ASYMPTOTIC FREEDOM IN EXTENDED CONFORMAL SUPERGRAVITIES

E.S. FRADKIN and A.A. TSEYTLIN
Department of Theoretical Physics, P.N. Lebedev Physical Institute, Academy of Sciences, Moscow 117924, USSR

Received 26 October 1981

We present the calculation of the one-loop β-function in extended conformal supergravities. $N = 1,2,3$ theories (free or coupled to the Einstein supergravities) are found to the asymptotically free (like the $N = 0$ Weyl theory) while the $N = 4$ theory becomes finite under some plausible hypothesis. The results support the possibility to solve the problem of ghosts in these theories. The obtained sequence of SU(N) β-functions appears to be in remarkable correspondence with that for gauged O(N) supergravity theories.

Conformal supergravities [1–6] are of physical interest mainly not on their own standing but coupled to a superconformal extension of the Einstein theory (like that of ref. [7]) ‡1

$$\mathcal{L}=\mathcal{L}_{SE}+\mathcal{L}_{SW}\,,\quad \mathcal{L}_{SE}=-(R|\phi|^2+6\,\partial_\mu\phi^*\partial_\mu\phi+\ldots)\,,\qquad (1)$$

$$\mathcal{L}_{SW}=\alpha^{-2}\{W-[(4-N)/4N]\,F^2_{\mu\nu}(A) - F^2_{\mu\nu}(V)+\ldots\}\qquad (2)$$

(here ϕ belongs to an appropriate scalar multiplet and A_μ and $V^i{}_{j\mu}$ are the axial and gauge fields of the N-extended superconformal theory [5,6]). This theory is renormalizable and does not contain tachyons (or instabilities on the classical level, see e.g. ref. [9]) if $\alpha^2 > 0$ but for $\alpha^2 > 0$ tree ghosts are present [e.g. A_μ, V_μ ‡2 as well as those coming from higher derivative terms in (2)]. However, a remarkable fact is that due to the supersymmetry the numbers of Fermi and Bose ghosts are equal (they fill a massive supermultiplet in addition to a physical multiplet, massless in the simplest case). This may lead to an effective decoupling of ghosts. Another possibility is that the problem of ghosts is cured by the account of radiative corrections. [10]. This can be realized if the theory is asymptotically free [11,8] or finite [12] in $\alpha^2 > 0$ [i.e. if the $\beta(\alpha)$ function is positive or zero ‡3].

Thus the knowledge of the one-loop β-function in N-extended conformal supergravities and hence (after the addition of the scalar multiplet contribution) in the total theory (1) seems to be very important. The general scheme of the corresponding calculation is the following. We consider the infinite part of the effective action I_∞ *in the gravitational sector* where only the metric has a non-trivial background. The one-loop relevant part of the action (2) can be written in the form

$$I=\sum_s \varphi_s \Delta_{p_s}\varphi_s\,,\qquad (3)$$

where the φ_s are the fields in the theory (including the Weyl graviton $h_{\mu\nu}$, the third derivative gravitino ψ_μ etc.) and Δ_p are the background metric dependent

‡1 We use euclidean formulation where the physical signs in the lagrangian are $+\frac{1}{2}(\partial_\mu\varphi)^2+\frac{1}{4}F^2_{\mu\nu}+\ldots$. Our notations are the same as in ref. [8]: $W=R^2_{\mu\nu}-\frac{1}{3}R^2$, $R_{\mu\nu}=R^\lambda{}_{\mu\lambda\nu}=\partial_\lambda\Gamma^\lambda_{\mu\nu}-\ldots$.

‡2 Note that these fields play a somewhat auxiliary role in the theory while the physical gauge fields are those of the Poincaré supergravity in (1).

‡3 In our notations [8]:

$$I=\alpha^{-2}I_0\,,\quad I_\infty=-\frac{1}{2}\frac{\beta}{(4\pi)^2}\log\frac{L^2}{\mu^2}\cdot I_0=\frac{B_4}{n-4}\,,$$

$$B_4=\frac{1}{(4\pi)^2}\int \overline{b}_4\,\mathrm{d}^4x\,,\quad \alpha^{-2}(L)=\alpha^{-2}(\mu)+\frac{\beta}{32\pi^2}\log\frac{L^2}{\mu^2}\,,$$

where I and I_∞ are the bare action and the infinite part of the effective action. Note that in this paper we omit boundary and quadratic divergences.

p-order differential operators [‡4]. The basic observation is that each term in (3) must be *background conformal invariant* (before gauge fixing) under (W_s is the Weyl weight of the field)

$$g'_{\mu\nu} = \lambda^2 g_{\mu\nu}, \qquad \varphi'_s = \lambda^{W_s}\varphi_s . \tag{4}$$

Choosing the background covariant and conformal invariant gauges (e.g. by assuming that the background metric has zero curvature scalar, cf. ref. [8]) we can diagonalize the highest derivative terms of the graviton, gravitino and the gauge fields operators Δ_p. The resulting expression for I_∞ is (see the third footnote)

$$I_\infty = \frac{1}{2}\sum_s (-1)^{\xi_s}(\log\det \mu^{-p_s}\Delta'_{p_s})_\infty = \frac{B_4}{n-4}, \tag{5}$$

$$\overline{b}_4 = \sum_s (-1)^{\xi_s}\,\overline{b}_4(\Delta'_{p_s}), \tag{6}$$

where $\xi_s = \pm 1$ if φ_s is a scalar or Majorana spinor and Δ'_{p_s} include all necessary ghost and "averaging over gauges" operators. Introducing the notation

$$\overline{b}_4 = \beta W, \tag{7}$$

we conclude that the total β-function is given by the sum of the Weyl graviton contribution β_h, N gravitino contributions β_ψ and the contributions of the gauge and the ordinary as well as the higher derivative "matter" fields in the theory. Some of these are well known (see e.g. ref. [13] and references therein)

$$\beta_0 = \tfrac{1}{60}, \quad \beta_{1/2} = \tfrac{1}{20}, \quad \beta_1 = \tfrac{1}{5}, \quad \beta_{1m} = \tfrac{13}{60}, \tag{8}$$

i.e. the results for one scalar, Majorana spinor and massless and massive vector fields. The Weyl graviton contribution was obtained in ref. [8]:

$$\beta_h = \tfrac{199}{15}. \tag{9}$$

Therefore the main problem is how to establish the gravitino counterpart β_ψ. It is sufficient to do the corresponsing calculation using the $N = 1$ conformal supergravity lagrangian [1,3]. Its one-loop gravitational sector relevant part can be written in the form

‡4 It can be proved that all mixings in the action are either absent (e.g. due to the structure of the invariance group) or do not contribute in the one-loop infinities (for example, the term $\overline{\psi}_\mu V_\mu \chi$ with χ being a Majorana spinor leads to the contribution in the infinities $\sim \mathrm{Sp}(\overline{V}_\mu\gamma_\nu V_\mu\gamma_\nu)$ which is zero due to the fact that S-invariance dictates $\gamma_\mu V_\mu = 0$).

(our spinor notations are the same as in refs. [1,3])

$$\mathcal{L}_1 = \alpha^{-2}(W - \tfrac{3}{4}F^2_{\mu\nu} + \overline{\psi}_\mu\Delta_{3\mu\nu}\psi_\nu - \tfrac{2}{3}\overline{\chi}\hat{D}\chi + \tfrac{1}{2}\overline{\psi}\hat{D}^3\psi), \tag{10}$$

where

$$\psi = \gamma_\mu\psi_\mu, \quad \chi = D_\mu\psi_\mu + \tfrac{1}{2}\hat{D}\psi,$$

$$(D_\mu)^\lambda_\rho = \delta^\lambda_\rho(\partial_\mu + \tfrac{1}{2}\sigma^{ab}\omega_{ab})_\mu + \Gamma^\lambda_{\rho\mu},$$

$$\Delta_{3\mu\nu} = -\hat{D}^3_{\mu\nu} + V^\alpha{}_{\mu\rho}(D_\alpha)_{\rho\nu}, \tag{11}$$

$$V^\alpha{}_{\mu\nu} = -R_{\rho[\mu}\epsilon_{\nu]\rho\alpha\sigma}\gamma_5\gamma_\sigma - \tfrac{1}{2}R_{\rho\alpha}\epsilon_{\mu\rho\nu\sigma}\gamma_5\gamma_\sigma + R_{\mu\nu}\gamma_\alpha - \tfrac{1}{2}\gamma_\rho R_{\rho\alpha}g_{\mu\nu} + \gamma_\rho R_{\rho(\mu}g_{\nu)\alpha} \tag{12}$$

(we assume for simplicity that $D_\lambda R_{\rho\mu} = 0$, $R = 0$). In view of the general coordinate, conformal and chiral invariances of the complete (untruncated) $N = 1$ theory one can choose the gauges $D_\mu h_{\mu\nu} = 0$, $h^\mu_\mu = 0$, $D_\mu A_\mu = 0$ and get the above mentioned gravitational and axial field contributions in the infinities [see (8), (9)]. Fixing the remaining Q- and S-supersymmetries ($\delta\psi_\mu = D_\mu\xi + \gamma_\mu\lambda$, ...) by the gauges

$$D_\mu\psi_\mu = \eta(x), \qquad \psi = \rho(x), \tag{13}$$

and averaging (see e.g. refs. [14,15] for the general procedure) over them with the suitable operators ($\hat{D}$ and $\hat{D}^3$) we can cancel the last two terms in (10). The resulting expression for the ψ_μ-part of the partition function is given by

$$Z_\psi = \frac{(\det\Delta_{3\mu\nu})^{1/2}}{\det\Delta_{gh}(\det\hat{D})^{1/2}(\det\hat{D}^3)^{1/2}}, \tag{14}$$

$$\Delta_{gh} = -\tfrac{4}{3}D_\mu(g_{\mu\nu} - \tfrac{1}{4}\gamma_\mu\gamma_\nu)D_\nu = -D^2 - R/12. \tag{15}$$

Taking for a moment $g_{\mu\nu} = \delta_{\mu\nu}$ in (14) we conclude that the third derivative gravitino lagrangian in (10) describes *eight* (and not six, or of "three gravitino", cf. refs. [1,16]) degrees of freedom just like the Weyl lagrangian descrobes *six* (and not four, or of "two gravitons") ones (see ref. [8]).

In order to establish the contribution of Δ_3(11) in the infinities one should multiply it by the operator $\Delta_{1\nu\rho} = -\hat{D}_{\nu\rho}$. The $\overline{b}_4$-coefficient for the resulting fourth-order operator

$$\Delta_{4\,\mu\nu} = (\Delta_3 \cdot \Delta_1)_{\mu\nu} = D^4_{\mu\nu} + V^{\alpha\beta}_{\mu\rho}(D_{(\alpha}D_{\beta)})_{\rho\nu} + U_{\mu\nu} ,$$

$$V^{\alpha\beta}_{\mu\rho} = -V^{(\alpha\gamma\beta)}_{\mu\rho} + 2\sigma \cdot R_{\mu\rho} g^{\alpha\beta} , \qquad (16)$$

$$\sigma \cdot R_{\mu\nu} \equiv \sigma^{ab} R_{ab\mu\nu} , \quad U_{\mu\nu} = \sigma \cdot R_{\mu\rho} \sigma \cdot R_{\rho\nu} + \dots ,$$

can be calculated using the Δ_4-algorithm presented in ref. [8] while the contributions of $\hat{\Delta}_2 = -\hat{D}^2$ in (14), Δ_{gh} in (15) and

$$\Delta^2_{1\,\mu\nu} = -D^2_{\mu\nu} + \tfrac{1}{4} R g_{\mu\nu} - \sigma \cdot R_{\mu\nu}$$

are easily obtained with the help of the well-known Δ_2-algorithm [14,17]. This yields the following results for β in (7)

$$\beta(\hat{\Delta}_2) = \beta(\Delta_{gh}) = -\tfrac{1}{10} , \quad \beta(\Delta^2_1) = \tfrac{34}{15} ,$$

$$\beta(\Delta_4) = \beta(\Delta_3) - \tfrac{1}{2}\beta(\Delta^2_1) = \tfrac{171}{30} , \qquad (17)$$

$$\beta_\psi = -\beta(\Delta_3) + 2\beta(\Delta_{gh}) + 2\beta(\hat{\Delta}_2) = -\tfrac{149}{30} .$$

It is worth noting that the gravitino gives the *negative* contribution in the gravitational (W) infinities while the ordinary matter fields (8) and the Weyl graviton (9) give the positive ones. Using the above results [eqs. (8), (9), (17)] we can finally write down the expressions for the total number of degrees of freedom and the one-loop β-function in the $N = 1$ conformal supergravity

$$N_{\rm I} = 6 - 8 + 2 = 0 , \quad \beta_{\rm I} = \beta_{\rm h} + \beta_\psi + \beta_1 = \tfrac{17}{2} . \qquad (18)$$

The value for $\beta_{\rm I}$ [and thus for $\beta_{\rm h}$ in (9)] was checked by the independent calculation of the $N = 1$ β-function in the A_μ-sector of the theory [i.e. substituting W by $-\frac{3}{4}F^2_{\mu\nu}$ in (7)] in ref. [18], where some questions of this paper are discussed in more detail.

In order to present the analogous results for the $N = 2,3,4$ extended theories we need first to establish the contributions in the infinities of some *unusual higher derivative matter fields* present in these theories. Let us begin with the antisymmetric tensor field $T_{\mu\nu}$ described by the lagrangian [19,6]

$$\mathcal{L}^{(0)}_{\rm T} = \partial_\mu T^-_{\mu\nu} \partial_\rho T^+_{\rho\nu} = (\partial_\mu T_{\mu\nu})^2 - \tfrac{1}{4}(\partial_\rho T_{\mu\nu})^2 , \qquad (19)$$

$$T^\pm_{\mu\nu} = T_{\mu\nu} \pm \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} T_{\rho\sigma}$$

(note that in the euclidean formulation we use $\epsilon_{\mu\nu\lambda\rho}$ and $T_{\mu\nu}$ are real). The conformal invariant [with $W_{\rm T} = +1$ in (4)] extension of (19) can be obtained either directly or by the comparison with the known $N = 2$ lagrangian in ref. [6]

$$\mathcal{L}_{\rm T} = D_\mu T^-_{\mu\nu} D_\rho T^+_{\rho\nu} - \tfrac{1}{2} R_{\mu\rho} T^-_{\mu\nu} T^+_{\rho\nu} . \qquad (20)$$

Introducing the natural new variables $\zeta_\mu = \xi_\mu - \eta_\mu$, $\zeta^*_\mu = \xi_\mu + \eta_\mu$,

$$T_{\mu\nu} = D_\mu \xi_\nu - D_\nu \xi_\mu + \epsilon_{\mu\nu\alpha\beta} D_\alpha \eta_\beta , \qquad (21)$$

we can rewrite (20) as

$$\mathcal{L}_{\rm T} = \zeta^*_\mu \Delta^{(\rm T)}_{4\,\mu\nu} \zeta_\nu - (D_\mu \zeta^*_\mu)(-D^2)(D_\nu \zeta_\nu) , \qquad (22)$$

where $\Delta_{4\,\mu\nu}$ has the same form as (16) but now

$$V^{\alpha\beta}_{\mu\nu} = g^{\alpha\beta} \bar{R}_{\mu\nu} - g_{\mu\nu} \bar{R}^{\alpha\beta} - 2\bar{R}^{(\alpha}_{(\mu} \delta^{\beta)}_{\nu)}$$

$$+ \bar{R}^{(\alpha}_{\sigma} \epsilon^{\beta)\sigma}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta} R , \qquad (23)$$

$$\bar{R}_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} R , \quad U_{\mu\nu} = \tfrac{1}{4} R g_{\mu\nu} .$$

One must carefully take into account the functional measure and gauge invariance introduced by (21). Averaging over the gauges $D_\mu \zeta_\mu = \gamma(x)$, $D_\mu \zeta^*_\mu = \gamma^*(x)$ we finally get the following partition function for (20)

$$Z_{\rm T} = \left(\frac{Z_1 \det \Delta_2 \cdot (\det \Delta_2)^{1/2}}{(\det \Delta^{(\rm T)}_{4\,\mu\nu})^{1/2}} \right)^2 , \quad \Delta_2 = -D^2 , \quad (24)$$

where $Z_1 = \det \Delta_2 [\det(-D^2_{\mu\nu} + R_{\mu\nu})]^{-1/2}$ is the one gauge field partition function. As a consequence, $T_{\mu\nu}$ describes *six* degrees of freedom: three physical and three ghost [one need the change $\eta \to i\eta$ in order to obtain (24)]. From (24) it follows that the contribution of $T_{\mu\nu}$ in the β-function (7) can be established again using the Δ_4-algorithm. The result is

$$\beta_{\rm T} = \tfrac{1}{10} . \qquad (25)$$

and surprisingly turns to be equal to that for *six* scalar fields, cf. (8) [note that the R^2-infinities of course cancel due to the conformal invariance of (20)].

Next let us consider the Majorana spinor field with the flat-space lagrangian $\bar{\Lambda}\hat{\partial}^3\Lambda$. The corresponding conformal invariant extension [$W_\Lambda = -\frac{1}{2}$ in (4)] can be found to be

$$\mathcal{L}_\Lambda = \bar{\Lambda} \Delta^{(\Lambda)}_3 \Lambda ,$$

$$\Delta^{(\Lambda)}_3 = \hat{D}^3 + (R_{\mu\nu} - \tfrac{1}{6} g_{\mu\nu} R)\, \gamma_\mu D_\nu . \qquad (26)$$

Multiplying $\Delta^{(\Lambda)}_3$ by $\hat{D}$ [in complete analogy with the case of the Δ_3 gravitino operator in (11)] and using the Δ_4-algorithm we are left with [cf. (17)]

$$\beta_\Lambda = -\tfrac{1}{60} \,. \qquad (27)$$

The last non-trivial contribution is given by the scalar field with the linearized lagrangian $C\Box^2 C$, the most general conformal invariant extension of which [$W_c = 0$ in (4)] can be shown to be

$$\mathcal{L} = C\Delta_4^{(c)}C \,,$$

$$\Delta_4^{(c)} = D^4 + (2R_{\mu\nu} - \tfrac{2}{3}g_{\mu\nu}R)\, D_\mu D_\nu + \tfrac{1}{2}\gamma C^2_{\lambda\mu\nu\rho} \,, \qquad (28)$$

where $C_{\lambda\mu\nu\rho}$ is the Weyl tensor ($C^2_{\lambda\mu\nu\rho} = 2W + \text{div}$) and γ is an arbitrary constant. This constant should be fixed by comparison with the complete nonlinear $N = 4$ lagrangian presently unknown (cf. ref. [6]). Therefore we are able only to make a hypothesis that the *true value of* γ *is* -1. This value seems to be the most natural one because it leads to the presence of the $W(1 - |C|^2)$ term in the $N = 4$ lagrangian reminiscent of the version of the $N = 4$ conformal supergravity with a manifest rigid SU(1,1) and extra local chiral U(1) invariances [6]. This choice is also justified *a posteriori* because it yields the *zero-β function in the $N = 4$ theory*, the result one could expect either from the fact that the $N = 4$ theory is the maximally extended one (cf. the discussion of other probably finite theories in ref. [20]) or from the correspondence with the situation in the O(N) supergravity theories [21,22] obvious for the cases $N = 1,2,3$ (see below). It is worth stressing that the field C appears *only in the $N = 4$* theory and thus the results for $N = 2$ and $= 3$ theories are independent of the above hypothesis. Using the Δ-algorithm for (28) with $\gamma = -1$ we get (the field C is real here)

$$\beta_c = -\tfrac{2}{15} - \gamma = \tfrac{13}{15} \,. \qquad (29)$$

Now it is possible to write down the results for the total number of degrees of freedom and for the one-loop β-function for the $N = 2,3,4$ extended conformal supergravities using their spectra presented in ref. [6] and eqs. (8), (9), (17), (25), (27) and (29) ($i, j = 1, ..., N$)

$N = 2$: $\{1\ h_{\mu\nu};\ 2\ \psi^i_\mu;\ 1\ A_\mu;\ 3$ SU(2) gauge fields $V^i{}_{j\mu}$;
2 spinors χ^i; 1 antisymmetric tensor $T^{ij}_{\mu\nu} = T^{[ij]}_{\mu\nu}\}$,

$$N_{\rm II} = 6 - 2\times 8 + 2 + 3\times 2 - 2\times 2 + 6 = 0 \,,$$

$$\beta_{\rm II} = \beta_{\rm h} + 2\beta_\psi + \beta_1 + 3\beta_1 + 2\beta_{1/2} + \beta_{\rm T} = \tfrac{13}{3} \,. \qquad (30)$$

$N = 3$: $\{1\ h_{\mu\nu};\ 3\ \psi^i_\mu;\ 1\ A_\mu;\ 8\ V^i{}_{j\mu};\ 1\ (\hat\partial^3)$-spinor Λ;
3 complex scalars E_i; 9 spinors χ^{ij}; 3 $T^{ij}_{\mu\nu}\}$,

$$N_{\rm III} = 6 - 3\times 8 + 2 + 2\times 8 - 3\times 2 + 3\times 2 - 9\times 2 + 3\times 6 = 0 \,,$$

$$\beta_{\rm III} = \beta_{\rm h} + 3\beta_\psi + \beta_1 + 8\beta_1 + \beta_\Lambda + 3\times 2\beta_0 + 9\beta_{1/2} + 3\beta_{\rm T} = 1 \,. \qquad (31)$$

$N = 4$: $\{1\ h_{\mu\nu};\ 4\ \psi^i_\mu;\ 15\ V^i{}_{j\mu}$; 1 complex $(\Box^2)$-scalar C;
$4(\hat\partial^3)$-spinors Λ_i; 10 complex scalars $E_{(ij)}$;
20 spinors $\chi^{ij}{}_k$; 6 $T^{ij}_{\mu\nu}\}$,

$$N_{\rm IV} = 6 - 4\times 8 + 15\times 2 + 2\times 2 - 4\times 3\times 2 + 2\times 10 - 20\times 2 + 6\times 6 = 0 \,,$$

$$\beta_{\rm IV} = \beta_{\rm h} + 4\beta_\psi + 15\beta_1 + 2\beta_{\rm c} + 4\beta_\Lambda + 20\beta_0 + 20\beta_{1/2} + 6\beta_{\rm T} = 0 \,. \qquad (32)$$

From these results [and (9) and (18)] we get the *decreasing* sequence of positive β_N-functions ($\tfrac{199}{15}$, $\tfrac{17}{2}$, $\tfrac{13}{3}$, 1, 0) and conclude that if $\alpha^2 > 0$ in (2) the $N = 1,2,3$ conformal supergravities are *asymptotically free* just like the ($N = 0$) Weyl theory [1,8] while $N = 4$ theory is finite. However, one can formally consider the situation where $\alpha^2 \equiv -g^2 < 0$ and so the gauge fields *are physical* in (2). Then it is possible to compare the behaviour of the one-loop $\beta(g)$ [$\equiv -\beta(\alpha)$] functions for the U(N) extended conformal and gauged O(N) extended De Sitter supergravities. One finds a remarkable and rather unexpected (in view of the fact that the O(N) β-functions are defined only on

Table 1

Fields	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$T_{\mu\nu}$	Λ	C
O(N)	$-\frac{87}{10}$	$+\frac{137}{60}$	$-\frac{1}{5}$	$-\frac{1}{20}$	$-\frac{1}{60}$	–	–	–
U(N)	$-\frac{199}{15}$	$+\frac{149}{30}$	$-\frac{1}{5}$	$-\frac{1}{20}$	$-\frac{1}{60}$	$-\frac{1}{10}$	$+\frac{1}{60}$	$-\frac{13}{15}$

Table 2

Fields	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$T_{\mu\nu}$	Λ	C
O(N)	0	$-\frac{13}{3}$	$+\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{12}$	–	–	–
U(N)	0	$-\frac{17}{2}$	$+\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{12}$	>0	<0	<0

shell) *correspondence.* Collecting the results of refs. [21,22] and the present paper (and also of ref. [18]) we find the following contributions of different fields in the $\beta(g)$-function in the gravitational and the gauge field sectors displayed in table 1 and table 2 respectively [note that our normalization is $\mathcal{L} = g^{-2}F_{\mu\nu}^2(V) + ...$] (all numbers in table 2 are to be multiplied by appropriate group invariants). The resulting sequences for $N = 0, ..., 4$ $\beta_N(g)$- functions are displayed in table 3 [note that the values for $N = 0,1$ in O(N) and $N = 0$ in U(N) cases make sense in the gravitational sector only]. The correspondence here is shifted ($\Delta N = 1$) probably due to the fact that O(N) $\subseteq$ U($N - 1$). Using tables 1 and 2 we see that the growth of $\beta_N(g)$ with N is due to the positive contribution of the gravitino (and Λ) in the gravitational sector or is due to the positive contributions of the gauge (and $T_{\mu\nu}$) fields in the gauge sector (remarkably enough $T_{\mu\nu}$ *supports* the asymptotic freedom in g) [‡5].

[‡5] Let us remark that the above correspondence holds also for topological infinities, see tables 4 and 5.

Table 3

N	0	1	2	3	4	5
O(N)	$-\frac{87}{10}$	$-\frac{77}{12}$	$-\frac{13}{3}$	$-\frac{5}{2}$	-1	0
U(N)	$-\frac{199}{15}$	$-\frac{17}{2}$	$-\frac{13}{3}$	-1	0	

Finally, let us note that the total β-function for (1) is obtained by summing that of the pure conformal supergravity with the contribution of the scalar multiplet in $\mathcal{L}_{SE}$ in (1). One can easily establish the latter, e.g. in the $N = 1$ case, by noting that the part of the $N = 1$ superconformal extension of the Einstein lagrangian [7] contributing in the $F_{\mu\nu}^2$- and W-type infinities is given by

$$\mathcal{L}'_{SE} = -6\mathcal{D}_\mu\phi^*\mathcal{D}_\mu\phi - \tfrac{1}{2}\bar{\chi}\hat{\mathcal{D}}_1\chi\,, \tag{34}$$

where

$$\mathcal{D}_\mu = D_\mu - \tfrac{1}{2}\mathrm{i}A_\mu\,, \quad \mathcal{D}_{1\mu} = D_\mu + \tfrac{1}{4}\mathrm{i}A_\mu\,,$$

and ϕ and χ belong to the $N = 1$ scalar multiplet. The infinities for (34) are obtained with the help of the Δ_2- algorithm [see also (8)]

$$\Delta\bar{b}_4 = \Delta\beta(W - \tfrac{3}{4}F_{\mu\nu}^2)\,, \quad \Delta\beta = \tfrac{1}{12} \tag{35}$$

(note that the $R\phi\phi^*$-term contribution in the divergences is suppressed due to the presence of W in the total lagrangian, cf. ref. [8]). As a consequence, the total β-function ($\beta_I + \Delta\beta$) is positive (and increasing). Thus all $N \leqslant 4$ superconformal theories obtained by coupling $\mathcal{L}_{SE}$ in (1) and $\mathcal{L}_{SW}$ in (2) are *asymptotically free* and not finite (cf. ref. [12]). However, there remains an interesting possibility that the total β-function for α^2 will be zero in a yet unknown $N > 4$ superconformal extension of (1), (2) [the pure $N > 4$ conformal supergravity will probably have *negative* $\beta(\alpha)$, cf. (18), (30)–(32)].

Table 4
Contributions of different fields in the topological infinities β_{top} ($\bar{b}_4 = \beta_{top} \cdot R^* \cdot R^*$).

Fields	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$T_{\mu\nu}$	Λ	C
O(N)	$\frac{212}{180}$	$-\frac{233}{720}$	$-\frac{13}{180}$	$\frac{7}{720}$	$\frac{1}{180}$	–	–	–
U(N)	$\frac{411}{180}$	$-\frac{173}{180}$	$-\frac{13}{180}$	$\frac{7}{720}$	$\frac{1}{180}$	$-\frac{2}{15}$	$\frac{21}{720}$	$\frac{1}{90} - \frac{1}{2}\gamma$

Table 5
$\beta_{\mathrm{top}}^{(N)}$-coefficients for O($N$) and U($N$) theories.

N	0	1	2	3	4	5
O(N)	$\frac{212}{180}$	$\frac{41}{12}$	$\frac{11}{6}$	0	-2	-4
U(N)	$\frac{411}{180}$	$\frac{5}{4}$	$-\frac{1}{24}$	$-\frac{3}{2}$	$-3-\gamma$	

References

[1] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D17 (1978) 3179.
[2] S. Ferrara and B. Zumino, Nucl. Phys. B134 (1978) 301.
[3] P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D19 (1979) 3166.
[4] S. Ferrara, M.T. Grisaru and P. van Nieuwenhuizen, Nucl. Phys. B138 (1978) 430;
B. de Wit and S. Ferrara, Phys. Lett. 81B (1979) 317.
[5] S. Ferrara, M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Nucl. Phys. B129 (1977) 125.
[6] E. Bergshoeff, M. De Roo and B. De Wit, Nucl. Phys. B182 (1981) 173.
[7] M. Kaku and P.K. Townsend, Phys. Lett. 76B (1978) 54.
[8] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. 104B (1981) 377; Lebedev Inst. preprint N 70 (1981).
[9] K.S. Stelle, Gen. Relat. Grav. 9 (1978) 353.
[10] J. Julve and M. Tonin, Nuovo Cimento 46B (1978) 137;
A. Salam and J. Strathdee, Phys. Rev. D18 (1978) 4480.
[11] E. Tomboulis, Phys. Lett. 97B (1980) 77;
B. Hasslacher and E. Mottola, Phys. Lett. 99B (1981) 221.
[12] A. Salam, Trieste preprint IC-81-61 (1981).
[13] G.W. Gibbons, Lecture Notes Math. 676 (1978) 513;
L. Parker, in: Recent developments in gravitation, eds. M. Levy and S. Deser (Plenum, New York, 1979).
[14] B.S. De Witt, Dynamical theory of groups and fields (Gordon and Breach, New York, 1965); in: Quantum Gravity, the Second Oxford Symp., eds. C. Isham, R. Penrose and S. Sciama (Oxford UP, London, 1981).
[15] N.K. Nielsen, Nucl. Phys. B140 (1978) 499;
R.E. Kallosh, Nucl. Phys. B141 (1978) 141.
[16] P. van Nieuwenhuizen, Phys. Rep. 68 (1981) 189.
[17] G. 't Hooft and M. Veltman, Ann. Inst. H. Poincaré A20 (1974) 69;
P.B. Gilkey, J. Diff. Geom. 10 (1975) 601.
[18] E.S. Fradkin and A.A. Tseytlin, β-function in conformal supergravities, Lebedev Inst. preprint (1981) N 213.
[19] B. de Wit and J.W. van Holten, Nucl. Phys. B155 (1979) 530.
[20] P.K. Townsend, Finite field theory, CERN preprint TH-3066 (1981)
[21] S.M. Christensen, M.J. Duff, G.W. Gibbons and M. Roček, Phys. Rev. Lett. 45 (1980) 161.
[22] T.L. Curtright, Phys. Lett. 102B (1981) 17.