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ON THE NEW DEFINITION OF OFF-SHELL EFFECTIVE ACTION

E.S. FRADKIN and A.A. TSEYTLIN

Department of Theoretical Physics, P.N. Lebedev Physical Institute, USSR Academy of Sciences, Leninsky Pr. 53, Moscow 117924, USSR

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We analyze the recently proposed definition of the off-shell, gauge-invariant, gauge-independent, effective action $\tilde{\Gamma}$, utilizing an invariant metric on the field space. It is shown how to establish correspondence between $\tilde{\Gamma}$ and the standard effective action, calculated in some particular (Landau-type) gauge. Several examples are explicitly discussed, including Yang-Mills theory, the effective potential in scalar QED, and one-loop quantum gravity. Generalization to the case of super-invariant theories (e.g. super-Yang-Mills and supergravity) is also presented.

1. Introduction and general framework

Recently Vilkovisky proposed a novel definition of an effective action in gauge theories [1]. It is reparametrization invariant, gauge invariant, and gauge independent even off the effective "mass shell" and thus may find applications in dynamical symmetry breaking, external field (or source) problems and especially in theories with a non-trivial configuration space metric, including quantum (super)gravity. Here we discuss the relation of the new effective action to the standard one and illustrate general statements on a number of well-known field theories. We also generalize the definition to the case of theories with fermions including supersymmetrical ones.

Let us first review Vilkovisky's definition in a slightly modified manner, clarifying some important points. Let $M=\{\varphi^i\}$ be a configuration (Bose field) space and $S(\varphi)$ be a *scalar* on M and an *invariant* of some global symmetry group G, i.e.

$$S'(\varphi')=S(\varphi),\quad \varphi'=f(\varphi),\qquad S(\tilde{\varphi})=S(\varphi),\qquad \tilde{\varphi}=g(\varphi).$$

Then the standard definition of the euclidean effective action (generator of irreducible vertices)*

$$\Gamma(\phi)=W(J(\phi))-J(\phi)\phi, \qquad \frac{\delta W}{\delta J}=\phi, \qquad \mathrm{e}^{-W}=\int \mathrm{d}\varphi\,\mu(\varphi)\mathrm{e}^{-S(\varphi)-\varphi J},$$

$$\mathrm{e}^{-\Gamma(\phi)}=\int \mathrm{d}\varphi\,\mu(\varphi)\mathrm{e}^{-S(\varphi)+(\varphi^i-\phi^i)\delta\Gamma/\delta\phi^i(\phi)}\equiv\int \mathrm{d}\eta\,\mathrm{e}^{-S(\phi+\eta)+\eta^i\Gamma_{,i}} \tag{1}$$

does not in general provide an off-shell ($\Gamma_{,i}\neq 0$) reparametrization-invariant and G-invariant result. Namely, $\Gamma'(\phi')\neq\Gamma(\phi)$ and $\Gamma(\tilde{\phi})\neq\Gamma(\phi)$, where $\Gamma(\phi)$ and $\Gamma'(f(\phi))$ are effective actions calculated according to (1) starting with $S(\varphi)$ and $S'(f(\phi))$ respectively. Suppose now that M is endowed with some symmetrical G-invariant connection Γ^i_{jk}. If we take $\sigma^i=\dot{\varphi}^i(0)$ to be the tangent vector at ϕ to the geodesic, connecting ϕ and φ, $\ddot{\varphi}^i+\Gamma^i_{jk}(\varphi(t))\dot{\varphi}^j\dot{\varphi}^k=0$, $\varphi(0)=\phi$, $\varphi(1)=\varphi$ (t is an affine parameter), then it will be a *vector* under transformations of ϕ and a *scalar* under transformations of φ and the following relations will be true

$$\sigma^k\nabla_k\sigma^i=-\sigma^i, \qquad \nabla_k\equiv\frac{\partial}{\partial\phi^k}+\Gamma^i_{jk}(\phi),$$

$$\eta^i\equiv\varphi^i-\phi^i=\sigma^i-\frac{1}{2!}\Gamma^i_{j_1j_2}(\phi)\sigma^{j_1}\sigma^{j_2}-\cdots-\frac{1}{n!}\Gamma^i_{j_1\cdots j_n}(\phi)\sigma^{j_1}\ldots\sigma^{j_n}+\cdots,$$

$$\Gamma^i_{j_1\cdots j_n}=\Gamma^i_{(j_1\cdots j_{n-1},j_n)}-(n-1)\Gamma^i_{m(j_2\cdots j_{n-1}}\Gamma^m_{j_1j_n)}, \tag{2}$$

$$\sigma^i(\varphi,\phi)=\eta^i+\tfrac{1}{2}\Gamma^i_{j_1j_2}|_\phi\eta^{j_1}\eta^{j_2}+\tfrac{1}{6}\left(\Gamma^i_{j_1j_2j_3}+\Gamma^i_{mj_2}\Gamma^m_{j_1j_3}\right)|_\phi\eta^{j_1}\eta^{j_2}\eta^{j_3}+\cdots. \tag{3}$$

Thus the *new* effective action

$$\mathrm{e}^{-\tilde{\Gamma}(\phi)}=\int \mathrm{d}\eta\exp\left[-S(\eta+\phi)+\sigma^i(\phi+\eta,\phi)\tilde{\Gamma}(\phi)_{,i}\right], \tag{4}$$

generating the same S-matrix as Γ in (1), will be off-shell reparametrization- and G-invariant. The next (purely technical, possible already in (1)) step is to change the quantum variable $\eta\to\sigma$ obtaining a manifestly "covariant" perturbation theory by noting that $S(\eta+\phi)=S(\phi)+\Sigma_{n=1}^{\infty}(1/n!)(\nabla_{j_1}\cdots\nabla_{j_n}S)\sigma^{j_1}\ldots\sigma^{j_n}$. The construction of $\tilde{\Gamma}$ in (4) depends (off shell) on the choice of the connection. For example, in the case of a riemannian manifold σ model, $S=\int \mathrm{d}^dx\,\frac{1}{2}g_{ij}(\varphi)\partial_\mu\varphi^i\partial_\mu\varphi^j$, it is natural to take Γ^i_{jk} to be the Christoffel connection for g_{ij}. It is in this particular case where an

* Here ϕ and $\eta=\varphi-\phi$ are the "background" and "quantum" fields and we use the "condensed notation" of ref. [2]. Eq. (1) is equivalent to other definitions of Γ within the background field method [3–6]. For simplicity we shall include contributions of local ($\log\mu\sim\delta(0)$) measures in the symbol $\mathrm{d}\eta$.

off-shell, invariant, effective action analogous to (4) (with the accent on the use of σ^i perturbation theory) was previously discussed in the literature [7–9]*. The expression (4) in its general form first appeared in [1].

Now let G contain local gauge transformations (g.t.), $\delta\varphi^i = R^i_\alpha(\varphi)\varepsilon^\alpha \equiv R^i(\varphi)$. Then if $\Gamma_{,i} \neq 0$, $\hat{S} = S - \eta^i\Gamma_{,i}$ in (1) is *not* invariant under *quantum* g.t. ($\delta\varphi = R(\varphi)$, $\delta\phi = 0$) and thus no gauge fixing is formally needed off shell. However, the resulting *perturbation theory* for Γ will be singular in the limit $\Gamma_{,i} \to 0$. That is why, the standard attitude is to give sense to (1) by choosing some gauge χ thus making the limit regular. The price which is paid is an off-shell gauge dependence of Γ, clear from the fact that one is averaging a non-gauge-invariant functional $e^{\eta^i\Gamma_{,i}}$. This observation prompts another suggestion to regulate the limit: one is to substitute $\eta^i\Gamma_{,i}$ by $\sigma^i(\varphi,\phi)\Gamma_{,i}$ *invariant under quantum g.t. thus restoring quantum gauge invariance of* $\hat{S}$ *in* (1) *off shell and as a result providing independence for the subsequent gauge fixing*. Also, we are to preserve the invariance under background g.t. ($\delta\varphi = 0$, $\delta\phi = R(\phi)$) spoiled by $\eta^i\Gamma_{,i}$ if the generators R^i_α are non-linear. A hint of how to construct σ^i is contained in the observation that a change of gauge is simply a change of parametrization of the space of group orbits $\mathcal{M} = \mathrm{M}/\mathrm{G}$ [2,12–14]. Integrating over $\mathcal{M}$ we are to preserve reparametrization invariance and thus, in view of the previous discussion, need a *connection* on $\mathcal{M}$. It can be readily constructed using the natural metric on $\mathcal{M}$ [12,13,1]. If g_{ij} is some G-invariant metric on M, i.e.

$$\mathcal{D}_i R_{j\alpha} + \mathcal{D}_j R_{i\alpha} = 0\,, \qquad R_{i\alpha} = g_{ij}R^j_\alpha\,, \qquad \mathcal{D}_i = \partial_i + \left\{ {k \atop ij} \right\}, \tag{5}$$

then

$$\bar{g}_{ij} = g_{ij} - R_{i\alpha}N^{\alpha\beta}R_{j\beta}\,, \qquad N_{\alpha\beta} = R^i_\alpha g_{ij} R^j_\beta\,, \qquad N^{\alpha\beta} = \left(N_{\alpha\beta}\right)^{-1}, \tag{6}$$

measures the distance between two orbits G_φ and $\mathrm{G}_{\varphi+\mathrm{d}\varphi}$**. Thus we define ($\bar{g}^{im} = g^{ik}g^{mj}\bar{g}_{kj}$)

$$\hat{\Gamma}^i_{jk} = \tfrac{1}{2}\bar{g}^{im}\left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right) \equiv \Gamma^i_{jk} + R^i_\alpha K^\alpha_{jk}\,,$$

* It was also proposed in ref. [10] to use analogous methods in the case of gravity considered as a non-linear realization of affine and conformal groups [11]. This approach, while stressing the "chiral" aspect of gravity, neglects the "gauge" one and thus is different from that of ref. [1] (see below).

** If $\mathcal{M}$ is coordinatized by embedding $\mathcal{M} \to \mathcal{M}' \subset \mathrm{M}$ with the help of the gauge condition $\chi_\alpha(\varphi,\phi) = 0$, then (6) can be extended on the whole M [1]: $\tilde{g}_{ij} = \bar{g}_{ij} + \chi_{\alpha,i}\chi_{\beta,j}C^{\alpha\beta}$ ($C_{\alpha\beta}$ is some group metric). The $\mathcal{M}'$-restriction $(\tilde{g})_{\mathcal{M}'} = PgP$, $P^i_j = \delta^i_j - g^{ik}\chi_{\alpha,k}\Delta^{-1\alpha\beta}\chi_{\beta,j}$, $\Delta_{\alpha\beta} = \chi_{\alpha,i}\chi_{\beta,j}g^{ij}$, is the metric of ref. [13]. Note that

$$\det\tilde{g} = \left(\det Q_{\alpha\beta}\right)^2 \times \left(\det C_{\alpha\beta}\right)^{-1} \times \left(\det N_{\alpha\beta}\right)^{-1} = \det(\tilde{g})_{\mathcal{M}'} \times \det\Delta_{\alpha\beta} \times \left(\det C_{\alpha\beta}\right)^{-1}.$$

Thus integrating over $\mathcal{M}'$ with the invariant measure $(\mathrm{d}\varphi)_{\mathcal{M}'}\sqrt{\det\tilde{g}_{\mathcal{M}'}} \times (\det N_{\alpha\beta})^{+1/2}$ we obtain the result of the standard covariant quantization.

where

$$\hat{\Gamma}^i_{jk} = \left\{ {i \atop jk} \right\} + T^i_{jk}, \qquad B^\alpha_i = N^{\alpha\beta} R_{i\beta},$$

$$T^i_{jk} = -2 B^\alpha_{(j} \mathfrak{D}_{k)} R^i_\alpha + R^p_\alpha \mathfrak{D}_p R^i_\beta B^\alpha_{(j} B^\beta_{k)}. \tag{7}$$

For simplicity we avoid explicit parametrization of $\mathfrak{M}$ and formally assume $\hat{\Gamma}^i_{jk}$ to be defined on the whole M (for a rigorous argument see [1]). Defining σ^i as in (2) one can prove that (i) terms $\sim R^i$ in $\hat{\Gamma}^i_{jk}$ drop from the product $\sigma^i \tilde{\Gamma}_{,i}$ if $\tilde{\Gamma}$ is background invariant, i.e. $R^i(\phi)\tilde{\Gamma}_{,i} = 0$ and thus (7) can be taken as the final form of the connection on M [1]; (ii) transformation rules under background and quantum g.t. are (recall that σ^i is a vector under $\delta\phi$ and a scalar under $\delta\varphi$)

$$\delta\sigma^i|_{\delta\phi = R(\phi)} = \left[R^i_\alpha(\phi)_{,j} \sigma^j + R^i_\gamma(\phi) \Sigma^\gamma_\alpha(\varphi, \phi) \right] \varepsilon^\alpha,$$

$$\delta\sigma^i|_{\delta\varphi = R(\varphi)} = R^i_\gamma(\phi) \Omega^\gamma_\alpha(\varphi, \phi) \varepsilon^\alpha, \tag{8}$$

providing both quantum and background invariances of $\sigma^i \tilde{\Gamma}_{,i}$ if $R^i(\phi)\tilde{\Gamma}_{,i} = 0$. It is worth stressing that (8) is valid only if the gauge algebra is *closed*, i.e. $E^{ji}_{\alpha\beta} = 0$ in

$$R^i_{\beta,j} R^j_\alpha - R^i_{\alpha,j} R^j_\beta = R^i_\gamma C^\gamma_{\alpha\beta} + S_{,j} E^{ji}_{\alpha\beta}. \tag{9}$$

To illustrate this point we note that according to (3) and (7)

$$\delta\sigma^i|_{\delta\phi} = \left[R^i_{\alpha,j} \sigma^j - R^i_\alpha(\phi) - \nabla_k R^i_\alpha \sigma^k + \cdots \right] \varepsilon^\alpha,$$

$$\delta\sigma^i|_{\delta\varphi} = \left[R^i_\alpha(\phi) + \nabla_k R^i_\alpha \sigma^k + \cdots \right] \varepsilon^\alpha,$$

where

$$\nabla_j R^i_\alpha = -R^k_{[\alpha} R^i_{\beta],k} B^\beta_j = \tfrac{1}{2}\left(R^i_\gamma C^\gamma_{\alpha\beta} + S_{,k} E^{ki}_{\alpha\beta} \right) B^\beta_j. \tag{10}$$

Thus we come to the following definition of a reparametrization-invariant, gauge-invariant, and gauge-independent off-shell effective action due to Vilkovisky [1]*

$$e^{-\tilde{\Gamma}(\phi)} = \int [d\eta]_\chi \exp\left[-S(\phi+\eta) + \sigma^i(\phi+\eta, \phi) \tilde{\Gamma}(\phi)_{,i} \right], \tag{11}$$

$$[d\eta]_\chi = d\eta\, \delta(\chi(\phi, \eta)) \det Q(\phi, \eta), \qquad Q_{\alpha\beta} = \frac{\delta\chi_\alpha}{\delta\eta^i} R^i_\beta(\phi+\eta), \tag{12}$$

* Here σ^i is constructed according to (2), (3), (7) and thus $\tilde{\Gamma}$ depends on the choice of the G-invariant metric g_{ij} on M. If G does not include local supersymmetry (see below), then g_{ij} is assumed to be local, i.e. independent of the derivatives of φ, so that transformation (3) is local in the limit $R^i_\alpha \to 0$.

where χ_α *is an arbitrary* (not necessarily background covariant) gauge. Eq. (11) is to be contrasted to the standard definition (cf. [3–6])

$$e^{-\Gamma_\chi(\phi)} = \int [d\eta]_\chi \exp\left[-S(\phi+\eta) + \eta^i \Gamma_{\chi,i}\right], \tag{13}$$

which gives an off-shell gauge-invariant (but χ-dependent) effective action *only* if (i) the generators R^i_α are linear (which is not true, e.g. in supergravity) and (ii) the gauge χ is covariant under simultaneous $\delta\varphi = R(\varphi)$, $\delta\phi = R(\phi)$ transformations (and $\chi(\varphi=\phi)=0$).

2. Gauge fixing and the relation to the standard effective action.

In view of the gauge independence of $\tilde{\Gamma}$ we are free to choose a particularly distinguished one, namely the "orthogonal" gauge

$$R^i_\alpha(\phi) g_{ij}(\phi) \sigma^j(\varphi,\phi) = 0. \tag{14}$$

Introducing the notation $\sigma_{\perp,\parallel} = \Pi_{\perp,\parallel}(\phi)\sigma$, $\Pi^i_{\parallel j} = R^i_\alpha B^\alpha_j$, $\Pi^i_{\perp j} = g^{im}\bar{g}_{jm}$, $\sigma^i_\parallel = R^i_\alpha(\phi)\rho^\alpha(\phi,\varphi)$, we conclude (cf. (8)) that under quantum g.t. $\delta\sigma_\perp = 0$, $\delta\rho^\alpha = \Omega^\alpha_\gamma \varepsilon^\gamma$, i.e. $\sigma^i_\perp$ is physical, while ρ^α is a spurious variable, which drops out from the exponent in (11). Thus $\rho = 0$ or (14) is really a natural gauge. Noting that according to (7) $T^i_{jk}(\phi)\sigma^j_\perp\sigma^k_\perp = 0$, we can put (assuming (14)) $T^i_{jk} = 0$ in (2) and thus in (3). Changing the variable $\eta \to \sigma$ we get

$$e^{-\tilde{\Gamma}(\phi)} = \int d\sigma\, \delta\left(R_{i\alpha}(\phi)\sigma^i\right) \det\bar{Q}(\phi,\sigma) e^{-\hat{S}(\phi,\sigma)}, \tag{15}$$

$$\hat{S} = S(\phi) + \sum_{n=1}^{\infty} \left(\mathfrak{D}_{j_1}\ldots\mathfrak{D}_{j_n}S\right)\sigma^{j_1}\ldots\sigma^{j_n} - \sigma^i\tilde{\Gamma}_{,i},$$

$$\bar{Q}_{\alpha\beta} = R_{i\alpha}(\phi)R^i_\beta(\phi+\eta(\sigma,\phi)) = R_{i\alpha}\left(R^i_\beta + R^i_{\beta,j}\sigma^j + \mathfrak{D}_k\left(R^i_{\beta,j}\right)\times\sigma^k\sigma^j + \cdots\right),$$

$$\det Q \times |\det\partial\eta/\partial\sigma| = \det\bar{Q}.$$

For example, at one loop

$$\tilde{\Gamma}(\phi) = S(\phi) - \tfrac{1}{2}\log\frac{\det N_{\alpha\beta}}{\det\Delta_{\perp ij}}, \qquad \Delta_\perp = \Pi_\perp\Delta\Pi_\perp,$$

$$\Delta_{ij} = \mathfrak{D}_i\mathfrak{D}_j S = S_{,ij} - \begin{Bmatrix} k \\ ij \end{Bmatrix} S_{,k}. \tag{16}$$

To establish correspondence with the standard definition (13), another (Landau-

De Witt or LD) gauge appears to be useful

$$R_{i\alpha}(\phi)\eta^i = 0, \qquad \text{i.e. } \eta^i_{\parallel} = 0. \tag{17}$$

Then from (3) and (7) we get

$$\sigma^i = \eta^i_\perp + \tfrac{1}{2}\left\{ {i \atop jk} \right\}\eta^j_\perp \eta^k_\perp$$

$$+ \tfrac{1}{6}\left(\left\{ {i \atop j_1\, j_2} \right\}_{,\, j_3} + \left\{ {i \atop mj_1} \right\}\left\{ {m \atop j_2 j_3} \right\} - 3\mathfrak{D}_{j_1} R^i_{\alpha} B^{\alpha}_m \left\{ {m \atop j_2 j_3} \right\}\right)\eta^{j_1}_\perp \eta^{j_2}_\perp \eta^{j_3}_\perp + \cdots. \tag{18}$$

Comparing (11) and (13) one can prove the following statement: *if there exists a parametrization in which* g_{ij} *is field independent, i.e.* $\left\{ {i \atop jk} \right\} = 0$ (*and thus* $\sigma^i = \eta^i_\perp$ *if* $\eta^i_\parallel = 0$) *then in this parametrization* $\tilde{\Gamma}(\phi) = \Gamma_{\text{LD}}(\phi)$. Therefore in order to obtain $\tilde{\Gamma}$ e.g. for a "Yang-Mills plus linear matter multiplets" theory one need only to calculate the standard effective action (13) in the LD gauge (17). If, however, g_{ij} is non-trivial (as it is in the case of gravity) then $\tilde{\Gamma} \neq \Gamma_{\text{LD}}$ already at one loop: fluctuation operators Δ differ by the local "correction term" $\left\{ {k \atop ij} \right\} S_{,k}$ (cf. (16)). Starting with two loops there are also differential corrections to the vertices (18). The resulting expressions for $\tilde{\Gamma}$ in the gauges (14) and (17) are of course the same. Let us now illustrate a calculation of $\tilde{\Gamma}$ in several characteristic examples.

3. Yang-Mills theory

In the ordinary parametrization we have: $\delta A^a_\mu = \mathfrak{D}^{ab}_\mu \varepsilon^b$,

$$\mathfrak{D}^{ab}_\mu = \delta^{ab}\partial_\mu + f^{abc}A^c_\mu, \qquad \gamma_{ab} = f_{acd}f_{bcd},$$

and thus

$$i \to (x_i, a_i, \mu_i), \qquad \alpha \to (x_\alpha, a_\alpha), \qquad g_{ij} \to \delta_{\mu_i\mu_j}\gamma^{a_i a_j}\delta(x_i - x_j),$$

$$\left\{ {i \atop jk} \right\} = 0, \qquad R^i_\alpha \to \mathfrak{D}^{a_i a_\alpha}_{\mu_i}\delta(x_i - x_\alpha),$$

$$N_{\alpha\beta} \to -\left(\mathfrak{D}^2_{x_\alpha}\right)^{a_\alpha a_\beta}\delta(x_\alpha - x_\beta), \qquad N^{-1}_{bc} = (-\mathfrak{D}^2)^{-1}_{bc},$$

i.e. (7) takes the form

$$\Gamma^i_{jk} \to \delta_{\mu_i\mu_j} f_{a_i a_j b} N^{-1}_{bc}(x_j - x_k)\mathfrak{D}^{ca_k}_{\mu_k}\delta(x_i - x_j)$$

$$+ \tfrac{1}{2}\mathfrak{D}^{db}_{\mu_i} f_{a_i dc} N^{-1}_{cp}(x_i - x_k)\mathfrak{D}^{pa_k}_{\mu_k} N^{-1}_{ce}(x_i - x_j)\mathfrak{D}^{ea_j}_{\mu_j} + (j \leftrightarrow k). \tag{19}$$

In the LD gauge (17), i.e. $\mathcal{D}_\mu(B)A_\mu=0$, we have

$$\tilde{\Gamma}(B)=\Gamma_{\mathrm{LD}}(B)=-\log\left\{\int \mathrm{d}A\,\delta\left(\mathcal{D}_\mu(B)A_\mu\right)\det\left[\mathcal{D}_\mu(B)\mathcal{D}_\mu(A+B)\right]\right.$$
$$\left.\times\exp\left[-S(A+B)+A\frac{\delta\tilde{\Gamma}}{\delta B}\right]\right\}.$$

This gauge is the $\alpha\to 0$ limit of $\mathcal{L}_g=(1/2\alpha)(\mathcal{D}_\mu(B)A_\mu)^2$ and thus can be used for explicit calculations (cf. [15]). At one loop

$$\tilde{\Gamma}(B)=\Gamma_{\alpha=0}=S(B)-\tfrac{1}{2}\log\frac{\det(-\mathcal{D}^2)}{\det\Delta_{1\perp}}, \tag{20}$$

where $\Delta_{1\mu\nu}=-\delta_{\mu\nu}\mathcal{D}^2-2F_{\mu\nu}$ and $\Pi_{\perp\mu\nu}=\delta_{\mu\nu}-\mathcal{D}_\mu\mathcal{D}^{-2}\mathcal{D}_\nu$. Only if $J_\nu\equiv\mathcal{D}_\mu F_{\mu\nu}=0$ does (20) coincide with the standard expression in the $\alpha=1$ gauge, because $\det\Delta_1=\det\Delta_{1\perp}\det(-\mathcal{D}^2)\times\det(\delta_{am}+\mathcal{D}^{-2}_{ab}f_{bcd}J^d_\mu\Delta^{-1cp}_{1\mu\nu}\mathcal{D}^{pm}_\nu)$ (for the $\alpha=0$ gauge result (20) see also [16])*. $\tilde{\Gamma}$ in (20) can be calculated, e.g. for a constant non-abelian B^a_μ with $\mathcal{D}_\mu F_{\mu\nu}\neq 0$ and thus applied to the problem of a vacuum as a minimum of $\tilde{\Gamma}$. Here it may be useful to clarify the meaning of the reparametrization invariance of $\tilde{\Gamma}$ (11): if we start with some unconventional variables, $A'^a_\mu=f(A)^a_\mu$, then g_{ij} is no longer trivial, $\left\{{i \atop jk}\right\}\neq 0$, and we are to use the general expression (11), e.g. in the gauge (17), giving some $\tilde{\Gamma}'(B')$. Then the reparametrization invariance implies $\tilde{\Gamma}'(B')=\tilde{\Gamma}(f^{-1}(B'))$.

4. Scalar electrodynamics

The aim of our next example (one-loop effective potential in scalar QED) is to illustrate the cancellation of gauge dependence in $\tilde{\Gamma}$ (and also to show that the calculation of $\tilde{\Gamma}$ in any gauge other then the LD one (17) is very cumbersome). Let us take the following class of gauges

$$\mathcal{L}=\tfrac{1}{4}F^2_{\mu\nu}+\tfrac{1}{2}\left(\mathcal{D}_\mu\varphi\right)^2+\frac{\lambda}{4!}\left(\varphi^a\varphi^a\right)^2,$$

$$\mathcal{L}_g=\frac{1}{2\alpha}\left(\partial_\mu A_\mu+\beta g\varepsilon_{ab}\phi_b\eta_a\right)^2,\qquad \eta^a=\varphi^a-\phi^a, \tag{21}$$

* Note that if one starts calculating $\tilde{\Gamma}$ without fixing any gauge but separating $A_\mu=A_{\perp\mu}+\mathcal{D}_\mu(B)\rho$, then at one loop $\delta^2S=A_\perp\Delta_1A_\perp+A_\perp K_1\rho+\rho K_2\rho$. Adding the correction term $\sim AAT\mathcal{D}F$, with T^i_{jk} given by (19), it is easy to check that it cancels the ρ-dependent terms restoring quantum gauge invariance. Separating $\int\mathrm{d}\rho$, e.g. by the gauge $\rho=0$ we again get (20).

where $\mathcal{D}_\mu \varphi^a = \partial_\mu \varphi^a + g\varepsilon^{ab}\varphi^b A_\mu$ and assume that the background fields are $B_\mu = 0$, $\phi^a = \text{const}$. At one loop (11) gives

$$\tilde{\Gamma}(\phi) = S(\phi) - \tfrac{1}{2}\log\det\Delta + \log\det\Delta_{\text{gh}},$$

$$\Delta_{ij} = \left(S + S_g\right)_{,ij} - \Gamma^k_{ij} S_{,k}, \qquad \Delta_{gh\alpha\beta} = \chi_{\alpha,i}(\phi) R^i_\beta(\phi). \tag{22}$$

The construction of the connection is straightforward: $\delta A_\mu = -\partial_\mu \varepsilon$,

$$\delta\varphi^a = g\varepsilon^{ab}\varphi^b\varepsilon, \qquad g_{ij} \to \left\{ g_{\mu\nu}, \delta^{ab} \right\}, \qquad R^i_\alpha \to \left\{ -\partial_{\mu_i}, g\varepsilon_{a_i b}\varphi_b \right\} \delta(x_i - x_\alpha),$$

$$N_{\alpha\beta} \to \left(-\Box + g^2\phi^2 \right) \delta(x_\alpha - x_\beta),$$

and thus

$$\Gamma^a_{\mu\nu} \mathcal{L}_{,a} = -\tfrac{1}{6}\lambda g^2 \phi^4 \partial_{x_\mu} N^{-2}_{x_\mu x_\nu} \partial_{x_\nu}$$

etc. where $\mathcal{L}_{,a} = \frac{1}{6}\lambda\phi^a\phi^2$. As a result $\Delta_{\mu\nu} = k^2\delta_{\mu\nu} + \cdots$,

$$\Delta_{a\mu} = ig\varepsilon_{ab}\phi_b C k_\mu, \qquad C = 1 - \beta/\alpha + \tfrac{1}{6}\lambda\phi^2 k^2 N^{-2}, \tag{23}$$

$$\Delta_{ab} = \left(k^2 + \tfrac{1}{2}\lambda\phi^2\right)\Pi^{\parallel}_{ab} + B\Pi^{\perp}_{ab},$$

$$B = k^2 + \left(\tfrac{1}{6}\lambda + \frac{\beta^2 g^2}{\alpha}\right)\phi^2 - \tfrac{1}{6}\lambda g^2\phi^4\left(2 - g^2\phi^2 N^{-1}\right)N^{-1},$$

$$N^{-1} = \frac{1}{k^2 + g^2\phi^2}, \qquad \Pi^{\parallel}_{ab} = \frac{\phi^a\phi^b}{\phi^2}.$$

Integrating over η^a, we get

$$\bar{\Delta}_{\mu\nu} = \Delta_{\mu\nu} - \Delta_{\mu a}\Delta^{-1}_{ab}\Delta_{b\nu} = \left(k^2 + g^2\phi^2\right)\Pi^{\perp}_{\mu\nu} + E\Pi^{\parallel}_{\mu\nu},$$

$$\Pi^{\parallel}_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}, \qquad E = \alpha^{-1}k^2 + g^2\phi^2 + \tfrac{1}{6}\lambda g^2\phi^4 N^{-2}k^2 - \frac{g^2\phi^2 k^2}{B}C^2, \tag{24}$$

and finally

$$\tilde{\Gamma} \equiv \int d^4x \left[\frac{\lambda}{4!}\phi^4 + \tilde{V}_1(\phi) \right], \qquad dk \equiv \frac{d^4k}{(2\pi)^4}$$

$$\tilde{V}_1 = -\tfrac{1}{2}\int dk \left[\log\left(k^2 + \tfrac{1}{2}\lambda\phi^2\right) + 3\log\left(k^2 + g^2\phi^2\right) + \log B + \log E \right.$$
$$\left. - 2\log\left(\alpha^{-1/2}\left(k^2 + \beta g^2\phi^2\right)\right) \right]$$
$$\equiv \tfrac{1}{2}\sum_n \int dk \log\left(k^2 + M_n^2\right) = \frac{\gamma}{64\pi^2}\phi^4\log\phi^2 + \cdots. \tag{25}$$

The coefficient γ has the following gauge-dependent value if calculated according to the standard recipe (13) (i.e. omitting N^{-1} terms in B, C, E in (25)) (cf. [17] for $\beta = 0$)

$$\gamma = 3g^4 + \tfrac{5}{18}\lambda^2 + \tfrac{1}{3}(2\beta - \alpha)\lambda g^2 . \tag{26}$$

To calculate γ for the "corrected" expression (25) one is to observe that $\gamma \sim \Sigma_n M_n^4$ and thus integrating $\int \mathrm{d}k \log(k^{2m} + A_1 k^{2m-2} + A_2 k^{2m-4} + \cdots)$ we only need to know $A_1^2 - 2A_2$. As a result, all α- and β-dependent terms cancel and we find

$$\tilde{\gamma} = 3g^4 + \tfrac{5}{18}\lambda^2 + \tfrac{2}{3}\lambda g^2 . \tag{27}$$

One could avoid the above involved calculations by working in the LD gauge (17), i.e. $\partial_\mu A_\mu + g\varepsilon_{ab}\phi_b\eta_a = 0$, where $\tilde{\Gamma} = \Gamma_{\mathrm{LD}}$, thus obtaining (27) as the $\alpha = 0$, $\beta = 1$ case of (26). It is interesting to note that it is possible to calculate V_1 according to (1) without fixing any gauge: we get (24) with $E \to \frac{1}{6}\lambda g^2\phi^2/(k^2 + \frac{1}{6}\lambda\phi^2)$ and thus

$$V_1 = -\tfrac{1}{2}\int \mathrm{d}k \left\{ \log\left(k^2 + \tfrac{1}{2}\lambda\phi^2\right) + 3\log\left(k^2 + g^2\phi^2\right) \right.$$
$$\left. + \left[\log\left(k^2 + \tfrac{1}{6}\lambda\phi^2\right) + \log\left[\left(\tfrac{1}{6}\lambda g^2\phi^4\right)/\left(k^2 + \tfrac{1}{6}\lambda\phi^2\right)\right]\right]\right\},$$

coincides with the result in the *unitary gauge* ($\gamma = 3g^4 + \frac{1}{4}\lambda^2$) after cancelling the infinite ($\sim L^4 \log\phi^2$) term by the local measure. Repeating the calculation with the "correction" term $\Gamma^i_{jk} S_{,k}$ (22), we see that it annihilates the longitudinal part of $\bar{\Delta}_{\mu\nu}$, restoring quantum gauge invariance and thus making it necessary to fix some gauge, e.g. $\partial_\mu A_\mu = 0$; then $\tilde{V} = -\frac{1}{2}\int \mathrm{d}k\,\{\log(k^2 + \frac{1}{2}\lambda\phi^2) + 3\log(k^2 + g^2\phi^2) + \log B_0 - \log k^2\}$, $B_0 = B(\beta = 0)$. The result for γ is again (27). Thus the result of the new "corrected" recipe is similar to the R gauge one and is different from that of the U gauge (or gauge non-fixing) prescription.

5. Quantum gravity

In the standard parametrization

$$\delta g_{\mu\nu} = \varepsilon^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda}\partial_\nu \varepsilon^\lambda + g_{\nu\lambda}\partial_\mu \varepsilon^\lambda, \quad R_{\mu\nu\rho} = 2g_{\rho(\mu}D_{\nu)}\delta(x_\mu - x_\rho), \quad D_\mu = \partial_\mu + \left\{ {\lambda \atop \mu\nu} \right\}.$$

The invariant (5) local (no $\partial_\lambda g_{\mu\nu}$) field-space metric is unique up to a real parameter [2,14] ($\lambda \neq -1/4$)

$$\mathrm{d}s^2 = \int \mathrm{d}^4x\, G^{\mu\nu\alpha\beta}\,\mathrm{d}g_{\mu\nu}\,\mathrm{d}g_{\alpha\beta}, \qquad G^{\mu\nu\alpha\beta} = \sqrt{g}\left(g^{\mu(\alpha}g^{\beta)\nu} + \lambda g^{\mu\nu}g^{\alpha\beta}\right). \tag{28}$$

Its Christoffel connection is

$$\left\{\begin{matrix}\mu\nu & & \lambda\kappa \\ & \rho\sigma & \end{matrix}\right\} = -\tfrac{1}{2}\Bigg[\delta_\rho^{(\mu}g^{\nu)(\lambda}\delta_\sigma^{\kappa)} + \delta_\rho^{(\lambda}g^{\kappa)(\mu}\delta_\sigma^{\nu)} - \tfrac{1}{2}g^{\mu\nu}\delta^\lambda_{(\rho}\delta^\kappa_{\sigma)} - \tfrac{1}{2}g^{\lambda\kappa}\delta^\mu_{(\rho}\delta^\nu_{\sigma)}$$
$$+\frac{1}{2(1+4\lambda)}g_{\rho\sigma}g^{\mu(\lambda}g^{\kappa)\nu} + \frac{\lambda}{2(1+4\lambda)}g_{\rho\sigma}g^{\mu\nu}g^{\lambda\kappa}\Bigg]. \tag{29}$$

The operator $N_{\alpha\beta}$ in (6) and the LD gauge (17) have the form

$$N_{\alpha\beta} = 2\left[-g_{\alpha\beta}D^2 - R_{\alpha\beta} - (1+2\lambda)D_\alpha D_\beta\right]\sqrt{g}\,\delta(x_\alpha - x_\beta), \tag{30}$$

$$R_{\alpha\beta\rho}G^{\alpha\beta\mu\nu}h_{\mu\nu} = 0, \qquad \text{i.e. } D_\rho\left(h^\rho_\alpha + \lambda\delta^\rho_\alpha h\right) = 0. \tag{31}$$

The expression for $\tilde{\Gamma}$ is then given by (11), or at one loop by (16), where now

$$\Delta^{\mu\nu\alpha\beta} = \hat{\Delta}^{\mu\nu\alpha\beta} - \left\{\begin{matrix}\mu\nu & & \alpha\beta \\ & \rho\sigma & \end{matrix}\right\}\frac{\delta S}{\delta g_{\rho\sigma}}, \qquad \hat{\Delta} = \frac{\delta^2 S}{\delta g^2}.$$

If $S = -\int d^4x(R - 2\Lambda_0)\sqrt{g}$, then $h\hat{\Delta}h = \frac{1}{2}h\Delta_0 h - [D_\mu(h^\mu_\alpha - \frac{1}{2}\delta^\mu_\alpha h)]^2$, where

$$\Delta_0 = -G_0D^2 + X_0, \qquad X_{0\mu\nu\alpha\beta} = 2C_{\mu(\alpha\beta)\nu} + \tfrac{1}{6}R\left(g_{\mu(\alpha}g_{\beta)\nu} - \tfrac{1}{4}g_{\mu\nu}g_{\alpha\beta}\right) - 2\Lambda_0 G_{0\mu\nu\alpha\beta}, \tag{32}$$

(here $C_{\mu\nu\alpha\beta}$ is the Weyl tensor and $G_0 = G(\lambda = -\frac{1}{2})$). Using (29) to establish the correction term and employing the condition (31) we get

$$\Delta = \tfrac{1}{2}(-\hat{G}D^2 + \hat{X}), \qquad \hat{G} = G(\hat{\lambda}),$$
$$\hat{\lambda} = -\tfrac{1}{2} - 2\left(\lambda + \tfrac{1}{2}\right)^2, \qquad E_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}R,$$
$$\hat{X}^{\mu\nu}_{\alpha\beta} = X^{\mu\nu}_{0\alpha\beta} + 2E^{(\mu}_{(\alpha}\delta^{\nu)}_{\beta)} - \tfrac{1}{2}g^{\mu\nu}E_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}E^{\mu\nu} - R\left(\frac{\lambda + \frac{1}{2}}{1+4\lambda}\right)\left(\delta^\mu_{(\alpha}\delta^\nu_{\beta)} - \tfrac{1}{4}g^{\mu\nu}g_{\alpha\beta}\right). \tag{33}$$

For simplicity we shall take $\lambda = -\frac{1}{2}$ (this value of λ in (28) was obtained in [1] by adding the condition $\hat{G}=G$). Then $\hat{G}=G_0$ and the one-loop result for $\tilde{\Gamma}$ is

$$\tilde{\Gamma}(g)=S(g)-\tfrac{1}{2}\log\frac{\det\left(-g_{\mu\nu}D^2-R_{\mu\nu}\right)}{\left(\Delta^{\mu\nu}_{\perp\rho\sigma}\right)}, \tag{34}$$

where $\Delta_\perp$ is Δ subjected to (31). Let us now compare the divergences of $\tilde{\Gamma}$ with those of the standard effective action in the gauges $\mathcal{L}_g=(1/2\alpha)[D_\mu(h^\mu_\nu+\lambda\delta^\mu_\nu h)]^2$ known for $\Lambda_0=0$ [18] and for $\Lambda_0\neq 0$ [19]. Introducing the notation:

$$\Gamma_\infty=-\tfrac{1}{2}\left(\tfrac{1}{2}L^4B_0+L^2B_2+B_4\log(L^2/\mu^2)\right),\qquad L\to\infty,$$

$$B_p=\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,b_p,\qquad b_0=2,\qquad b_2=\rho_1R+\rho_2\Lambda_0,$$

$$b_4=\tfrac{53}{45}R^*R^*+\beta_2E^2_{\mu\nu}+\tfrac{1}{3}\beta_3R^2+\beta_4R\Lambda_0+\beta_5\Lambda_0^2, \tag{35}$$

we have for $\lambda=-\frac{1}{2}$ and $\alpha=1$ (De Donder gauge) and $\alpha=0$ (31) the cases

	β_2	β_3	β_4	β_5	ρ_1	ρ_2
$\alpha=1$:	$\frac{7}{10}$	$\frac{23}{40}$	$-\frac{26}{3}$	20	$-\frac{23}{3}$	20
$\alpha=0$:	$\frac{7}{10}$	$\frac{63}{40}$	-8	12	$-\frac{17}{3}$	12

(36)

Observing that the only coefficient which changes after adding the correction term is β_2 we conclude that the infinite part of $\tilde{\Gamma}$ is given by (35) with $\beta_3,\beta_4,\beta_5,\rho_1,\rho_2$ being as in (36) for $\alpha=0$. The value of β_2 can be found by direct diagram computation or by using the algorithm for non-diagonal operators developed in [20]. Moreover, if the background geometry is of the Einstein type, $R_{\mu\nu}=\Lambda g_{\mu\nu}$ (i.e. $E_{\mu\nu}=0$) then the correction term vanishes and $\tilde{\Gamma}$ coincides with the standard one in the $\alpha=0$, $\lambda=-\frac{1}{2}$ gauge.

It should be understood that $G^{\mu\nu\alpha\beta}$ in (28) is not related to a particular structure of gravitational action (it can be used for the construction of $\tilde{\Gamma}$ for any covariant action, e.g. of the $R+R^2$ type). Thus $G^{\mu\nu\alpha\beta}$ does not actually govern the full dynamics of Einstein quantum gravity. At the same time, it is important to take into account the non-trivial geometry of $g_{\mu\nu}$ configuration space dictated by $G^{\mu\nu\alpha\beta}$, in the quantization of this theory.

6. Supersymmetrical theories

Let us first formally generalize the definition of $\tilde{\Gamma}$ (11) to the case when fields and gauge parameters can be either Bose or Fermi. We shall take all derivatives to be right ones and use two kinds of invariant contractions: $\xi^k\eta_k$ and $\xi_k\eta^k$, $\xi_k=(-)^k\xi_k$

(cf. [21, 22]). Then $ds^2 = d\varphi^i g_{i\hat{m}} d\varphi^m$, $dS = S_{,\hat{k}} d\varphi^k$, $\delta\varphi^i = R^i_{\hat{\alpha}}\varepsilon^\alpha$, $S_{,\hat{k}} R^k_{\hat{\alpha}} = 0$, and we get the analog of (7) in the form

$$\Gamma^i{}_{\hat{k}\hat{m}} = \left\{ {i \atop \hat{k}\hat{m}} \right\} + T^i{}_{\hat{k}\hat{m}}, \qquad \Gamma^i{}_{k\hat{m}} = (-)^k \Gamma^i{}_{\hat{k}\hat{m}},$$

$$\left\{ {i \atop \hat{m}\hat{k}} \right\} = \tfrac{1}{2} g^{in} \left[g_{n\hat{m},\hat{k}} + g_{n\hat{k},\hat{m}} (-)^{km} - g_{m\hat{k},\hat{n}} (-)^{n(1+m+k)+m} \right],$$

$$T^i{}_{\hat{m}\hat{k}} = -R^i{}_{\hat{\alpha};\hat{m}} B^\alpha{}_{\hat{k}} (-)^{m\alpha} - R^i{}_{\hat{\alpha};\hat{k}} B^\alpha{}_{\hat{m}} (-)^{k(\alpha+m)}$$
$$+ \tfrac{1}{2} R^i{}_{\hat{\alpha};\hat{n}} R^n{}_{\hat{\beta}} B^\beta{}_{\hat{m}} B^\alpha{}_{\hat{k}} (-)^{\alpha m} + \tfrac{1}{2} R^i{}_{\hat{\alpha};\hat{n}} R^n{}_{\hat{\beta}} B^\alpha{}_{\hat{m}} B^\beta{}_{\hat{k}} (-)^{\beta(m+\alpha)}, \qquad (37)$$

where

$$g_{m\hat{k}} g^{kn} = \delta_m{}^n, \qquad N_{\hat{\alpha}\hat{\beta}} = R^i{}_{\hat{\alpha}} g_{i\hat{m}} R^m{}_{\hat{\beta}} (-)^{i\alpha},$$

$$B^\alpha{}_{\hat{k}} = g_{k\hat{m}} R^m{}_{\hat{\beta}} N^{\beta\alpha} (-)^{k(\alpha+1)}, \qquad \xi^i_{;\hat{m}} = \xi^i_{,\hat{m}} + (-)^{ik} \xi^k \left\{ {i \atop k\hat{m}} \right\}.$$

Constructing $\sigma^i = \eta^i + \frac{1}{2}\eta^m \Gamma^i{}_{m\hat{k}} \eta^k (-)^{im} + \cdots$ as in (2), (3), we finally get (11) with $\tilde{\Gamma}_{,\hat{k}} \sigma^k$. The LD gauge (17) now is $R^i{}_{,\hat{\alpha}} g_{i\hat{m}} \eta^m (-)^{i\alpha} = 0$ (for example, in the spinor QED: $\partial_\mu A_\mu + iea\bar{\psi}_{\text{quant}} \psi_{\text{back}} + \text{h.c.} = 0$, $[a] = \text{cm}$).

Consider now a theory which is supersymmetry invariant (or "Q-invariant"). To preserve off-shell supersymmetry we need a Q-invariant metric on the field space. The construction of such metric in terms of component fields appears to be *non-trivial* (Q-algebra mixes φ and $\partial\varphi$). For example, in the Wess-Zumino model ($\varphi^i \to \varphi, \psi, F$) no local metric can be obtained starting with $\int d^4x (d\varphi\, d\varphi^* + a\, d\psi\, d\psi + \cdots)$, while the correct Q-invariant metric is $ds^2_{\text{wz}} = \int d^4x (d\varphi\, dF - \frac{1}{2} d\psi\, d\psi + \text{h.c.})$ (for notation see [23]). Its field independence implies the equivalence of (1) and (4) and thus the Q-invariance of $\Gamma(\varphi, \psi, F)$. At the same time, no analog of ds^2_{wz} exists for a component gauge multiplet ($A_\mu, \lambda, \mathcal{D}$). Thus we are confronted with a problem of construction of the gauge-invariant super-invariant component effective action, e.g., for a super-Yang-Mills theory. A natural solution is provided by superspace generalization of the definitions (11), (37) of $\tilde{\Gamma}$: one simply substitutes space by superspace, fields by superfields (e.g. chiral or real), and gauge transformations by supergauge ones. Hence in order to get a supergauge-invariant, supergauge-independent (and superfield reparametrization-invariant) off-shell effective action $\tilde{\Gamma}_s$ we only need to find the super-gauge-invariant (and super Poincaré-invariant) metric on the space of the superfields. Given $\tilde{\Gamma}_s$ in terms of background superfields we can choose a physical (WZ) background supergauge solving the above problem. More explicitly, e.g. in the WZ model, $ds^2_{\text{wz}} = \int dz (d\Phi\, d\Phi\, \delta(\bar{\theta}) + \text{h.c.})$, $dz = d^4x\, d^2\theta\, d^2\bar{\theta}$, $\Phi(z) =$

$\varphi+\sqrt{2}\,\theta\psi+\theta\theta F$, etc. while in the $N=1$ super-Yang-Mills case

$$ds^2_{\text{SYM}}=\int dz\,\text{tr}(de^V de^{-V})=\int dz\,g_{ab}(V)dV^a dV^b, \tag{38}$$

where $V^+=V$, $V_{ij}=V^a t^a_{ij}$ (t^a are generators of the gauge group G), $W_\alpha=-\frac{1}{4}\bar{D}^2 e^{-V}D_\alpha e^V$. This metric (and $S=\int dz(W^\alpha W_\alpha\delta(\bar\theta)+\text{h.c.})$) is invariant under supergauge transformations

$$e^{V'}=e^{-i\Lambda^+}e^V e^{i\Lambda},\qquad \text{or}\qquad \delta V=iL_{V/2}\left[(\Lambda+\Lambda^+)+\coth(L_{V/2})(\Lambda-\Lambda^+)\right],$$

$$L_A\ldots=[A,\ldots],\qquad \text{i.e.}\qquad \delta V^a=R^a{}_b(V)\Lambda^b+\text{h.c.}$$

Given a group G, one first finds $g_{ab}(V)$ (being the metric on the homogeneous space G_C/G_R) and $R^a{}_b(V)$, then one calculates the connection (37) and finally establishes the form of the effective action $\tilde\Gamma_s$ (being gauge independent, it can be evaluated either in the WZ or in manifestly supersymmetric quantum supergauges). It should be stressed that the use of the non-trivial metric (38) is essential for the construction of the gauge invariant $\tilde\Gamma_s$ in view of the non-linearity of $R^a{}_b$. At this point one observes the analogy between this recipe and the special "background-quantum" splitting used in [24,25] for particular super YM and supergravity models (cf. the relation of refs. [7,8]). The advantage of the new approach is in its generality, quantum-gauge independence, and its reparametrization invariance.

Finally, let us discuss the case of $N=1$ supergravity. Working in superspace, a natural candidate for a super invariant metric is $\int dz(E_A{}^N E_B{}^M+\cdots)dE_M{}^A dE_N{}^B$. However, the supervielbein $E_M{}^A$ is a constrained variable and therefore dE implies a shift of H *and* ∂H, H^M being the unconstrained (dynamical) superfield of Ogievetsky-Sokatchev or Siegel-Gates. The desired superinvariant metric on the (H,ϕ) space (ϕ is an auxiliary chiral superfield) can be taken in the form

$$ds^2_{\text{SG}}=\int dz\left[E\,dH^M g_{M\hat N}(H,\partial H,\partial^2 H)\,dH^N+E^{1/3}d\phi\,d\phi\,\delta(\bar\theta)+\text{h.c.}\right], \tag{39}$$

where

$$g_{M\hat N}=E_M{}^A\eta_{A\hat B}E_N{}^B(-)^{N(1+B)},\qquad E=\text{sdet}\,E_M{}^A,$$

and $E_M{}^A$ is built from H, ∂H and $\partial^2 H$ according to the rules of ref. [26]:

$$E_A=E_A{}^M D_M,\qquad E_\alpha=e^{-H}D_\alpha e^H,\qquad H=H^M D_M,\ \text{etc.}$$

A peculiar feature of supergravity is that the metric necessarily depends on *derivatives* of fields. All the following construction of the invariant action $\tilde\Gamma_s$ can be done as above. Note that (39) is not an extension of a simplest general-coordinate-invariant metric for component fields

$$\mathrm{d}s^2 = \int \mathrm{d}^4x \sqrt{g}\left\{\left[e^\mu_b e^\nu_a - \tfrac{1}{2}(1+t)e^\mu_a e^\nu_b + t g^{\mu\nu} g_{ab}\right]\mathrm{d}e^a_\mu \mathrm{d}e^b_\nu + a_1 \mathrm{d}\bar{\psi}_\mu \mathrm{d}\psi_\nu g^{\mu\nu}\right.$$

$$\left. + a_2(\mathrm{d}S\,\mathrm{d}S + \mathrm{d}P\,\mathrm{d}P - \mathrm{d}A_a \mathrm{d}A_a)\right\}, \qquad (t = \text{parameter}),$$

which is sufficient, if we do not demand supersymmetry of $\tilde{\Gamma}$ (e.g. computing it in the gravitational sector). The corresponding LD gauges (17) are: $D_\mu(h^\mu_a - \frac{1}{2}e^\mu_a h) = 0$, $e^\nu_{[a} h_{b]\nu} = 0$, $D_\mu \psi_\mu = 0$. Even in this case σ^i depends on all powers of the quantum auxiliary fields and thus one cannot integrate them out from the very beginning.

In conclusion we remark that the new effective action may be useful in non-polynomial theories describing coupling of matter with (super) gravity. The non-trivial configuration-space metric of the "pure" theory is then "extended" to the matter fields sector.

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Note added in proof:

The standard definition of effective action within the background field method was recently discussed also in ref. [27].

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