\varepsilon}_{\mu\nu}(x_{12}) \sim \frac{1}{\varepsilon}(x^2_{12})^{-1+\varepsilon}g_{\mu\nu}(x_{12})|_{\varepsilon \ll 1} \simeq \frac{1}{\varepsilon}D^{A,\,\mathrm{long}}_{\mu\nu}(x_{12}) + D^{A,\,\mathrm{tr}}_{\mu\nu}(x_{12}) + \cdots \tag{6.17}$$

and defines the scalar product on an invariant subspace M^{tr}

$$\{j^{\mathrm{tr}}, j^{\mathrm{tr}}\}_1 = \int \mathrm{d}x_1\,\mathrm{d}x_2\, j^{\mathrm{tr},\varepsilon}_\mu(x_1) D^{A^\varepsilon}_{\mu\nu}(x_{12}) j^{\mathrm{tr},\varepsilon}_\nu(x_2)|_{\varepsilon=0}\,. \tag{6.18}$$

The invariant regularization of the fields $j^{\mathrm{tr}}_\mu \to j^{\mathrm{tr},\varepsilon}_\mu$ is described below. Consider the longitudinal kernel $D^{A,\,\mathrm{long}}_{\mu\nu}$. It has the form (2.28) and defines the invariant scalar product on the space $\tilde{M}$ of equivalence classes $\{\tilde{j}_\mu\}$, see Eq. (2.22),

$$\{\tilde{j}, \tilde{j}\}_2 = \int \mathrm{d}x_1\,\mathrm{d}x_2\, \tilde{j}_\mu(x_1) D^{A,\,\mathrm{long}}_{\mu\nu}(x_{12}) \tilde{j}_\nu(x_2)\,. \tag{6.19}$$

All the Green functions of the field A_μ^{long} are longitudinal, while the Green function of the field $\tilde{A}_\mu$ depend on the choice of realization of the representation $\tilde{Q}_A$. To different equivalence classes $\{\tilde{A}_\mu\} \subset \tilde{M}_A$, different realizations correspond. In the Section 5.4 we have dealt with the realization which corresponds to a transversal representative: $\tilde{A}_\mu(x) \to A_\mu^{\text{tr}}(x)$. Its propagator coincides with the transversal component of the kernel (6.17):

$$D_{\mu\nu}^{A;\,\text{tr}}(x_{12}) = \langle \tilde{A}_\mu^{\text{tr}}(x_1)\tilde{A}_\nu^{\text{tr}}(x_2)\rangle \sim (\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\ln x_{12}^2 . \tag{6.20}$$

The conformal transformations in this case are non-local and different from the transformations of the field A_μ^{long}.

Another realization of the representation $\tilde{Q}_A$, in which the transformations are local and have the usual form (2.9), has been studied in Sections 2–4. In this case, the propagator $\langle \tilde{A}_\mu \tilde{A}_\nu\rangle$ coincides with the kernel (6.16)

$$D_{\mu\nu}^{A^\varepsilon}(x_{12}) = \langle \tilde{A}_\mu(x_1)\tilde{A}_\nu(x_2)\rangle \sim \frac{1}{\varepsilon}(x_{12}^2)^{-1+\varepsilon} g_{\mu\nu}(x_{12}) , \tag{6.21}$$

singular at $\varepsilon = 0$. Unlike Eq. (6.20), it has a longitudinal part which is singular for $\varepsilon = 0$ and does not contribute to conformally invariant contractions (6.18). Naturally, the invariant scalar product on M_j^{tr} does not depend on the choice of realization

$$\int dx_1\, dx_2\, j_\mu^{\text{tr},\varepsilon}(x_1) D_{\mu\nu}^{A^\varepsilon}(x_{12}) j_\nu^{\text{tr},\varepsilon}(x_2)|_{\varepsilon=0} = \int dx_1\, dx_2\, j_\mu^{\text{tr}}(x_1) D_{\mu\nu}^{A;\,\text{tr}}(x_{12}) j_\nu^{\text{tr}}(x_2) . \tag{6.22}$$

The regularization (6.21) is more preferable technically.

The energy–momentum tensor and the metric field may be considered analogously. Let us couple the pair of irreducible fields $T_{\mu\nu}^{tr}$ and $\tilde{T}_{\mu\nu}$ to the pair of invariant propagators

$$\langle T_{\mu\nu}^{\text{tr}}(x_1) T_{\rho\sigma}^{\text{tr}}(x_2)\rangle \quad \text{and} \quad \langle \tilde{T}_{\mu\nu}(x_1)\tilde{T}_{\rho\sigma}(x_2)\rangle .$$

They are identified with the kernels of invariant scalar products on the spaces $\tilde{M}_h$ and M_h^{long}, see Eq. (2.91). Accordingly, for even $D \geq 4$ one has a pair of kernels

$$D_{\mu\nu\rho\sigma}^{T;\,\text{tr}}(x_{12}) \quad \text{and} \quad D_{\mu\nu\rho\sigma}^{T^\varepsilon}(x_{12}) , \tag{6.23}$$

invariant under the transformation (2.4). The propagator of the tensor $T_{\mu\nu}^{\text{tr}}$ coincides with the first of them

$$D_{\mu\nu\rho\sigma}^{T;\,\text{tr}}(x_{12}) = \langle T_{\mu\nu}^{\text{tr}}(x_1) T_{\rho\sigma}^{\text{tr}}(x_2)\rangle \sim H_{\mu\nu,\rho\sigma}^{\text{tr}}(\partial^x)\Box^{(D-4)/2}\delta(x_{12}) . \tag{6.24}$$

The propagator of the field $\tilde{T}_{\mu\nu}$ depends on the realization of the representation $\tilde{Q}_T$, which acts on equivalence classes $\{\tilde{T}_{\mu\nu}\} \subset \tilde{M}_T$, see Eq. (2.90). In a non-local realization of the previous section this propagator is longitudinal

$$D_{\mu\nu\rho\sigma}^{T,\,\text{long}}(x_{12}) = \langle \tilde{T}_{\mu\nu}^{\text{long}}(x_1)\tilde{T}_{\rho\sigma}^{\text{long}}(x_2)\rangle = C_T H_{\mu\nu,\rho\sigma}^{\text{long}}(\partial^x)\Box^{(D-4)/2}\delta(x_{12}) . \tag{6.25}$$

The propagator of total energy–momentum tensor $T_{\mu\nu}(x)$ equals to sum of these two expressions. In the local realization (2.4) the propagator of the field $\tilde{T}_{\mu\nu}$ demands regularization

$$D_{\mu\nu\rho\sigma}^{T^\varepsilon}(x_{12}) = \langle \tilde{T}_{\mu\nu}(x_1)\tilde{T}_{\rho\sigma}(x_2)\rangle_\varepsilon \tag{6.26}$$

and is given by the expression (6.12). The transition from Eq. (6.25) to Eq. (6.26) is performed through the transformation inside the equivalence class $\tilde{T}^{\text{long}}_{\mu\nu}$, $\tilde{T}_{\mu\nu} \subset \{\tilde{T}_{\mu\nu}\}$. Both kernels

$$D^{T,\,\text{long}}_{\mu\nu\rho\sigma}(x_{12}) \quad \text{and} \quad D^{T^\varepsilon}_{\mu\nu\rho\sigma}(x_{12}) \tag{6.27}$$

define the same scalar product on an invariant subspace $M^{\text{long}}_h \subset M_h$ (see (2.8)):

$$(h^{\text{long}}, h^{\text{long}}) = \int \mathrm{d}x_1\, \mathrm{d}x_2\, h^{\text{long}}_{\mu\nu}(x_1) D^{T,\,\text{long}}_{\mu\nu\rho\sigma}(x_{12}) h^{\text{long}}_{\rho\sigma}(x_2)$$

$$= \int \mathrm{d}x_1\, \mathrm{d}x_2\, h^{\text{long},\varepsilon}_{\mu\nu}(x_1) D^{T^\varepsilon}_{\mu\nu\rho\sigma}(x_{12}) h^{\text{long},\varepsilon}_{\rho\sigma}(x_2)|_{\varepsilon=0}\,. \tag{6.28}$$

The irreducible fields $\tilde{h}_{\mu\nu}$ and $h^{\text{long}}_{\mu\nu}$ correspond to the kernels

$$D^{h^\varepsilon}_{\mu\nu\rho\sigma}(x_{12}) \quad \text{and} \quad D^{h,\,\text{long}}_{\mu\nu\rho\sigma}(x_{12}) \tag{6.29}$$

which are invariant under the transformation (2.10). The first one is singular at $\varepsilon = 0$ and is given by the expression

$$D^{h^\varepsilon}_{\mu\nu\rho\sigma}(x_{12}) \sim \frac{1}{\varepsilon}(x_{12}^2)^\varepsilon \left[g_{\mu\rho}(x_{12}) g_{\nu\sigma}(x_{12}) + g_{\mu\sigma}(x_{12}) g_{\nu\rho}(x_{12}) - \frac{2}{D}\delta_{\mu\nu}\delta_{\rho\sigma} \right]$$

$$\simeq \frac{1}{\varepsilon} P^{\text{long}}_{\mu\nu,\rho\sigma}(\partial^x) \ln x_{12}^2 + \cdots \tag{6.30}$$

In analogy with Eqs. (6.18) and (6.19), the kernels $D^{h^\varepsilon}_{\mu\nu\rho\sigma}$ and $D^{h,\,\text{long}}_{\mu\nu\rho\sigma}$ define invariant scalar products on the spaces M^{tr} and $\tilde{M}_T$, respectively (see Eqs. (2.89) and (2.90)). The propagator of the field $\tilde{h}_{\mu\nu}$ in the local realization coincides with Eq. (6.30), while in the realization of the previous section, with the transversal part of this expression, finite for $\varepsilon = 0$; see Eq. (6.22) for comparison.

6.4. *The equivalence conditions for higher green functions of the current and the energy–momentum tensor*

Consider the higher Green functions of the fields (6.4). These functions can be expressed through the green functions of irreducible fields

$$\langle \tilde{A}_\mu \varphi \ldots \varphi^+ \rangle, \langle A^{\text{long}}_\mu \varphi \ldots \varphi^+ \rangle \quad \text{and} \quad \langle j^{\text{tr}}_\mu \varphi \ldots \varphi^+ \rangle, \langle \tilde{j}_\mu \varphi \ldots \varphi^+ \rangle\,. \tag{6.31}$$

Here we assume the invariance of the Green functions $\langle \tilde{A}_\mu \varphi \ldots \varphi^+ \rangle$, $\langle \tilde{j}_\mu \varphi \ldots \varphi^+ \rangle$ under the local transformations (2.9) and (2.3). Such a realization of the representations $\tilde{Q}_A$ and $\tilde{Q}_j$ was used in Sections 2–4.

In this realization, the Green functions $\langle A_\mu \varphi \ldots \varphi^+ \rangle$ and $\langle j_\mu \varphi \ldots \varphi^+ \rangle$ may be represented as sums of pairs of terms

$$\langle \tilde{A}_\mu(x) \varphi \ldots \varphi^+ \rangle + \langle A^{\text{long}}_\mu(x) \varphi \ldots \varphi^+ \rangle \quad \text{and} \quad \langle j^{\text{tr}}_\mu(x) \varphi \ldots \varphi^+ \rangle + \langle \tilde{j}_\mu(x) \varphi \ldots \varphi^+ \rangle\,. \tag{6.32}$$

Single components of these sums have different partial wave expansions. For example, the functions $\langle j^{tr}_\mu \ldots \rangle$ are decomposed into the set of transversal functions (2.65), while the functions $\langle \tilde{j}_\mu \ldots \rangle$, into the set (2.68). These sets are mutually orthogonal. Hence each term in the second sum

(6.32) may be readily identified in terms of partial wave expansions. It was used in Section 2 for the derivation of the conditions (2.77) which retain only the second term in the sum (6.32).

In non-local realization, Eqs. (6.5) and (6.6), the total Green functions of the fields (6.4) may also be expressed through the functions (6.31). Taking into account that the functions $\langle A_\mu^{\rm long}\varphi \dots \varphi^+\rangle$ are longitudinal and $\langle j_\mu^{\rm tr}\varphi \dots \varphi^+\rangle$ are transversal, we obtain

$$\langle A_\mu(x)\varphi \dots \varphi^+\rangle = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\langle \tilde{A}_\nu(x)\varphi \dots \varphi^+\rangle + \langle A_\mu^{\rm long}(x)\varphi \dots \varphi^+\rangle$$

$$\langle j_\mu(x)\varphi \dots \varphi^+\rangle = \langle j_\mu^{\rm tr}(x)\varphi \dots \varphi^+\rangle + \frac{\partial_\mu\partial_\nu}{\Box}\langle \tilde{j}_\nu(x)\varphi \dots \varphi^+\rangle\,.$$

Consider the equivalence conditions (2.26). The operator relations (2.27) may be rewritten in terms of kernels $D_{\mu\nu}^{j^\varepsilon}$ and $D_{\mu\nu}^{A^\varepsilon}$ which are singular at $\varepsilon = 0$:

$$\langle \tilde{A}_\nu(x)\varphi \dots \varphi^+\rangle = \int {\rm d}y\, D_{\mu\nu}^{j^\varepsilon}(x-y)\,\langle j_\nu^{{\rm tr},\varepsilon}(y)\varphi \dots \varphi^+\rangle|_{\varepsilon=0}\,, \tag{6.33}$$

$$\langle \tilde{j}_\nu(x)\varphi \dots \varphi^+\rangle = \int {\rm d}y\, D_{\mu\nu}^{A^\varepsilon}(x-y)\,\langle A_\nu^{{\rm long},\varepsilon}(y)\varphi \dots \varphi^+\rangle|_{\varepsilon=0}\,, \tag{6.34}$$

Let us represent the Green functions (6.31) inn terms of partial wave expansions. The expansion of the function $\langle A^{\rm long}\varphi \dots \varphi^+\rangle$ includes a set of longitudinal three-point functions which are derived from Eq. (2.69) by the change $d \to D - d$. Denote it as $B^{l,{\rm long}}_{\mu,\mu_1\dots\mu_s}$:

$$B^{l,{\rm long}}_{\mu,\mu_1\dots\mu_s}(x_1x_2x_3) = \langle \Phi^l_{\mu_1\dots\mu_s}(x_1)\varphi(x_2)A_\mu^{\rm long}(x_3)\rangle \sim \{(l-d-s),1\}\,\Delta^l_A(x_1x_2x_3)$$
$$\sim \partial_\mu^{x_3}\left[\lambda^{x_1}_{\mu_1\dots\mu_s}(x_2x_3)(x_{12}^2)^{-(l+d-s)/2}\left(\frac{x_{23}^2}{x_{13}^2}\right)^{(l-d-s)/2}\right]. \tag{6.35}$$

The functions $\langle \tilde{A}_\mu\varphi \dots \varphi^+\rangle$ are decomposed into the set of functions

$$B^l_{1\mu,\mu_1\dots\mu_s}(x_1x_2x_3) = \langle \Phi^l_{\mu_1\dots\mu_s}(x_1)\varphi(x_2)\tilde{A}_\mu(x_3)\rangle = \{A, B\}\,\Delta^l_A(x_1x_2x_3)\,, \tag{6.36}$$

where the notation $\{A, B\}$ was introduced in Eq. (1.83),

$$\Delta^l_A(x_1x_2x_3) = (x_{12}^2)^{-(l+d-s)/2}(x_{23}^2)^{(l-d-s)/2}(x_{13}^2)^{-(l-d-s)/2}\,. \tag{6.37}$$

Under a suitable choice of coefficients A, B in Eq. (6.36) these functions are related by the equivalence relation with the functions (2.65)

$$B^l_{2\mu,\mu_1\dots\mu_s}(x_1x_2x_3) = \int {\rm d}x_4\, D_{\mu\nu}^{A^\varepsilon}(x_{34})C^{l,{\rm tr},\varepsilon}_{2\nu,\mu_1\dots\mu_s}(x_1x_2x_4)|_{\varepsilon=0}\,. \tag{6.38}$$

Here we have used the conformally invariant regularization of the function (2.65). This is done through the substitution $l_j \to l_j^\varepsilon = D - 1 + \varepsilon$, which is equivalent to introducing of the factor $(x_{12}^2)^{-\varepsilon/2}(x_{13}^2x_{23}^2)^{\varepsilon/2}$ into the expression (2.66). Analogously, the functions (6.35) are related by the equivalence condition with the functions (2.68):

$$C^l_{1\mu,\mu_1\dots\mu_s}(x_1x_2x_3) = \int {\rm d}x_4\, D_{\mu\nu}^{j^\varepsilon}(x_{34})B^{l,{\rm long},\varepsilon}_{\nu,\mu_1\dots\mu_s}(x_1x_2x_4)|_{\varepsilon=0}\,. \tag{6.39}$$

The regularized functions $B^{l,\mathrm{long}}_{\mu,\mu_1\ldots\mu_s}$ are derived by the substitution $l_A \to l^\varepsilon_A = 1-\varepsilon$, which is equivalent to introducing of the factor $(x^2_{12})^{\varepsilon/2}(x^2_{13}x^2_{23})^{-\varepsilon/2}$ into Eq. (6.37).

Relations (6.38) and (6.39) are the analogues of the amputation conditions (1.35) for the case of pairs of fields $\tilde{A}_\mu$, j^{tr}_μ and $\tilde{j}_\mu$, A^{long}_μ, respectively. Taking into account that partial wave expansions of the functions (6.31) include just one term (of the two terms of Eq. (1.86)), as well as the relations (6.38) and (6.39), one can conclude that the equivalence conditions (6.33) and (6.34) are reduced to the equality of kernels for corresponding conformal partial wave expansions.

Finally, let us write down the relations inverse to the relations (6.38) and (6.39):

$$C^{l,\mathrm{tr}}_{2\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \int \mathrm{d}x_4\, D^{j,\,\mathrm{tr}}_{\mu\nu}(x_{34}) B^{l}_{2\nu,\mu_1\ldots\mu_s}(x_1x_2x_4)\,, \tag{6.40}$$

$$B^{l,\mathrm{long}}_{\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \int \mathrm{d}x_4\, D^{A,\,\mathrm{long}}_{\mu\nu}(x_{34}) C^{l}_{1\nu,\mu_1\ldots\mu_s}(x_1x_2x_4)\,. \tag{6.41}$$

They are related with the equivalence conditions for the Green functions (6.31) written in the form (2.27).

All that has been said evidently admits a generalization to the case of the energy–momentum tensor and the metric field. In particular, the equivalence condition (2.108) may be written either in the form given by Eqs. (2.109) and (2.113), or, in analogy with Eqs. (6.33) and (6.34), in the form

$$\tilde{h}_{\mu\nu}(x) = \int \mathrm{d}y\, D^{h^\varepsilon}_{\mu\nu\rho\sigma}(x-y) T^{\mathrm{tr},\varepsilon}_{\rho\sigma}(y)|_{\varepsilon=0}\,,$$
$$\tilde{T}_{\mu\nu}(x) = \int \mathrm{d}y\, D^{T^\varepsilon}_{\mu\nu\rho\sigma}(x-y) h^{\mathrm{long},\varepsilon}_{\rho\sigma}(y)|_{\varepsilon=0}\,. \tag{6.42}$$

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Appendix A.

$$\pi^{-d}\frac{\Gamma(d)\Gamma(D-d)}{\Gamma(D/2-d)\Gamma(d-D/2)}\int \mathrm{d}x_3\,(x^2_{13})^{-d}(x^2_{23})^{-D+d} = \delta(x_{13})\,. \tag{A.1}$$

$$\int \mathrm{d}x_4 \left(\frac{1}{2}x^2_{14}\right)^{-\delta_1}\left(\frac{1}{2}x^2_{24}\right)^{-\delta_2}\left(\frac{1}{2}x^2_{34}\right)^{-\delta_3}\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_4x_2)$$
$$= (2\pi)^{D/2}\frac{\Gamma(D/2-\delta_1)\Gamma(D/2-\delta_2+s)\Gamma(D/2-\delta_3)}{\Gamma(\delta_1+s)\Gamma(\delta_2)\Gamma(\delta_3)}$$
$$\times\left(\frac{1}{2}x^2_{12}\right)^{-D/2+\delta_3}\left(\frac{1}{2}x^2_{13}\right)^{-D/2+\delta_2}\left(\frac{1}{2}x^2_{23}\right)^{-D/2+\delta_1}\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2)\,, \tag{A.2}$$

where $\delta_1+\delta_2+\delta_3 = D$.

$$\int dx_4 \left(\frac{1}{2}x_{14}^2\right)^{-\delta_1}\left(\frac{1}{2}x_{24}^2\right)^{-\delta_2}\left(\frac{1}{2}x_{34}^2\right)^{-\delta_3} q^{x_4}_{\mu_1\ldots\mu_s}(x_1|x_2x_3)$$

$$= (2\pi)^{D/2}\frac{\Gamma(D/2-\delta_1)\Gamma(D/2-\delta_2)\Gamma(D/2-\delta_3)\Gamma(D-\delta_1+s-1)}{\Gamma(\delta_1+s)\Gamma(\delta_2+s)\Gamma(\delta_3+s)\Gamma(D-\delta_1-s)}$$

$$\times\left(\frac{1}{2}x_{23}^2\right)^{-D/2+\delta_1}\left(\frac{1}{2}x_{13}^2\right)^{-D/2+\delta_2+s}\left(\frac{1}{2}x_{12}^2\right)^{-D/2+\delta_3+s}\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3)\,, \tag{A.3}$$

where $\delta_1+\delta_2+\delta_3+s=D$,

$$q_{\mu_1\ldots\mu_s} = q_{\mu_1}\ldots q_{\mu_s} - \text{traces},\ q^{x_4}_\mu(x_1|x_2x_3) = g_{\mu\nu}(x_{14})\lambda^{x_4}_\nu(x_2x_3)\,.$$

$$\int dx_4 \left(\frac{1}{2}x_{34}^2\right)^{-\delta_1}\left(\frac{1}{2}x_{24}^2\right)^{-\delta_2}\left(\frac{1}{2}x_{14}^2\right)^{-\delta_3}\lambda^{x_2}_\mu(x_3x_4)\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_4)$$

$$= (2\pi)^{D/2}\frac{\Gamma(D/2-\delta_1+s)\Gamma(D/2-\delta_2)\Gamma(D/2-\delta_3)}{\Gamma(\delta_1)\Gamma(\delta_2+1)\Gamma(\delta_3+s)}\times\left\{\left(\frac{D}{2}-\delta_1\right)\lambda^{x_2}_\nu(x_3x_1)\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2)\right.$$

$$\left.+\frac{1}{2x_{12}^2}\left[\sum_{k=1}^{s} g_{\mu\mu_k}(x_{12})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_3x_2) - \text{traces}\right]\right\}\left(\frac{1}{2}x_{12}^2\right)^{-D/2+\delta_1}\left(\frac{1}{2}x_{13}^2\right)^{-D/2+\delta_2}\left(\frac{1}{2}x_{23}^2\right)^{-D/2+\delta_3}, \tag{A.4}$$

where $\delta_1+\delta_2+\delta_3=D$.

To derive the relations (A2)–(A3) we have used the result of Ref. [57].

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