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QUANTUM PROPERTIES OF HIGHER DIMENSIONAL AND DIMENSIONALLY REDUCED SUPERSYMMETRIC THEORIES

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We discuss the relation of quantum properties of d-dimensional and dimensionally reduced theories with a special emphasis on the use of a regularization accounting for power divergences. The main examples are $N=1$, $d=10$ super Yang-Mills (SYM^1_{10}) and $N=1$, $d=11$ supergravity (SG^1_{11}) and their reductions. In particular, we understand the vanishing of the one-loop β-function in SYM^4_4 (zero conformal anomalies in SG^8_4) as a consequence of the absence of L^6 (L^7) power divergences in SYM^1_{10} (SG^1_{11}). The heat kernel expansion coefficients b_{2k}, $k=0,\ldots,3$ are found to be zero for SG^1_{11} (as well as for SYM^1_{10} and SG^1_{10}) and thus the one-loop finiteness of maximal SGs in $d \leqslant 7$ is explicitly demonstrated. We also present the expressions for the one-loop constant gauge field effective lagrangian and scalar effective potential in the SYM^4_4 theory and analyse the problem of $N=4$ supersymmetry breaking.

1. Introduction

Recently it was understood that a number of interesting four-dimensional theories may be obtained by a reduction of higher dimensional ones, most notably, from $N=1$, $d=10$ super Yang-Mills (SYM^1_{10}) [1] and $N=1$, $d=11$ supergravity (SG^1_{11}) [2] (see e.g. [3–8]). Two points of view are possible on the relation of d-dim and 4-dim theories: (A) our world is described by a *complete* d-dim theory, while its apparent four dimensionality is only a low-energy phenomenon and can be attributed to a particular structure of d-dim vacuum space (e.g., $\mathrm{M}^4 \times \mathrm{N}^{d-4}$, N being a compact manifold of a "scale" R, for example, a group G [9] or a coset G/K space [10, 11]); (B) dimensional reduction is a technical tool in analysis of 4-dim theories (useful, e.g., in constructing various supergravities [3, 12]). Believing in (A), one can classically consider the observed light particles in M as massless modes of "Fourier" expansion in internal coordinates. However, at the quantum level it is necessary to include the infinity of massive modes ($M_n \sim 1/R$). The reason is that the equality of d-dim (Γ_d) and 4-dim (Γ_4) quantum effective actions (the latter defined to be that of

d-dim theory rewritten on terms of 4-dim fields) holds only when all modes are taken into account*. Being interested only in low-energy processes we may try to define some effective light particles theory, integrating over massive modes. The resulting (non-local) effective lagrangian $\mathcal{L}_{\text{light}}$ *does not* coincide with the classical zero mode lagrangian $\mathcal{L}^{(0)}$, following by reduction from d-dim classical action. Naturally, it is tempting to connect both expressions in some approximation. This program may seem analogous to constructing "effective field theories" (see e.g. [13]): if the external momenta $p_i \ll \min(M_n)$, then $\mathcal{L}_{\text{light}}$ is given by the infinite sum of local terms, with those of dim > 4 being suppressed by $1/M$ factors. However, there is one *essential* difference: d-dim theories are generally *non-renormalizable* while the analysis in [13] assumes renormalizability of the complete theory (evident for a *finite* number of heavy states). That is why it seems impossible to describe consistently the low-energy sector of quantized d-dim theory with the help of some quantized 4-dim one. Thus one can *a priori* criticize the attempts [14, 7, 8] [*if* viewed from the point of (A)] to construct a unified theory starting with power counting non-renormalizable d-dim gauge theories. However, two improvements are possible: (A′) it may happen that d-dim theory is finite *on shell* (either in each order of loop expansion or after its summation, cf. [15]); (A″) one may pass to a more sophisticated picture where d-dim theory is only an intermediate stage, being a low-energy ($\alpha' \to 0$) limit of some finite or renormalizable theory of (super) strings in d dimensions (see e.g. [16] and references therein). Disregarding (A″) in this paper, let us point out that the existence of finite $d > 4$-dim theories is doubtful at present. Namely, according to superstring (supergraph) power counting rules [17, 16] ([18]) we expect UV finiteness of maximally extended SYM and SG d-dim theories (following by reduction from SYM^1_{10} and SG^1_{11}) for the following number of loops:

$$\text{SYM}: l < \frac{4}{d-4} \qquad \left(l < \frac{6}{d-4}\right),$$

$$\text{SG}: l < \frac{6}{d-2} \qquad \left(l < \frac{14}{d-2}\right), \tag{1.1}$$

i.e. SYM^1_{10} is already infinite in the first loop [while SG^1_{11} is infinite for $l \geqslant 1$ ($l \geqslant 2$)]. We will confirm this conclusion by demonstrating the presence of one-loop on-shell quadratic divergence in SYM^1_{10}. Once again we see that the approaches [7, 8], starting with SYM^1_{10} are to be considered as unsatisfactory at the quantum level. The

* It is worth stressing that $\Gamma_d = \Gamma_4$ is valid for infinite as well as for finite parts; to avoid possible contradictions (cf. [6]) one is to observe that $(\Gamma_4)_\infty$ = (divergences for every mode) + additional divergences which are due to *infinite sums* of finite parts of partial 4-dim effective actions for separate modes.

following remark may be useful concerning the reasoning in ref. [8]. Here the probable finiteness of SYM_4^4 (i.e. the *simplest* reduction of SYM_{10}^1) was implicitly treated as an indication of some good quantum properties of SYM_{10}^1 itself. The latter theory was then used as a starting point for a different (coset) reduction. Finally, it was conjectured that the resulting "realistic" $d=4$ theory may thus be distinguished from the point of divergences. It should, however, be stressed that different reductions *a priori* have *different* quantum behaviour* and it is the simplest supersymmetry preserving one which is singled out by its UV properties. Different reductions do have the same UV limit but only when *all* (massive) modes are taken into account**. But then we again confront the problem of infinities of complete d-dim theory.

Therefore, let us now turn to the second point of view (B). Here we are to quantize a 4-dim theory marked by the fact that its classical action can be obtained by a reduction of some d-dim one. Then a natural question is about information, concerning the quantum theory, which can be gained from this circumstance. To provide an answer we are to relate the effective action $\Gamma_4^{(r)}$ of our 4-dim theory to that (Γ_d) of the initial d-dim one. In view of the above discussion, the latter is defined only for a fixed UV cut-off L. Then it is clear that shrinking the radii of compact dimensions to zero we get $\lim_{R\to 0}(\Gamma_d)_{\text{fixed } L} = \Gamma_4^{(r)}$ (after proper rescalings of couplings and wave functions to absorb all R^{d-4} factors). Generally, this implies that the study of infinities of Γ_d may tell us something about those of $\Gamma_4^{(r)}$. However, care is needed in interpreting the above equality. A subtle point is that one should use different *cut-offs* for four (L_4) and $d-4$ (L_{d-4}) dimensions, relating the limits $L_{d-4}\to\infty$ and $R\to 0$ e.g. by $L_{d-4}\sim 1/R$ (because $L_{d-4}\to\infty$ *after* $R\to 0$ is senseless). Now it is evident that in order to relate the infinities of Γ_d and $\Gamma_4^{(r)}$ it is necessary to use a power-divergence preserving regularization*** (i.e. one cannot choose the standard dimensional or ζ-function regularizations in d dimensions). In this connection let us comment on the proposal [22] to understand the finiteness of SYM_4^4 by relating the absence of certain ("$F\Box^3F$") *logarithmic* counter-terms for SYM_{10}^1 to that of "F^2" ones for SYM_4^4. From our point of view this relation cannot be justified. In fact, the simplest counter-example is provided by YM in an odd number of dimensions. This theory is free from one-loop logarithmic divergences but the reduced theory has a non-zero one-loop β-function. It turns that it is the "L^{d-4}-type" d-dim counter-term which is related to the logarithmic one of 4-dim theory. Another important clarification is that it is not sufficient to study only d-dim

* For example, the coset reduction generally changes the number of degrees of freedom and in this respect is analogous to truncation, which is known to modify quantum results (e.g. $\beta_{1\text{-loop}}=0$ in SYM_4^4 but $\beta_{1\text{-loop}}\neq 0$ in SYM_4^2).

** One should also bear in mind that the initial gauge group of d-dim theory (e.g. E_8 in [7, 8]) will be restored in the high-energy limit of the final $d=4$ theory only after summing contributions of all modes.

*** The use of such a regularization naturally solves possible paradoxes discussed in [6]. Note also that regularizations accounting for power divergences were recently discussed, e.g., in [19–21].

invariant counter-terms because reduction *breaks* d-dim symmetry (e.g. $O(d) \to O(4) \times O(d-4)$) and thus the infinities of the reduced theory cannot be expressed solely in terms of d-dim invariants. The moral is that one must be cautious in applying d-dimensional considerations in the reduced theory. But having properly understood the connection of d-dim and 4-dim results for some particular reduction one can use d-dim theory to facilitate the analysis. This appears to be especially efficient in supersymmetric theories where different background sectors are tied together by supersymmetry and hence one can do calculations in a suitable one, common for reduced and d-dim theories.

The aim of this paper (based mainly on the point of view (B)) is: (i) to study some properties of higher dimensional theories; (ii) to connect them with those of reduced theories; (iii) to study some new features of SYM_4^4, illustrating the advantages of the $d=10$-dimensional approach. In sect. 2 we prove the "lemma" establishing the connection of one-loop results in d-dim and (simply) reduced theories. Then we discuss the correspondence of (one and higher loop) infinities with special emphasis on the use of a gauge-invariant, power-divergence preserving regularization. Sect. 3 is devoted to the one-loop analysis of SYM_4^4 using SYM_{10}^1 as a guide. First, in subsect. 3.1 we show the absence of L^{10-2k}, $k \leqslant 3$, one-loop infinities (i.e. the vanishing of $b_{2k} \sim F^k$ counter-terms) in SYM_{10}^1 and relate the $b_4 = 0$ property of SYM_{10}^1 to that of the zero one-loop β-function in SYM_4^4. We also observe that $b_4(YM_{26}) = 0$ and remark on the connection with string theory. It turns out, however, that $b_8(SYM_{10}^1) \neq 0$ (subsects. 3.2, 3.3) and so there are at least L^2 (on-shell) infinities in ten dimensions (implying logarithmic ones in the $d=8$ reduction of SYM_{10}^1). Thus we *explicitly* demonstrate the absence of one-loop finiteness of SYM_{10}^1. The above conclusions are in accordance with power-counting results (1.1) and also with those of one-loop four-particle amplitude calculations in maximal SYM_d theories [17] (see also [18]), which showed its UV finiteness for $d < 8$*. Next, in subsect. 3.2, we calculate the SYM_4^4 one-loop homogeneous abelian gauge field effective action, which is distinguished by its UV finiteness (as compared to corresponding expressions for QED [23] and YM theory [24], cf. also [25,26]). However, supersymmetry does not cure the well-known IR instability of a constant background (see e.g. [27]), which is due to the negative mode of the gauge field operator. In subsect. 3.3 we analyse the one-loop effective potential for scalars. The problem appears to be closely connected with that of the effective action for a constant non-abelian gauge potential background ($A_i \neq 0$, $i = 5, \ldots, 10$) in SYM_{10}^1. In particular, this provides an understanding of the IR instability (cf. [28]) of the final (off-shell) answer. Here we also show the absence of perturbative SYM_4^4 supersymmetry breaking and discuss a possibility to construct a realistic model starting with SYM_4^4 with softly broken supersymmetry (this theory still has zero

* The vanishing of one-loop SYM_4^4 k-particle ($k \leqslant 3$) amplitude (or the corresponding Green function in a suitable gauge) can be related to $b_{2k} = 0$, while its non-zero value for $k = 4$ indicates $b_8 \neq 0$.

β-function and thus should be contrasted to those of refs. [7, 8] which do not provide soft supersymmetry breaking).

The main topic of sect. 4 is the study of one-loop divergences in higher dimensional supergravities. We start (in subsect. 4.1) with pure d-dim gravity and prove that it is *one-loop on-shell infinite for any* $d>4$ by computing the b_{2k} ($\sim R^k$) coefficients of L^{d-2k} divergences for $k \leqslant 3$. Then we comment on infinities in reduced Kaluza-Klein theories. Subsect. 4.2 deals with the gravitational sector b_{2k} calculation for antisymmetric tensor gauge fields which are known to be present in supergravities. Fairly general results are presented, clarifying the subject of "quantum (in)equivalence" and extending the previous work [29]. Further, in subsect. 4.3 we solve the *non-trivial* problem of the background field method quantization of the gravitino in $d>4$ dimensions. We explicitly construct the "standard" gauge where the gravitino operator takes its simplest form and thus, e.g., it is straightforward to compute its b_{2k} coefficients. Finally, all is prepared for the discussion of supergravities (subsect. 4.4). We start with the maximal one, SG^1_{11}, which appears to be free from $L^{11}, \ldots, L^5$ one-loop divergences, i.e. has $b_{2k}=0$, $k \leqslant 3$. Though the algorithm for b_8 is not presently available (cf. [30]) it seems very probable that $b_8(SG^1_{11}) \neq 0$, implying the absence of one-loop finiteness of this theory (and thus invalidating the conjecture about the on-shell finiteness of SG^1_{11} made in [5, 6]). These results for b_{2k} are in agreement with *superstring* counting rules (1.1) and also with one-loop four-particle amplitude calculations for maximal SG_ds [17] ([18]), which indicate the presence of UV infinities for $d \geqslant 8$*. At the same time, we find *one-loop finiteness of maximal supergravities* in $d \leqslant 7$ dimensions. To appreciate the non-triviality of this result (based on cancellations and not merely on the non-existence of possible on-shell invariants) one is to recall the absence of finiteness of pure gravity in $d>4$. For example, the SG^4_6 theory, corresponding from the 4-dim point of view to SG^8_4 plus the infinity of its "massive copies", appears to be one-loop finite and thus may *serve as a basis for (at least at one loop) consistent complete Kaluza-Klein theory*. Returning to four dimensions, we recognize the $b_4=0$ property of SG^1_{11} as the vanishing of conformal anomalies (or topological infinities) in the version of SG^8_4, obtained by dimensional reduction (cf. [31–33]). Finally, we prove that $b_{2k}=0$, $k \leqslant 3$ is valid also for SG^1_{10} but raise doubt about the suggestion [22] (see also [16]) that $d=4$ reduction of this *non-maximal* supergravity (with $d=4$ reducible supersymmetry) may be (at least) one-loop finite.

Some speculations and concluding remarks are gathered in sect. 5. Appendix A gives our notations and some useful identities while appendix B contains γ-matrix relations in d dimensions.

* Note that b_8 gives a logarithmic counter-term in $d=8$.

2. Correspondence of divergences in d-dimensional and dimensionally reduced theories

Let us start with the one-loop approximation which can generally be represented as

$$Z=\int \mathrm{d}\phi\, \mathrm{e}^{-(1/2)\phi\Delta_2\phi}, \qquad \Gamma=\tfrac{1}{2}\log\det\Delta_2, \tag{2.1}$$

$$\Delta_2=-\frac{1}{\sqrt{g}}\mathcal{D}_M g^{MN}\sqrt{g}\,\mathcal{D}_N+X, \tag{2.2}$$

where $\mathcal{D}_M=\partial_M+A_M$ and $M, N=1,\dots,d$ (for notation see appendix A). Using the well-known expansion [9, 34, 30]

$$(\mathrm{tr}\,\mathrm{e}^{-s\Delta_2})_{s\to 0}\simeq\sum_{p=0}^{d}s^{(p-d)/2}\mathcal{Q}_p, \tag{2.3}$$

$$\mathcal{Q}_p=B_p+C_p, \qquad B_p=\int_M b_p\sqrt{g}\,\mathrm{d}^d x, \qquad B_{2k+1}=0,$$

$$C_p=\int_{\partial M}c_p\sqrt{\gamma}\,\mathrm{d}^{d-1}x, \tag{2.4}$$

we get the infinite part of the effective action

$$\Gamma_\infty=-\tfrac{1}{2}\int_\varepsilon^\infty\frac{\mathrm{d}s}{s}\mathrm{tr}\,\mathrm{e}^{-s\Delta_2}=\sum_{p=0}^{d}\frac{\varepsilon^{(p-d)/2}}{p-d}\mathcal{Q}_p. \tag{2.5}$$

More explicitly,

$$\Gamma_\infty=-\left(\frac{1}{d}L^d\mathcal{Q}_0+\cdots+\frac{1}{d-p}L^{d-p}\mathcal{Q}_p+\cdots+\tfrac{1}{2}\mathcal{Q}_d\log\frac{L^2}{\mu^2}\right), \tag{2.6}$$

where $L=\varepsilon^{-1/2}\to\infty$ is the "proper time" cut-off. This formula is valid for odd as well as for even dimension d. If $d=2k+1$ we conclude that there are no *volume* logarithmic infinities. However, this *does not* imply (even if $\partial M=0$, i.e. $\mathcal{Q}_{2k+1}=0$) the one-loop finiteness of any theory in M^{2k+1} because of possible "power-type" divergences. Accounting for these (L^p) terms (in any appropriate regularization) appears necessary for establishing the relation of counter-terms in d-dimensional and reduced theories.

In this paper we shall neglect surface infinities, assuming $\partial M^d = 0$. As for the volume terms in (2.4), only the following four coefficients are presently explicitly known [30]:

$$\bar{b}_p \equiv (4\pi)^{d/2} b_p, \qquad \bar{b}_0 = \operatorname{tr} \mathbb{1},$$

$$\bar{b}_2 = \operatorname{tr}\left(\mathbb{1} \cdot \tfrac{1}{6} R - X\right), \tag{2.7}$$

$$\bar{b}_4 = \operatorname{tr}\Big[\mathbb{1}\left(\tfrac{1}{180} R^2_{MNPQ} - \tfrac{1}{180} R^2_{MN} + \tfrac{1}{72} R^2 + \tfrac{1}{30}\mathcal{D}^2 R\right)$$

$$+ \tfrac{1}{12} F^2_{MN} + \tfrac{1}{2} X^2 - \tfrac{1}{6} R X - \tfrac{1}{6}\mathcal{D}^2 X\Big], \tag{2.8}$$

$$\bar{b}_6 = \operatorname{tr}\Big\{\mathbb{1} \cdot \Big[\tfrac{1}{15120} R^{MN}_{PQ} R^{PQ}_{KS} R^{KS}_{MN}$$

$$+ \tfrac{1}{3240}\left(R^{MN}_{PQ} R^{PQ}_{KS} R^{KS}_{MN} - 2 R^{MN}_{PQ} R^{QS}_{NK} R^{KM}_{SP}\right)\Big]$$

$$- \tfrac{1}{120}\left(\mathcal{D}_M F_{NK}\right)^2 - \tfrac{1}{45} F_{MN} F_{NK} F_{KM} + \tfrac{1}{72} R_{MNKP} F_{MN} F_{KP}$$

$$+ \tfrac{1}{12} X \mathcal{D}^2 X - \tfrac{1}{6} X^3 - \tfrac{1}{12} X F^2_{MN} - \tfrac{1}{180} X R^2_{MNKP}\Big\}, \tag{2.9}$$

where in (2.9) we assumed $R_{MN} = 0$, omitted total derivative terms and used the relations of appendix A (for complete expression see [30]). Thus (2.7)–(2.9) give the algorithm to compute one-loop L^d, L^{d-2}, L^{d-4} and L^{d-6} divergences in any d-dimensional theory.

Now let us consider a dimensionally reduced theory, assuming the simplest reduction when $M^d = M^n \times S^1 \times \cdots \times S^1$ and only "zero modes" in "internal" coordinates are retained, i.e.

$$\partial_i \phi = 0, \qquad i = 1, \ldots, d-n, \tag{2.10}$$

is assumed in the classical action and in the path integral. This condition implies the breaking of the d-dimensional general covariance group to the product of an n-dimensional one and $[U(1)]^{d-n}_{\text{local}} \times SL(d-n, R)_{\text{global}}$ (and also $O(d) \to O(n) \times [U(1)]^{d-n}$), see e.g. [3]. That is why we may use the standard Kaluza-Klein parametrization of g_{MN} in terms of the n-dimensional metric $g_{\mu\nu}$, vectors and scalars

$$g^{MN}\sqrt{g_{(d)}} = \left[\begin{array}{c|c} g^{\mu\nu}\sqrt{g} & -B^{i\nu}\sqrt{g} \\ \hline -B^{j\mu}\sqrt{g} & \left(\varphi^{ij} + g^{\mu\nu} B^i_\mu B^j_\nu\right)\sqrt{g} \end{array}\right] \tag{2.11}$$

(here we used the rescaled metric: $g_{\mu\nu}=\lambda^{1/n-2}\bar{g}_{\mu\nu}$, $\lambda=\det\bar{\varphi}_{ij}$,

$$g_{MN}=\left[\begin{array}{c|c}\bar{g}_{\mu\nu}+\bar{\varphi}_{ij}B^i_\mu B^j_\nu & \bar{\varphi}_{ij}B^i_\mu\\ \hline \bar{\varphi}_{ij}B^j_\nu & \bar{\varphi}_{ij}\end{array}\right],\qquad \varphi_{ij}=\lambda^{1/n-2}\bar{\varphi}_{ij},$$

$g=\det g_{\mu\nu}$). The classical action in (2.1) now takes the form $\frac{1}{2}(\int d^{d-n}x)\int d^n x\sqrt{g}\,\phi\tilde{\Delta}_2\phi$, with the analog of Δ_2 (2.2) being

$$\tilde{\Delta}_2=-\frac{1}{\sqrt{g}}\tilde{\mathfrak{D}}_\mu g^{\mu\nu}\sqrt{g}\,\tilde{\mathfrak{D}}_\nu+\tilde{X}, \tag{2.12}$$

$$\tilde{\mathfrak{D}}_\mu=\partial_\mu+\tilde{A}_\mu,\qquad \tilde{A}_\mu=A_\mu-B^i_\mu A_i,$$

$$\tilde{X}=\lambda^{-1/n-2}X-\varphi^{ij}A_iA_j, \tag{2.13}$$

(we put $A_M=(A_\mu,A_i)$ and assume that $\partial_i X=0$, $\partial_i g_{MN}=0$). As a result, the divergences of the (first reduced and then quantized) theory are given by (2.5)–(2.9) with $d\to n$, $g_{MN}\to g_{\mu\nu}$, etc.

It is easy to understand that the correspondence to d-dimensional divergences is established by comparing $\mathfrak{A}_p$'s *for equal* p*. Thus $L^k(\log L^2)$ infinities in n dimensions are counterparts for $L^{d-n+k}(L^{d-n})$ in d dimensions. Comparing $b_p(\Delta_2)$ and $b_p(\tilde{\Delta}_2)$ for (2.2) and (2.12) we conclude that in general they do not coincide. It is important to observe that in view of the explicit breaking of d-dimensional symmetry by the reduction (2.10), the counter-terms (i.e. the b_p) *in reduced theory cannot be written only in terms of d-dim covariant objects*. Hence the analysis of *only* d-covariant counter-terms in d-dimensional theory *is not sufficient* for obtaining information about counter-terms of reduced theory (cf. [22]). Nevertheless, it is possible to prove the equality of $\bar{b}_p$ in d and n dimensions under some special choice of background fields. Namely, the following "*lemma*" is true: if

$$A_M=\{A_\mu,0\},\qquad g_{MN}=\left[\begin{array}{c|c}g_{\mu\nu} & 0\\ \hline 0 & \delta_{ij}\end{array}\right], \tag{2.14}$$

then $\bar{b}_p(\Delta_2)=\bar{b}_p(\tilde{\Delta}_2)$. Thus, in order to compute the counter-terms of the reduced theory in the case of (2.14), one may first calculate the corresponding ones in d dimensions and *then* substitute (2.14). This non-trivial observation is based on *universality* (no explicit dependence on d, [cf. (2.7)–(2.9)]) of the $\bar{b}_p$ coefficients [30]. It provides essential simplifications in calculations when quantum fields have d-dimensional indices: we need not do reduction for quantum fields (and therefore

* The $(d-n)$ volume can always be trivially factorized if divergences are local.

may use natural d-dimensional gauges, etc.). It should be understood that this lemma is valid *only* for the simplest reduction (2.10), which does not involve indices of fields (and thus "conserves" the number of degrees of freedom). Different reductions (for example, the coset space one [10,11,14]) may lead to quite different quantum theories with counter-terms having no natural connection with d-dimensional ones.

Now let us make several remarks on a possible higher loop generalization of the above discussion. The most adequate framework is again the background field method and coordinate space heat kernel technique (for recent progress see [35,36]). With one-loop experience in mind, we need a (gauge-invariant) regularization preserving power-type divergences. A natural candidate is a generalization of the "proper-time" one in (2.5): given a diagram with k internal lines we may represent all k propagators as $G(x_1, x_2) = \int_\varepsilon^\infty \mathrm{d}s \langle x_1 | \mathrm{e}^{-s\Delta_2} | x_2 \rangle$, thus finally obtaining a gauge-invariant expression like (cf. [36]) $\int_\varepsilon^\infty \ldots \int_\varepsilon^\infty \mathrm{d}s_1 \ldots \mathrm{d}s_k \, J(s_1, \ldots, s_k/g, A)$, where g, A are background fields and $L = \varepsilon^{-1/2} \to \infty$. This procedure can be straightforwardly implemented for the calculation of counter-terms (at least at the two-loop level) by generalizing various earlier background field method results [35–37] on the d-dimensional case. Another appropriate regularization is a modification of the standard dimensional one. The main idea is to consider d as a parameter taking any integer value $\bar{d} = 0, 1, 2, \ldots, d$ and to sum all corresponding infinities in ordinary dimensional regularization, as if applied in $0, 1, \ldots, d$ dimensions. For example, $\mathcal{Q}_p$ in (2.6) can be considered as a (logarithmic) counter-term in dimensional regularization for $\bar{d} \to p$, because $([1/(\bar{d}-p)] L^{\bar{d}-p})_{\bar{d} \to p} \to \frac{1}{2} \log L^2$ (cf. [20,21]). Thus to N-loop order we shall have*

$$\Gamma_\infty = \sum_{l=0}^{N} \hbar^l \sum_{k=1}^{l} \sum_{p=0}^{d} \left[\frac{a_k^{(l,d)}}{(\bar{d}-p)^k} \right]_{\bar{d} \to p} .$$

The analogous expression with an explicit (dimension-1) cut-off L reads

$$\Gamma_\infty = \sum_{l=0}^{N} \hbar^l \sum_{k=1}^{l} \left(\sum_{p=0}^{d-1} L^{d-p} \cdot \mathcal{Q}_p^{(l,k)} + \log(L^2/\mu^2) \mathcal{Q}_d^{(l,k)} \right)^k , \tag{2.15}$$

where the $\mathcal{Q}$s depend on background fields and no renormalization was carried out (except neglecting non-local parts in $\mathcal{Q}$s). As a result, simply on dimensional grounds, any possible relation of infinities in d- and n-dimensional theories must

* This should be compared with the ordinary $d = 4$ dimensional regularization result

$$\Gamma_\infty = \sum_{l=0}^{N} \hbar^l \sum_{k=1}^{l} \left[\frac{a_k^{(l)}}{(\bar{d}-4)} \right]_{\bar{d} \to 4} .$$

have the form

$$\left\{ L^{d-n+k} \sum_{r<s} \alpha_r (\log L^2)^r \right\}_d \to \{ L^k (\log L^2)^s \}_n, \qquad k = 0, \dots, N \cdot d. \quad (2.16)$$

However, it seems difficult to establish any relation between $\mathcal{Q}$'s without information about interaction terms in the action and structure of indices of quantized fields ϕ.

In the above analysis we assumed the same cut-off L for all dimensions in d-dimensional theory, which was then supposed to be related to the cut-off in reduced theory. It is possible to trace the emergence of reduced theory infinities by employing different cut-offs for n and $d-n$ momenta ($|p_\mu| \leqslant L_n$, $\mu = 1, \dots, n$, $|p_i| \leqslant L_{d-n}$, $i = 1, \dots, d-n$). Then the structure of divergences is given by (2.15) under the substitution $L^p \to \sum_{k=0}^{p} C_{k,p} (L_n)^k (L_{d-n})^{p-k}$, $\log L^2 \to \log L_n^2 \times \log L_{d-n}^2 + \cdots$, etc. Imposing the reduction condition (2.10) on the d-dim path integral we get $\prod_i \delta^{(d-n)}(p_i)$ factors in all loop integrations. As a consequence, the infinities of reduced theory as viewed from d dimensions, will take the form

$$\Gamma_\infty^{(r)} \sim \hbar (L_{d-n})^{d-n} \times \left[(L_n)^n + \cdots + \log L_n^2 \right] + \hbar^2 (L_{d-n})^{2(d-n)}$$
$$\times \left\{ \left[(L_n)^n + \cdots + \log L_n^2 \right]^2 + (L_n)^n + \cdots + \log (L_n)^2 \right\} + \cdots .$$

Finally, we are to absorb all L_{d-n} factors by the wave function and (or) coupling constant redefinitions. This remark provides explicit illustration of correspondence (2.16) in the case of regularization, breaking initial d-dim symmetry. It seems, however, that no constructive algorithm, relating d-dim and n-dim counter-terms, exists (beyond one- or at least two-loop level) if we start with a quantum theory having *unbroken* d-dimensional symmetry.

3. Super Yang-Mills: results for $d = 4$ as they follow from $d = 10$

3.1. ONE-LOOP INFINITIES

As is well-known [1], the lagrangian of $N = 4$, $d = 4$ super Yang-Mills theory [containing 1 vector, 6 (pseudo)scalars, and 4 Majorana spinors, all in the adjoint representation of some gauge group G] can be obtained by simple reduction (2.10) from that of SYM_{10}^1 in ten dimensions (we omit possible (?) auxiliary fields):

$$\mathcal{L}_{10} = \frac{1}{4 g_{(10)}^2} (F_{MN}^a)^2 + i \bar{\psi}^a \hat{\mathcal{D}} \psi^a, \qquad (3.1)$$

where

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + f^{abc} A^b_M A^c_N,$$

$$M, N = 1, \ldots, 10, \qquad a, b = 1, \ldots, \dim G,$$

$\hat{\mathcal{D}} = \gamma_M \mathcal{D}_M$, γ_M are 32×32 Dirac matrices (see appendix B) and ψ is a Majorana-Weyl spinor. After the reduction $A_M = \{A_\mu, A_i\}$ the first term in (3.1) takes the form

$$\mathcal{L}_4^{(\mathrm{Bose})} = \frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{2g^2}\left(\mathcal{D}_\mu A^a_i\right)^2 + \frac{1}{4g^2}\left(F^a_{ij}\right)^2,$$

$$F^a_{ij} = f^{abc} A^b_i A^c_j. \tag{3.2}$$

It is straightforward to quantize (3.1) in the one-loop approximation assuming that only A_M has a non-trivial background:

$$Z = \frac{\det \Delta_0}{\sqrt{\det \Delta_1}} \times \left[\sqrt{\det \Delta_{1/2}}\right]^{1/4}, \tag{3.3}$$

$$\Delta_0 = -\mathcal{D}^2, \qquad \Delta_{1MN} = -g_{MN}\mathcal{D}^2 - 2F_{MN},$$

$$\Delta_{1/2} = -\left(\hat{\mathcal{D}}\right)^2 = -\mathcal{D}^2 - \tfrac{1}{2}\gamma_{MN}F_{MN}. \tag{3.4}$$

We have chosen the background gauge $\mathcal{D}_M A^{(q)}_M = \xi(x)$, used matrix notations (A.2) and taken into account the Majorana-Weyl constraint* on ψ. It is now easy to calculate the L^{d-2k}, $k \leqslant 3$ infinities in d dimensions using (2.6)–(2.9). Introducing the notations

$$\bar{b}_0 = \beta_0, \qquad \bar{b}_4 = \beta_1 \tfrac{1}{12} \operatorname{tr} F^2_{MN},$$

$$\bar{b}_6 = -\beta_2 \cdot \tfrac{1}{60} \operatorname{tr}\left(\mathcal{D}_M F_{MN}\right)^2 - \beta_3 \tfrac{1}{72} \operatorname{tr}\left(F_{MN}F_{NK}F_{KM}\right), \tag{3.5}$$

and using (A.3), (A.4), (B.9), we get $b_2 = 0$ and ($\nu = 2^{[d/2]}$)

	β_0	β_1	β_2	β_3
Δ_0	1	1	1	1
Δ_1	d	$d-24$	$d-40$	d
$\Delta_{1/2}$	$-\nu$	-2ν	-4ν	$-\nu$

(3.6)

* Note that for a Majorana spinor $\int \mathrm{d}\psi\, \mathrm{e}^{\bar{\psi}\hat{\mathcal{D}}\psi} = [\sqrt{\det \hat{\mathcal{D}}^2}\,]^{1/2}$; note also that our gauge contains quantum scalars after the reduction: $\mathcal{D}_\mu A^{(q)}_\mu + [A^{(c)}_i, A^{(q)}_i] = \xi(x)$.

Let us now establish total results for a d-dimensional system of one gauge vector, N_0 scalars and $N_{1/2}$ spinors (all in the adjoint representation):

$$\bar{b}_p=\bar{b}_p^{(1)}+N_0\bar{b}_p(\Delta_0)-\frac{1}{\gamma}N_{1/2}\bar{b}_p(\Delta_{1/2}),\qquad \bar{b}_p^{(1)}\equiv\bar{b}_p(\Delta_1)-2\bar{b}_p(\Delta_0),\quad (3.7)$$

where $\gamma=1,2,4$ for Dirac, Majorana and Majorana-Weyl spinors, respectively. Imposing the condition

$$\bar{b}_0=\bar{b}_2=\bar{b}_4=\bar{b}_6=0,\qquad (3.8)$$

we get three equations:

$$\beta_0=\beta_3=(d-2)+N_0-(\nu/\gamma)N_{1/2}=0,$$

$$\beta_1=(d-26)+N_0+2(\nu/\gamma)N_{1/2}=0,$$

$$\beta_2=(d-\tfrac{1}{2})+N_0+4(\nu/\gamma)N_{1/2}=0,\qquad (3.9)$$

with a unique solution $d+N_0=10$, $(\nu/\gamma)N_{1/2}=8$, corresponding to SYM_{10}^1 ($N_0=0$, $d=10$, $N_{1/2}=1$, $\gamma=4$) and all its reductions. Thus the condition of finiteness in $d=4$ uniquely fixes SYM_4^4. Eq. (3.8) implies the absence of leading $L^{10},\dots,L^4$ one-loop divergences in ten dimensions. Applying the lemma (2.14), we now understand the known one-loop finiteness* of SYM_4^4 as a consequence of some property ($b_4=0$) of SYM_{10}^1. Moreover, we get $b_6(\mathrm{SYM}_4^4)=0$, which is connected with the vanishing of one-loop correction in the 3-point function in SYM_4^4 (in a proper supersymmetric background gauge [38]).

Eq. (3.8) also holds for another possible reduction, SYM_6^2, implying its off-shell one-loop finiteness in six dimensions [cf. (2.6)]. This (power-counting non-renormalizable) theory provides an example of *off-shell* cancellations due to supersymmetry. When treated from the four-dimensional point of view, it contains SYM_4^4 as well as the infinite number of its "massive analogs". Interestingly enough, we see that the inclusion of all massive states does not disturb the one-loop finiteness of SYM_4^4**.

One more observation following from (3.6), (3.9) is the *absence of L^{22} divergences in pure Yang-Mills theory in $d=26$ ($b_4=0$)*, establishing, through the lemma, the vanishing of the one-loop gauge coupling β-function in the corresponding $d=4$ reduced theory (which contains YM_4+22 scalars in the adjoint representation). This "zero" is of non-supersymmetric nature and can be understood by exploiting

* Here we get finiteness in the gauge field sector, which, however, is sufficient to conclude about finiteness for a general background in view of irreducible supersymmetry.

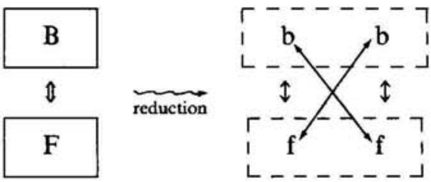
** As a by-product of (3.6) and (3.9) we get $b_6=0$ on shell ($\mathcal{D}_M F_{MN}=0$) in *any* supersymmetric theory [notice that β_3 in (3.6) is equal to the number of degrees of freedom β_0]. This does not, however, mean its on-shell finiteness in $d=6$ because of possible quadratic divergences (if $b_4\neq 0$).

the connection with the open sector of the Bose string model (see e.g. [16] and references therein). Namely, YM_{26} coincides with the $\alpha' \to 0$ limit of this model in $d = 26$ [39, 16]. By this statement we express a possibility of describing tree string amplitudes by the gauge field "effective" lagrangian ($a_i = \text{const}$)

$$\mathcal{L} = \operatorname{tr}\Big\{ a_1 F_{MN}^2 + \alpha'\Big[a_2 (\mathcal{D}_M F_{MN})^2 + a_3 F_{MN} F_{NK} F_{KM} \Big] + O(\alpha'^2) \Big\} . \quad (3.10)$$

Quantizing this theory and utilizing the fact that all non-trivial divergences of *one-loop* open string amplitudes can be absorbed solely by α' renormalization, we conclude that a_1 (and also a_2, a_3) must be *finite* to one-loop order. However, it seems impossible to generalize this reasoning to higher loops because of apparent difficulties in the quantum Bose string model (cf. [16])*. Turning to the supersymmetric open string theory, which is probably a renormalizable one [16] and has SYM_{10}^1 as its $\alpha' \to 0$ limit [1, 17], we recognize $b_4(SYM_{10}^1) = 0$ as its one-loop consequence and may also conjecture the *absence of F_{MN}^2 type infinities in SYM_{10}^1 to any loop order*. It is this latter property (and *not* that of some class of logarithmic counter-terms in $d = 10$, cf. [22]) that implies the vanishing of $F_{\mu\nu}^2$ infinities (and *thus zero β-function*) in the reduced SYM_4^4 theory [cf. (2.16) and discussion in sect. 2].

The examples of YM_{26} and SYM_{10}^1 are useful to illustrate the following general observation concerning the interplay of dimensional reduction and supersymmetry. While supersymmetry "glues" together all sectors of the theory, different fields initially related by d-dim symmetry are completely independent after reduction. It is only in *supersymmetric* dimensionally reduced theory that *all* fields are mutually connected. This can be expressed by the following diagram (we assume that supersymmetry is irreducible)



(3.11)

where arrows stand for supersymmetry, relating Bose and Fermi fields and hence Bose (and Fermi) fields among themselves. As a consequence, no initial relations (e.g. between couplings) following from the d-dim lagrangian, survive quantization if it is not for supersymmetry. For example, one can readily check that *there is* a one-loop renormalization of the scalar potential in the reduced analog of YM_{26}**.

* This is in agreement with a non-zero value of two-loop gauge field β-function in the ($YM_4 + 22$ adjoint scalars) theory (see e.g. [40]).

** O(26), which connected A_μ and A_i, is broken by reduction. Quantum corrections respect only O(22) symmetry and thus induce $(A_i^2)^2$ invariant in addition to the $(F_{ij}^a)^2$ potential in (3.2).

We conclude this section with several remarks. The results (3.8) for SYM_4^4 are in agreement with the existence of $N=4$ sum rules [41, 42]: $b_{2k} \sim \sum_\lambda (-1)^{2\lambda} d(\lambda) \lambda^k = 0$, $k<4$. The absence of a $k=4$ sum rule gives a hint for $b_8 \neq 0$ to be confirmed in the next section. Next, let us note that the formal use of d-dimensional notations was already appreciated in the 3-loop calculation of the β-function in SYM_4^N in [43]. Their one-loop result $\beta_1 \sim (d-10)$ clearly coincides with ours [cf. (3.9)]. Finally, we once more want to stress that $d=10$ results (3.8) imply the corresponding ones for $d=4$ *only* for the simplest supersymmetry preserving reduction (2.10). Quantum properties of differently reduced theories (cf. [7, 8]) are to be studied independently in four dimensions and are expected to be worse than to those of SYM_4^4.

3.2. CONSTANT GAUGE FIELD EFFECTIVE LAGRANGIAN

Here we are going to consider the effective action, corresponding to (3.3), supposing the following background for $d=10$ potential ($\mu, \nu = 1, \dots, 4$):

$$A_\mu^a = -\tfrac{1}{2} F_{\mu\nu} X_\nu n^a, \qquad (n^a)^2 = 1, \qquad A_i = 0, \tag{3.12}$$

and taking, for simplicity, the gauge group to be SU(2). The one-loop effective action for a homogeneous background is given by (see e.g. [23, 44], cf. (2.5))

$$\Gamma^{(1)} = -\tfrac{1}{2}\hbar \int_0^\infty \frac{\mathrm{d}s}{s} \sum_k \mathrm{tr}\, \mathrm{e}^{-\Delta_2^{(k)} s} = -\frac{\hbar}{2} \frac{V_d}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}s}{s^{1+d/2}} \Phi(s), \tag{3.13}$$

where V_d is the d-dimensional volume and Φ depends on background fields. It is easy to understand (cf. the lemma and (2.14)) that for (3.12) Φs *are the same* in d-dimensional and reduced theories. Thus the only difference in Γs is in the s-integration measure

$$\mathcal{L}_{10}^{(1)} = -\frac{\hbar}{2(4\pi)^5} \int_0^\infty \frac{\mathrm{d}s}{s^6} \Phi(s), \qquad \mathcal{L}_4^{(1)} = -\frac{\hbar}{2(4\pi)^2} \int_0^\infty \frac{\mathrm{d}s}{s^3} \Phi(s). \tag{3.14}$$

Therefore we may use $d=10$ theory to calculate* Φ.

Let us introduce the following notations (our signature is euclidean):

$$J_1 = \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu}, \qquad J_2 = \tfrac{1}{4} F_{\mu\nu} F^*_{\mu\nu}, \qquad F^*_{\mu\nu} = \tfrac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}, \tag{3.15}$$

$$F_{1,2}^2 = J_1 \pm \sqrt{J_1^2 - J_2^2}\,. \tag{3.16}$$

Note that $J_1 = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{H}^2) \geqslant J_2 = (\boldsymbol{E}\boldsymbol{H})$ and thus the eigenvalues $\langle \pm F_1, \pm F_2 \rangle$ of $F_{\mu\nu}$

* We note in passing that Φ is gauge independent [(3.12) satisfies the classical field equations].

are real (the transition to pseudo-euclidean notations is given by $E \to iE$). The total expression for Φ follows from (3.3)

$$\Phi = \Phi_1 - \tfrac{1}{4}\Phi(\Delta_{1/2}), \qquad \Phi_1 \equiv \Phi(\Delta_1) - 2\Phi(\Delta_0). \tag{3.17}$$

After some standard calculations (cf. [23–24]) we have (sh ≡ sinh, ch ≡ cosh)

$$\Phi_0 \equiv \Phi(\Delta_0) = c\frac{sF_1}{\operatorname{sh} sF_1}\cdot\frac{sF_2}{\operatorname{sh} sF_2}, \qquad c(\mathrm{SU}(2)) = 2, \tag{3.18}$$

$$\Phi(\Delta_{1/2}) = \nu \operatorname{ch} sF_1 \cdot \operatorname{ch} sF_2 \cdot \Phi_0, \qquad \nu = 2^{[d/2]}, \tag{3.19}$$

$$\Phi_1 = (d-2)\Phi_0 + 4(\operatorname{sh}^2 sF_1 + \operatorname{sh}^2 sF_2)\Phi_0. \tag{3.20}$$

Here we used

$$\Phi_0 = \operatorname{tr}\left[\det\frac{sF}{\operatorname{sh} sF}\right]^{1/2}, \qquad \Phi(\Delta_{1/2}) = \operatorname{tr} \mathrm{e}^{(1/2)\gamma_{MN}\cdot F_{MN}\cdot s}\Phi_0,$$

$$\Phi_1 = [\operatorname{tr} \mathrm{e}^{2F_{MN}s} - 2]\Phi_0,$$

with $F_{MN} = \langle F_{\mu\nu}, 0\rangle$, and took $(\gamma_M)_{M=1,2,3,4} = (\gamma_\mu)_4 \otimes \mathbf{1}_8$. The final result reads

$$\Phi(s) = 4(\operatorname{ch} sF_1 - \operatorname{ch} sF_2)^2\Phi_0, \tag{3.21}$$

and thus, e.g., in four dimensions,

$$\mathcal{L}_4^{(1)} = -\frac{\hbar}{2}\frac{4c}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}\frac{sF_1}{\operatorname{sh} sF_1}\frac{sF_2}{\operatorname{sh} sF_2}(\operatorname{ch} sF_1 - \operatorname{ch} sF_2)^2. \tag{3.22}$$

This expression is remarkable for its ultraviolet (UV) finiteness, evident from small-s expansion of (3.21):

$$\Phi(s)|_{s\to 0} \simeq 4c\left[s^4(J_1^2 - J_2^2) - \tfrac{1}{90}s^8(J_1^2 - J_2^2)(3J_1^2 + J_2^2) + \mathrm{O}(s^{10})\right]. \tag{3.23}$$

This expansion also gives information about total b_p coefficients, calculated on the background (3.12). In view of the obvious formula [cf. (2.3), (2.5), (3.13)]

$$(\Phi(s))_{s\to 0} \simeq \sum_{k=0}^{\infty} s^k \bar{b}_{2k}, \tag{3.24}$$

and noticing that $\Phi(s)$ is even in s, we get

$$\bar{b}_p = 0, \qquad p \leqslant 6, \qquad \bar{b}_8 = \tfrac{1}{4}c\left[\left(F_{\mu\nu}F_{\mu\nu}\right)^2 - \left(F_{\mu\nu}F^*_{\mu\nu}\right)^2\right],$$
$$\bar{b}_{12} = 0, \qquad \bar{b}_{4k+2} = 0, \qquad \bar{b}_{4k} \neq 0, \qquad k \geqslant 4. \tag{3.25}$$

The $b_8 \neq 0$ result implies the quadratic divergence of the $d = 10$ effective lagrangian (3.14) [see also (2.6)]. This explicitly demonstrates that SYM^1_{10} is *not finite* in one loop. Note that the absence of logarithmic divergence of $\mathcal{L}^{(1)}_{10}$ ($b_{10} = 0$) is probably an artifact of our choice of background (3.12) [recall that (1.1) predicts one-loop logarithmic infinities in $d = 10$].

Let us now discuss several properties of our effective lagrangian (3.22). As is evident from (3.15), (3.16), it vanishes if* $F_{\mu\nu} = \pm F^*_{\mu\nu}$. This fact is a particular case of the absence of one-loop radiative corrections on the (non-abelian) self-dual gauge field background in any supersymmetric gauge theory (which is due to supersymmetry relations between non-zero eigenvalues of scalar, spinor and vector operators, cf. [45, 25])**. Next, let us comment on the $s \to \infty$ infrared (IR) behaviour of (3.22). Expanding it in powers of F we get a series of divergent terms $\sum_{k=1}^{\infty} \mu^{-4k} F^{2k+2}$, where μ is the IR cut-off (note, that the first F^4 term is related to the four-particle amplitude also found to be IR divergent [17, 18]). We conclude that supersymmetry, though providing UV finiteness, does not improve the singular IR behaviour, characteristic to all massless theories, except for the fact that it cancels logarithmic IR divergences. Given a particular constant background problem, one could, of course, consider the above power-type IR divergences as artificial ones, absent after summation of the series. This is really true for the partial scalar (3.18) and spinor (3.19) contributions in the effective action. However, the integral over s for the gauge field contribution is divergent for $s \to \infty$ if $F \neq \pm F^*$. This is a manifestation of the gauge field IR instability of the background (3.12) (cf. [24, 27]), which should not, of course, be confused with "ordinary" IR divergences. This instability originates from the "anomalous magnetic moment" term $(-2F_{MN})$ in Δ_1 in (3.4) [or the second term in (3.20)] and may be attributed to the negative mode of Δ_1. Thus we are to introduce some IR cut-off or to rotate the contour of integration ($s \to is$) for the divergent part of the integrand in (3.22). The second recipe leads to an imaginary part in the effective lagrangian, implying the decay of the "vacuum" (3.12) by gauge particle pair creation (cf. [27]). As a result, supersymmetry does not also solve the problem of IR instability of constant abelian gauge field configurations. This could be expected from the observation that it is the *total* (regularized) number of one-loop fluctuation modes $B_4 = N_+ + N_- + N_0$ which is equated to zero by supersymmetry,

* It is also invariant under duality transformation $F \to \pm F^*$, which is connected with helicity conservation in the corresponding amplitudes (cf. [25]).

** Zero modes also cancel in $N = 4$ SYM theory [45].

and thus it is still possible to have a non-zero number of negative modes. The above conclusions may be considered as arguments in favour of the analogy in IR behaviour of YM_4 and SYM_4^4, e.g., implying the confinement (in spite of zero β-function) in the second theory.

To illustrate the discussion let us now calculate (3.22) explicitly, assuming, e.g., that $\boldsymbol{E} \neq 0$, $\boldsymbol{H} = 0$. Then the total one-loop corrected lagrangian is

$$\mathcal{L}_4 = \frac{1}{2g^2} E^2 - \frac{\hbar a}{4\pi^2} E^2 \equiv \frac{1}{2g_{\rm eff}^2} E^2, \tag{3.26}$$

where

$$a = I_1 + 2I_0 - 2I_{1/2}, \qquad I_{1,0,1/2} = \int_0^\infty \frac{\mathrm{d}x}{x^2}\left\{ \operatorname{sh} x;\, \frac{1}{\operatorname{sh} x};\, \operatorname{cth} x \right\},$$

where I_1 is to be defined by $x \to ix$. In view of the UV finiteness of (3.22) there are no usual $E^2 \log E$ terms in (3.26) and thus there is no regularization ambiguity in the constant a. Calculation of the integrals yields (cf. [46])

$$\frac{1}{g_{\rm eff}^2} = \frac{1}{g^2} + \frac{\hbar}{8\pi^2}\left[24\zeta'(-1) - 8\zeta(-1)\ln 2 + i\pi\right]. \tag{3.27}$$

Note that if we had employed the IR cut-off by inserting an $e^{-\mu^2 s}$ factor in (3.22), then the imaginary part of (3.27) would appear through $\lim_{\mu\to 0} E^2 \log[(\mu^2 - E)/(\mu^2 + E)]$. It should, of course, be understood that the simple structure of (3.26) is due to the condition $H = 0$, while in general (3.22) will non-trivially depend on the dimensionless combination E/H.

3.3. EFFECTIVE POTENTIAL FOR SCALARS

Our aim in this subsection is to consider the effective action for a constant scalar background

$$A_i = \text{const}, \qquad A_\mu = 0, \tag{3.28}$$

illustrating the efficiency of the $d = 10$ approach and pointing out some general facts about (super)symmetry breaking in SYM_4^4 theory. According to (3.2) the *classical* scalar potential and its absolute minima are given by

$$V_0 = \frac{1}{4g^2} F_{ij}^a F_{ij}^a, \qquad F_{ij} = \left[A_i, A_j\right] = 0 \tag{3.29}$$

(we assume the gauge group G to be compact and six internal dimensions to be space-like so that all scalars are physical and $V_0 \geqslant 0)^\star$. Note that in our case $F_{ij}=0$ does not imply "triviality" of A_i because after reduction A_i's transform homogeneously under the gauge transformations. All vacua (3.29) do not break supersymmetry ($V_0=0$)** but may spontaneously break the gauge symmetry, giving masses to some fields, grouped in $N=4$ multiplets. For example, if we take $A_i=n_iA$, $n_i^2=1$, then for G = SU(5) and a suitable matrix A we get the standard SU(3) × SU(2) × U(1) symmetry breaking as (with the help of one adjoint Higgs multiplet) in the Georgi-Glashow model [48]. The mass matrix, corresponding to $F_{ij}=0$, is the same for scalars, vectors and spinors:

$$M^{2ab}=f^{amk}f^{bck}A_i^mA_i^k. \tag{3.30}$$

One can show that the *minimal* number of *massless* $N=4$ multiplets which survive the tree level gauge symmetry breaking is equal to the rank r of G ($r=n$ for SU($n+1$), SO($2n$), E_n, etc.). Therefore, the potential (3.29) admits a "realistic" *gauge* symmetry breaking but still there are the problems of spin degeneracy (unbroken supersymmetry), of energy degeneracy ($V_0=0$) and also of scale degeneracy of vacua ($F_{ij}=0$ does not fix the scale of A_i). As is known for $N=1$ supersymmetric theories, the first problem cannot be dynamically solved within the loop expansion [49, 50], the second one may probably be solved by inclusion of supergravity couplings (see e.g. [51]), while the third difficulty may be cured by the effect of dimensional transmutation (cf. [52]). It appears that none of these problems can be resolved (at least within perturbation theory) in UV finite and thus scale invariant pure SYM_4^4 theory, and thus they are confronted also at the level of quantum effective potential. One may hope that coupling SYM_4^4 to (conformal $N=4$) supergravity can improve the situation. This coupling may probably be mimicked by introducing scalar (and spinor) mass terms and thus softly breaking supersymmetry at the tree level. We shall discuss the effect of soft breaking on the example of one-loop effective potential below.

Turning to the evaluation of the quantum effective potential let us first note that from SYM_{10}^1 point of view, the choice of gauge background (3.28) implies the breaking of $d=10$ Lorentz symmetry O(10) → O(4) × K, K ⊆ O(6) and in this respect plays a role analogous to (and is well suited for) dimensional reduction. The SYM_{10}^1 effective action calculation for (3.28) is formally analogous to that in YM_4 theory for a constant non-abelian gauge potential background [28]. However, some

* The classical equations, corresponding to V_0, $[A_i,[A_i,A_j]]=0$, coincide with the Yang-Mills equations in $(d-4)$-dimensional gauge theory in the case of constant gauge potentials. They can have "non-trivial" ($F_{ij}\neq 0$) real solutions only if G is non-compact and (or) the $(d-4)$-space signature is pseudo-euclidean (cf. [47]).

** In view of the form of supersymmetry transformations $\delta\psi=-(1/4\sqrt{2})F_{MN}\gamma_{MN}\varepsilon$ it is obvious that $F_{ij}\neq 0$ is a necessary condition for supersymmetry breaking (for the appearance of a Goldstone fermion).

care is needed to extract the SYM_4^4 result we are interested in. Namely, for (3.28) in $d=10$ we have $\mathfrak{D}_{10}^2 = \Box_{10} + 2A_i\partial_i + A_i^2$, while one is to drop all ∂_i terms to make contact with the reduced theory. As a consequence, the relation of exact $d=10$ and $d=4$ effective actions will be more complicated then in (3.14). With this clarification in mind, we can start with the $d=10$ expression (3.3) but omit all ∂_i terms in the operators (3.4), which therefore take the form

$$\Delta_0 = -\Box + M^2, \qquad (M^2)^{ab} = f^{aec}f^{bed}A_i^cA_i^d,$$

$$\Delta_{1/2} = -\Box + M_{1/2}^2, \qquad \left(M_{1/2}^2\right)^{ab}_{\alpha\beta} = \delta_{\alpha\beta}M^{2ab} - \tfrac{1}{2}(\gamma_{ij})_{\alpha\beta}F_{ij}^{ab}, \qquad (3.31)$$

$$\Delta_{1MN} = \left\{\Delta_{1\mu\nu} = \delta_{\mu\nu}\Delta_0;\ \Delta_{1i\mu} = 0;\ \Delta_{1ij} = -\delta_{ij}\Box + M_{0ij}^2\right\},$$

$$\left(M_0^2\right)^{ab}_{ij} = \delta_{ij}M^{2ab} - 2F_{ij}^{ab}, \qquad (3.32)$$

where $\gamma_i = \delta_{iM}\gamma_M$. The one-loop effective lagrangian is given by (3.14) where now

$$\Phi(s) = \mathrm{tr}\left(\mathrm{e}^{-M_0^2 s} + 2\mathrm{e}^{-M^2 s} - \tfrac{1}{4}\mathrm{e}^{-M_{1/2}^2 s}\right) \qquad (3.33)$$

(the trace goes over a, b; i, j; α, β). The formal integration yields

$$V_1 \equiv \mathfrak{L}_4^{(1)} = \frac{\hbar}{64\pi^2}\mathrm{tr}\left(M_0^4\log M_0^2 + 2M^4\log M^2 - \tfrac{1}{4}M_{1/2}^4\log M_{1/2}^2\right). \qquad (3.34)$$

Here we made use of

$$\int_{L^{-2}}^{\infty}\frac{\mathrm{d}s}{s^3}\mathrm{e}^{-M^2 s} = \tfrac{1}{2}L^4 - M^2L^2 - \tfrac{1}{2}M^4\left(\gamma_0 - \tfrac{3}{2}\right) - \tfrac{1}{2}M^4\log\left(M^2/L^2\right), \qquad \gamma_0 = 0.57\ldots$$

and the fact of cancellation of all infinities proved in subsect. 3.1 [see (3.38)] which here appears to be a consequence of the following $N=4$ "mass sum rules" (cf. [42, 53]), obvious from (3.31), (3.32) [see also (3.33), (3.24)]*

$$\bar{b}_{2k} = \frac{(-1)^k}{k!}\mathrm{tr}\left(M_0^{2k} + 2M^{2k} - \tfrac{1}{4}M_{1/2}^{2k}\right) = 0, \qquad k < 4. \qquad (3.35)$$

Let us now discuss several properties of the effective potential (3.34). (a) It is *scale invariant*, i.e. can be written in terms of logarithms of *dimensionless* variables**. For

* Note that, as in (3.25), $b_8 \neq 0$, again implying the presence of a quadratic divergence in $\mathfrak{L}^{(1)}(\mathrm{SYM}_{10}^1)$ for the background (3.28).

** Let us also mention in passing that the effective potential (3.34) is *gauge independent* (independent of the parameter of the covariant gauge); this can be understood, e.g., as a consequence of the absence of the gauge coupling and wave function renormalization.

example, if $A_i = m\bar{A}_i$ for dimensionless $\bar{A}_i$, then the total potential has the form

$$V = m^4\left\{\frac{1}{4g^2}\bar{F}_{ij}^2 + \frac{\hbar}{64\pi^2}\left(\bar{M}_0^4 \log \bar{M}_0^2 + \cdots\right)\right\}$$

[the proof is based on (3.35)]. Hence the potential calculated on the extrema defined by $\partial V/\partial m = 0$, $\partial V/\partial \bar{A}_i = 0$, is always *equal to zero* (and thus there is no supersymmetry breaking). This simple argument can be generalized to *all loops*, assuming that the zero β-function property [38] (and thus scale invariance) of SYM_4^4 persists to more than three loops. This connection of the absence of supersymmetry breaking and scale invariance can be seen also from the following reasoning: if $\beta = 0$ then $T^\mu_\mu = 0$ but $\langle T^{\mu\nu}\rangle_0 = Vg^{\mu\nu}$ and thus $V = 0$, which implies the absence of supersymmetry breaking (cf. [50]). Turning this argument around we may say that a non-perturbative dynamical breaking of SYM_4^4 supersymmetry (if possible at all) must be *accompanied* by a breaking of scale invariance of this theory. (b) One can easily check that the classical minima (3.19) are also the *solutions* of effective equations, i.e. $V(F_{ij} = 0) = 0$* (this is the analog of the $N = 1$ supersymmetry result, saying that if the classical potential is zero at some point then the effective potential is also zero at that point [49]). (c) Combining the observations made in (a) and (b) we conclude that *all* effective minima of $V = V_0 + V_1$ are *exhausted* by the classical ones (3.29). Really, suppose that $V(A)$ is a *regular* function of A_i^a, which is known to have two properties: (i) if $(\partial V/\partial A)_{\tilde{A}} = 0$, then $V(\tilde{A}) = 0$, and (ii) there exists A_0 such that $V(A_0) = 0$. Then a simple theorem of analysis states that $\tilde{A}$ *coincides* with A_0. This argument is valid, of course, to any order in loop expansion. Thus all possible patterns of gauge symmetry breaking are only the classical ones (with $F_{ij} = 0$) discussed above. (d) Our next remark is about the IR instability of the background (3.28), i.e. *the negative modes of* M_0^2 (for $F \neq 0$), giving the imaginary part $\sim \mathrm{tr}[\pi M_0^4 \times \theta(-M_0^2)]$ to (3.34). These negative modes can be proved to be present always when $F_{ij} \neq 0$, but the simplest way to predict their appearance is to recall the analogous instability (negative modes of the gauge field Δ_1 operator) of the constant non-abelian gauge potential background in YM_4 [27, 28]. Indeed, the scalar operator in (3.32) appears as a part of the $d = 10$ gauge field one and thus the origin of instability is the same: the "anomalous magnetic moment" term "$-2F_{MN}$" (see also subsect. 3.2). This instability in our case is, however, a pure off-shell effect, because we already know that $F_{ij} = 0$ for all solutions of effective equations.

Let us now suppose that we added scalar and spinor mass terms in the classical lagrangian (3.2)

$$\Delta\mathcal{L} = \frac{1}{2g^2}\mu_{0ij}^2 A_i^a A_j^a + \mu_{1/2\alpha\beta}\bar{\psi}_\alpha^a \psi_\beta^a \tag{3.36}$$

* This follows from (3.31), (3.32) and the zero total number of degrees of freedom in the theory, or from the invariance of (3.34) under $F_{ij} \to -F_{ij}$. The latter property implies that only b_{4k} coefficients ($k > 3$) are non-zero for background (3.28) [cf. (3.25)].

[here g^2 is needed in view of scalar kinetic term normalization in (3.2)]. They produce *soft* supersymmetry breaking, i.e. do not spoil finiteness of the theory in the sense that gauge coupling *β-function is still zero* and no field-dependent quadratic divergences are induced (cf. [54]). Moreover, it is possible to cancel logarithmic "mass" renormalizations by properly adjusting μ_0 and $\mu_{1/2}$. A remarkable fact is that we can now solve the previous problems of masses and vacua degeneracies and thus (in principle) construct *a realistic unified model starting with* SYM_4^{4}*. The important consequence of the presence of bare scalar mass terms is a possibility of avoiding negative modes of the scalar one-loop operator and hence of obtaining a well-defined (real) effective potential which is now not forbidden to have non-trivial effective minima. Then we are in position to study the question of dynamical gauge symmetry breaking and to search for a solution of gauge hierarchy problem, generating some well-separated effective scales, e.g. $A_i^2 \sim \mu_0^2 \exp(\pm c/g^2)$ or $A_1/A_2 \sim \exp(\pm c/g^2)$ (for a recent proposal of its solution in the framework of unified models based on $N=1$ softly broken supersymmetry [55], see ref. [56]).

It is not the aim of this paper to present such an analysis. That is why we shall consider only illustrative examples, calculating (3.34) explicitly for two simple scalar backgrounds. First of all, let us note that in order to get a finite (up to A-independent divergent constant) expression for the effective potential, one is to relate μ_0 and $\mu_{1/2}$ [cf. (3.35)]: $\Sigma_i \mu_{0ii}^2 = \frac{1}{4}\Sigma_\alpha \mu_{1/2\alpha\alpha}^2$. This condition is assumed in the following. We start with the simplest ($F_{ij}=0$) O(6)-preserving background: $A_i^a = n_i A^a$, $n_i^2 = 1$, and take $G = SU(2)$, $\mu_{0ij}^2 = \delta_{ij}\mu^2$, $\mu_{1/2\alpha\beta}^2 = \frac{3}{4}\mu^2\delta_{\alpha\beta}$. Diagonalizing (3.30) $M^2 = \{0, A^2, A^2\}$, $A^2 = A_a A_a$ we get the following expression for the one-loop effective potential (3.34):

$$V = \mu^4 \left\{ \frac{1}{2g^2} x + \frac{\hbar}{64\pi^2} \left[3(1+x)^2 \log(1+x) + x^2 \log x \right.\right.$$
$$\left.\left. - 4\left(x+\tfrac{3}{4}\right)^2 \log\left(x+\tfrac{3}{4}\right) \right] \right\} + \text{const}, \qquad (3.37)$$

where $x = A^2/\mu^2$. The $A \neq 0$ extrema of V satisfy

$$x \log \frac{x(x+1)^3}{\left(x+\frac{3}{4}\right)^4} + 3\log\frac{x+1}{x+\frac{3}{4}} = -\frac{4\pi^2}{\hbar g^2},$$

and thus are absent if $x \geqslant 0$. Taking formally $x \leqslant -1$ (i.e. $\mu^2 < 0$) it is possible to find a unique solution *if* g^2 is some suitable *large* number (which is generally

* This proposal seems superior to those of refs. [7, 8], where a coset reduction of SYM_{10}^1 was used to generate some $d=4$ theory with broken supersymmetry but with (very probably) bad quantum properties.

considered not to be a good choice). We observe stability of $A_i = 0$ classical solution, which can be related to the fact of UV finiteness of (3.37) [i.e. to the finite $x \to \infty$ limit of the quantum term in (3.37)]. Our next example (now with $F_{ij} \neq 0$) is provided by the following background ($G, \mu_0, \mu_{1/2}$ are as above):

$$A_i^a = A\left[\delta_i^a - (\xi + 1)n^a n_i\right], \qquad a = 1,2, \qquad i = 1,2, \qquad n_i^2 = 1, \qquad A, \xi = \text{const}, \tag{3.38}$$

all other components being zero. Then $F_{ij}^{ab} = \varepsilon^{ab}\varepsilon_{ij}A^2 \cdot \xi$, $M_{ab}^2 = A^2[\xi^2\delta_{ab} - (\xi^2 - 1)n_a n_b]$. The eigenvalues of mass matrices in (3.31), (3.32) (where we put $\gamma_i =$ (Pauli matrices) $\otimes \mathbb{1}_{16}$) are given by

$$M^2 = \{A^2\xi^2;\, A^2; 0\}, \qquad M_0^2 = \{4 \times (A^2\xi^2 + \mu^2), 4 \times (A^2 + \mu^2);$$

$$\times 2 \times (A^2\lambda_0^+ + \mu^2), 2 \times (A^2\lambda_0^- + \mu^2), 6 \times \mu^2\};$$

$$M_{1/2}^2 = \{32 \times (A^2\lambda_{1/2}^+ + \tfrac{3}{4}\mu^2), 32 \times (A^2\lambda_{1/2}^- + \tfrac{3}{4}\mu^2), 32 \times (\tfrac{3}{4}\mu^2)\}, \tag{3.39}$$

where

$$\lambda_0^\pm = \tfrac{1}{2}\left(\xi^2 + 1 \pm \sqrt{(\xi^2 - 1)^2 + 16\xi^2}\right), \qquad \lambda_{1/2}^+ = 1 + \xi^2, \qquad \lambda_{1/2}^- = 0.$$

It is now easy to establish the effective potential, using (3.34), (3.35),

$$V = \mu^4\left\{\frac{1}{2g^2}(x + y + xy) + \frac{\hbar}{32\pi^2}\left[2(y+1)^2\log(y+1)\right.\right.$$

$$+ 2(x+1)^2\log(x+1) + (x\lambda_0^+ + 1)^2\log(x\lambda_0^+ + 1) + (x\lambda_0^- + 1)^2\log(x\lambda_0^- + 1)$$

$$\left.\left. + y^2\log y + x^2\log x - 4(x + y + \tfrac{3}{4})^2\log(x + y + \tfrac{3}{4})\right]\right\} + \text{const}, \tag{3.40}$$

where $x = A^2/\mu^2$, $y = x\xi^2$. One may easily check that the $O(\hbar)$ part of (3.40) [as well as of (3.37)] can be rewritten in terms of logs of ratios of polynomials having equal degrees and thus is bounded when $x, y \to \infty$. The $\mu \to 0$ limit of (3.40) serves as a good illustration of the statements made above concerning the pure SYM_4^4 case. If $\mu^2 < A^2\lambda_0^-$ we get the already discussed negative modes. That is why we are to relate x and ξ properly in order to preserve the reality of (3.40). The analysis of extrema of (3.40) shows that no solutions exist for an arbitrary value of g^2 (except the classical one $x = y = 0$). At the same time it is possible to find the approximate minima if g^2

is sufficiently large, while $\xi \gg 1$, $x < \frac{1}{3}$ and $y < 1$. Thus we get dynamical SU(2) symmetry breaking and three different mass scales μ^2, $\mu^2 x$, $\mu^2 y$. However, this is not an "exponential" [$\sim \exp(c/g^2)$] hierarchy, which, in fact, seems to be *impossible* if we demand the UV finiteness of the potential. It remains to be seen whether it can be generated, if we omit the relation between μ_0 and $\mu_{1/2}$ and consider a realistic example of $G = SU(5)$ or E_8.

4. Supergravities in $d \leqslant 11$: one-loop divergences and anomalies

The subject of this section is analysis of infinities [i.e. calculation of b_p coefficients in (2.6), (2.4)] in higher dimensional supergravity theories on a d-dimensional gravitational background. These theories are known to contain antisymmetric tensor gauge fields and gravitinos along with the graviton itself. One-loop results, therefore, will be given by a sum of separate contributions of all these fields. Finally, using the lemma of sect. 2 we will be able to obtain some information concerning the corresponding reduced theories. We shall use the following notations:

$$\bar{b}_0 = N, \qquad \bar{b}_2 = \rho R, \tag{4.1}$$

$$\bar{b}_4 = \alpha_1 R^2_{MNPQ} + \alpha_2 R^2_{MN} + \alpha_3 R^2 + \alpha_4 \mathcal{D}^2 R, \tag{4.2}$$

$$\bar{b}_6 = \sigma_1 I_1 + \sigma_2 E, \tag{4.3}$$

where in (4.3) we used the definitions (A.5), (A.7), assumed that $R_{MN} = 0$ (we will compute b_6 only on mass shell) and omitted all total derivative terms [as was already done in (2.9)]. It should be understood that $R^*R^* \equiv R^2_{MNPQ} - 4R^2_{MN} + R^2$ [E in (A.7)] is the integrand of the Euler number (A.8) and thus may be neglected for topologically trivial backgrounds *only* if $d = 4$ [$d = 6$]. Let us also recall that only one (two) of the first three invariants in (4.3) are independent (while I_1 and E vanish on shell) when $d = 2$ ($d = 3$). One more remark is that $I_2 = 0$ if $d = 4$ and $R_{MN} = 0$ [57] and hence $E(d = 4) = I_1$.

4.1. GRAVITY IN d DIMENSIONS

The background field method one-loop quantization of gravity in d dimension can be done straightforwardly in the gauge $\mathcal{D}_M(h_{MN} - \frac{1}{2} g_{MN} h) = \xi_N(x)$,

$$Z = \frac{\det \Delta_g}{\sqrt{\det \Delta_h}}, \qquad \Delta_{g_{MN}} = -g_{MN}\mathcal{D}^2 - R_{MN}, \tag{4.4}$$

$$\Delta_h = -P \cdot \mathcal{D}^2 + X, \qquad P^{MN}_{PQ} = \delta^M_{(P}\delta^N_{Q)} - \tfrac{1}{2} g^{MN} g_{PQ},$$

$$X^{MN}_{PQ} = 2R_{P(MN)Q} - 2\delta^{(M}_{(P} R^{N)}_{Q)} + g_{MN} R_{PQ} + g_{PQ} R_{MN} + P^{MN}_{PQ} \cdot R. \tag{4.5}$$

We see that eqs. (4.4), (4.5) are universal in the chosen gauge, i.e. are the same as, e.g., in the $d=4$ case. Now we are to use eqs. (2.7)–(2.9) (with $(F_{MN})^{PQ}_{KS} = 2\delta^{(P}_{(K}R^{Q)}_{S)MN}$, etc.) providing the following results for $\bar{b}^{(h)}_p = \bar{b}_p(\Delta_h) - 2\bar{b}_p(\Delta_g)$ [cf. (4.1)–(4.3)]:

$$N = \tfrac{1}{2}d(d-3), \qquad \rho = \tfrac{1}{12}(-5d^2+9d-48), \tag{4.6}$$

$$\alpha_1 = \tfrac{1}{180}\frac{d(d-3)}{2} - \tfrac{1}{12}(d-18), \qquad \alpha_2 = \tfrac{1}{360}(-d^2+543d-3600), \tag{4.7}$$

$$\alpha_3 = \tfrac{1}{144}(43d^2-303d+696), \qquad \alpha_4 = \tfrac{1}{60}(-4d^2+7d-40), \tag{4.8}$$

$$\sigma_1 = \frac{1}{15120}\frac{d(d-3)}{2}, \qquad \sigma_2 = \frac{1}{3240}\frac{d(d-3)}{2} - \frac{(d+30)}{180}, \tag{4.9}$$

which are in agreement with the previously known ones for $d=4$ (b_2 [15,19]; b_4 [58,59]) and for $d=6$ (σ_1 in b_6 [60,59])*. Evidently, only N, α_1, σ_1 and σ_2 are gauge independent (they contribute on shell). Recalling the meaning of b_p [cf. (2.6)] we immediately conclude that: (i) $d=2$, 3 and 4 gravities are one-loop finite on shell (the latter up to topological divergence); (ii) all gravities in $d>4$ dimensions have at least L^{d-4} and L^{d-6}($\log L^2$ for $d=6$) divergences on shell. Our results (4.6)–(4.9) give *complete* expressions for infinities in $d \leqslant 7$ theories (according to (2.6) the weakest linear infinity in $d=2k+1$ case is governed by b_{2k}). Put in this perspective the one-loop finiteness of $d=4$ gravity appears to be an accident. Thus one should strongly believe in the "uniqueness" of $d=4$ number for higher loops [contradicting complete universality of the one-loop expressions (4.4), (4.5)] in order to start with two-loop calculations.

An interesting question connected with quantization of d-dimensional gravity is about divergences in reduced Kaluza-Klein (KK) theories, following by reduction from the classical lagrangian ($\sim R$) in d dimensions (see e.g. [9,3]). *A priori* one may hope that these theories have better quantum behaviour as compared to some general system of gravity, scalars and vectors (known to be one-loop infinite even on shell [58,61]) because the classical mass shell of reduced theory coincides with that of the d-dimensional one ($R_{MN}=0$). This conjecture is not, however, supported by the above statement on the absence of finiteness in higher dimensions**. Strictly speaking, this result cannot be immediately applied to the reduced theory (it says only that the *total* KK theory, i.e. with all massive states, is infinite). Moreover, as follows from the discussion in sect. 2, divergences of reduced theory generally cannot

* It should be pointed out that $\sigma_2(d=6)$ was not computed in refs. [60,59] because their authors assumed that the integral of E is zero [cf. (A.9)]. However, it is erroneous to omit E in the local expression for $d=6$ gravitational conformal anomaly $T^\mu_\mu = \bar{b}_6(x)$ (compare with the $d=4$ expression $T^\mu_\mu = \bar{b}_4 = \alpha_1 R^*R^* + \cdots$).

** The presence of R^2_{MNPQ} on shell infinities is a hint for e.g. F^4(B) divergences in reduced theory [for notations see (2.11)].

be written only in terms of d-dimensional objects. That is why one must do an explicit calculation (in four dimensions) to settle the question. It seems worth presenting the outcome of this calculation of divergences in the simplest reduced KK theory, following from $d=5$ gravity

$$\mathcal{L}=\left(-\frac{1}{k^2}R+\frac{1}{4g^2}F_{\mu\nu}^2(\mathrm{B})\mathrm{e}^{2a\varphi}+\tfrac{1}{2}\partial_\mu\varphi\partial_\mu\varphi\right)\sqrt{g}\,, \tag{4.10}$$

where $a=\frac{1}{2}\sqrt{3}$, $k^2=16\pi G$. Let us consider the following background: $g_{\mu\nu}\neq\delta_{\mu\nu}$, $B_\mu\neq 0$, $\varphi=0$. Then the quantum scalar field contribution in the one-loop infinities (which should be added to that of the Einstein-Maxwell system [61]) is given by

$$\Delta\bar{b}_4=\tfrac{1}{6}a^2\left(\mathcal{D}_\mu F_{\mu\nu}\right)^2-\tfrac{1}{3}a^2T_{\mu\nu}R_{\mu\nu}+4a^2T_{\mu\nu}^2$$
$$+\tfrac{1}{6}a^4\left(F_{\lambda\rho}F_{\mu\rho}\right)^2+\tfrac{7}{12}a^4\left(F_{\mu\nu}^2\right)^2,\qquad T_{\mu\nu}\equiv F_{\mu\lambda}F_{\nu\lambda}-\tfrac{1}{4}g_{\mu\nu}F^2. \tag{4.11}$$

Thus the theory is still infinite on shell. It has even worse divergence structures [the last two terms in (4.11)] because of breaking of duality invariance by the vector-scalar coupling in (4.10). The lessons we can draw from this example are the following: (i) the fact that some theory is obtained by dimensional reduction does not by itself imply an improvement of situation with infinities; (ii) the inclusion of scalar-vector couplings does not by itself imply better quantum behaviour. As a consequence, one-loop on shell finiteness of SG_4^N ($N\leqslant 8$) is due to (i) one-loop on-shell finiteness of $d=4$ gravity and (ii) *irreducible* supersymmetry, connecting all sectors of the S-matrix with the finite gravitational one (i.e. to the fact that R^2 invariants are built up to superinvariants) and should not be specially attributed to the possibility of obtaining these theories from SG_{11}^1 by a reduction (and truncation)*. This conclusion is in agreement with the observation made in subsect. 3.1, that it is supersymmetry which provides finiteness of SYM_4^4 as compared to (YM_4 + 22 scalars) theory, both theories being reductions of SYM_{10}^1 and YM_{26} correspondingly. To reiterate, even if the $(g_{\mu\nu}-B_\mu)$ sector of (4.26) was finite we would have troubles in the scalar one, because of breakdown of the initial d-dimensional symmetry by the reduction. The following conjecture suggests itself in this connection: if it is possible to construct a theory with some symmetry connecting gravity with a *finite* number N of *Bose* fields (such theories with $N=\infty$ are, of course, known, being complete KK theories as seen from lower dimensions, cf. [11]), then it may have an improved one-loop on-shell quantum behaviour characteristic to supergravities.

4.2. ANTISYMMETRIC TENSOR GAUGE FIELDS

The correct recipe for quantization of the antisymmetric tensor gauge field $A_{M_1\ldots M_n}$ ($n<d$) with the lagrangian $\mathcal{L}\sim(\mathcal{D}_{[N}A_{M_1\ldots M_n]})^2$ on a d-dimemsional gravi-

* In particular, it is supersymmetry that provides duality invariance of scalar-vector interactions in SGs.

tational background is given by (cf. [62, 29])

$$Z^{(n)} = \prod_{k=0}^{n} [\det \Delta_{n-k}]^{-(1/2)(-1)^k (k+1)}, \tag{4.12}$$

where the Hodge-DeRham operators Δ_p are defined as follows:

$$(\Delta_p)^{N_1 \dots N_p}_{M_1 \dots M_p} = -\delta^{N_1}_{[M_1} \dots \delta^{N_p}_{M_p]} \mathcal{D}^2 + \sum_{i=1}^{p} R^{N_i}_{[M_i} \delta^{N_1}_{M_1} \dots \delta^{N_p}_{M_p]}$$
$$- \sum_{j>i=1}^{p} R^{N_i N_j}_{[M_i M_j} \delta^{N_1}_{M_1} \dots \delta^{N_p}_{M_p]} \tag{4.13}$$

($\delta^{N_{i,j}}_{M_{i,j}}$ are omitted in sums). For example, $\Delta_0 = -\mathcal{D}^2$, $\Delta_{1MN} = -g_{MN}\mathcal{D}^2 + R_{MN}$, etc. To compute the infinite part of (4.12) one must first establish $b_p(\Delta_k)$ and then to find the total result, which according to (4.11), is

$$\bar{b}_p^{(n)} = \sum_{k=0}^{n} \bar{b}_p(\Delta_{n-k}) C^k_{-2}, \tag{4.14}$$

where we introduced binomial coefficients C^r_s obeying $(n, k = 1, 2, \dots,\ r, s = 0, \pm 1, \dots)$

$$C^n_r = \frac{r(r-1)\dots(r-n+1)}{n!}, \qquad C^{-n}_r = 0, \qquad C^0_r = 1, \tag{4.15}$$

$$C^n_k = C^{k-n}_k, \qquad C^{n+k}_n = 0, \qquad C^n_{r+s} = \sum_{k=0}^{n} C^{n-k}_r C^k_s, \tag{4.16}$$

$$C^k_{-1} = (-1)^k, \qquad C^k_{-2} = (-1)^k (k+1), \qquad C^{n-m}_{d-2-p} = \sum_{k=0}^{n-m} C^{n-m-k}_{d-p} C^k_{-2}. \tag{4.17}$$

Employing again (2.7)–(2.9), (4.1)–(4.3), we get

$$N = C^n_{d-2}, \qquad \rho = \tfrac{1}{6} C^n_{d-2} - C^{n-1}_{d-4}, \tag{4.18}$$

$$\alpha_1 = \tfrac{1}{180} C^n_{d-2} - \tfrac{1}{12} C^{n-1}_{d-4} + \tfrac{1}{2} C^{n-2}_{d-6}, \tag{4.19}$$

$$\alpha_2 = -\tfrac{1}{180} C^n_{d-2} + \tfrac{1}{2} C^{n-1}_{d-4} - 2 C^{n-2}_{d-6}, \tag{4.20}$$

$$\alpha_3 = \tfrac{1}{72} C^n_{d-2} - \tfrac{1}{6} C^{n-1}_{d-4} + \tfrac{1}{2} C^{n-2}_{d-6}, \qquad \alpha_4 = \tfrac{1}{30} C^n_{d-2} - \tfrac{1}{6} C^{n-1}_{d-4},$$

$$\sigma_1 = \tfrac{1}{15120} C^n_{d-2}, \qquad \sigma_2 = \tfrac{1}{3240} C^n_{d-2} - \tfrac{1}{180} C^{n-1}_{d-4} + \tfrac{1}{6} C^{n-2}_{d-6} - \tfrac{2}{3} C^{n-3}_{d-8}. \tag{4.21}$$

If $d=4$ and $n \leqslant 3$ our results for b_4 are in agreement with those of ref. [29]. To restore the analogous expressions for $b_p(\Delta_n)$ $(n \leqslant \frac{1}{2}p)$ one should simply substitute $d \to d+2$ in (4.18)–(4.21).

In view of duality invariance of Δ_n we have $\det \Delta_n = \det \Delta_{d-n}$ and $b_p(\Delta_n|d) = b_p(\Delta_{d-n}|d)$. However, in general $Z^{(n)} \neq Z^{(d-2-n)}$. Let us make a precise statement of this "*quantum inequivalence*"*

$$\bar{b}_p^{(n)}(d) = \bar{b}_p^{(d-2-n)}(d), \qquad p<d,$$

$$\bar{b}_p^{(n)}(d) - \bar{b}_p^{(d-2-n)}(d) \begin{cases} = (-1)^n (n+1-d/2) H_p(d), & p \geqslant d, \quad d = \text{even}, \\ \neq 0, \quad d = \text{odd}, \end{cases} \tag{4.22}$$

where

$$H_p(d) \equiv \sum_{k=0}^{d} (-1)^k \bar{b}_p(\Delta_k|d) \begin{cases} =0, & p<d \\ = \mathfrak{E}_d, & p=d \\ \neq 0, & p>d, \end{cases} \tag{4.23}$$

for $d=$ even and $H_p(d)=0$ for $d=$ odd. $\mathfrak{E}_d$ in (4.22) is the integrand of the Euler number (A.8). The proof of (4.22) is evident from (4.14), while (4.23) is justified in [30]. To provide understanding of these relations it is useful to consider them as consequences of the general formulas $(p=2m)$

$$\bar{b}_p(\Delta_n|d-2) = \bar{b}_p^{(n)}(d), \qquad p<d,$$

$$\bar{b}_p(\Delta_n|d) = \sum_\alpha a_{\alpha,p} I_{\alpha,p}, \qquad \bar{b}_p^{(n)}(d) = \sum_\alpha a_{\alpha,p}^{(n)} I_{\alpha,p}, \tag{4.24}$$

$$a_{\alpha,p} = \sum_{k=0}^{p/2} \gamma_{\alpha,p,k} \cdot \left(C_{d-2k}^{n-k} + C_{d-2k}^{d-n-k} \right),$$

$$a_{\alpha,p}^{(n)} = \sum_{k=0}^{p/2} \gamma_{\alpha,p,k}^{(n)} \cdot C_{d-2-2k}^{n-k}, \qquad p \gtrless d, \tag{4.25}$$

where $I_{\alpha,p}$ are "$R^{p/2}$" curvature invariants and $\gamma_{\alpha,p,k}$ are *universal* numbers (cf. (4.18)–(4.21)). Now it is clear that $n \leftrightarrow d-2-n$ equivalence in (4.22) is based on the property $C_{d-2-2k}^{n-k} = C_{d-2-2k}^{d-2-n-k}$ which holds only for $d-2-2k>0$. Therefore,

* Comparing different terms in b_ps for $d=2,3$ one is to remember that not all invariants (e.g. in b_4) are independent and thus these relations hold only for complete b_p coefficients.

the $d=p$ anomaly in (4.22), (4.23) is due to the last $(C_{d-2-p}^{n-p/2})$ term in (4.25). Let us give some explicit examples of this anomaly: if $d=4$ we get the results of ref. [29] (see also [32, 5]): $\bar{b}_4^{(3)}-0=-2\mathcal{E}_4$, $\bar{b}_4^{(2)}-\bar{b}_4^{(0)}=\mathcal{E}_4$; for $d=6$ we have: $\bar{b}_4^{(5)}-0=-3\mathcal{E}_6$, $\bar{b}_6^{(4)}-\bar{b}_6^{(0)}=\mathcal{E}_6$, $\bar{b}_6^{(3)}-\bar{b}_6^{(1)}=-\mathcal{E}_6$ [these relations can be also obtained using (4.21) (A.7), (A.9)]. Next, to illustrate (4.22), (4.23) for $p>d$ we put $d=4$, $p=6$; then $\bar{b}_6^{(3)}-0\neq 0$, $\bar{b}_6^{(2)}-\bar{b}_6^{(0)}\neq 0$, as follows, e.g., from (4.21)*.

Let us mention the following useful check of d and n dependence in (4.25). One first observes that $A_{M_1\ldots M_n}$ can be described as a system of fields in $m<d$ dimensions: $\{A_{\mu_1\ldots\mu_n}; (d-m)A_{\mu_1\ldots\mu_{n-1}};\ldots; C_{d-m}^{n}A\}$, $\mu=1,\ldots,m$, and thus can employ the formula

$$\bar{b}_p^{(n)}(d)=\sum_{k=0}^{n} C_{d-m}^{k}\bar{b}_p^{(n-k)}(m). \tag{4.26}$$

Then the consistency of (4.25) and (4.26) follows simply from the "sum rule" in (4.16). It is interesting also to note that using (4.26) one can compute $b_p^{(n)}(d)$ given $b_p^{(k)}(d)$ for all $k<n$ *and* $b_p^{(n)}(m)$ for some $m<d$. Our final remark is that this "reduction" procedure works also in the calculation of the gravitational contribution (subsect. 4.1). Indeed, $g_{MN}\to\{g_{\mu\nu}; (d-m)B_\mu; \frac{1}{2}(d-m)(d-m+1)\varphi\}$ (cf. (2.11)) and so

$$\bar{b}_p^{(h)}(d)=\bar{b}_p^{(h)}(m)+(d-m)\bar{b}_p^{(1)}(m)+\tfrac{1}{2}(d-m)(d-m+1)\bar{b}_p^{(0)}(m). \tag{4.27}$$

Thus, all we need to know are gravitational, vector and scalar contributions, e.g., for $m=4$. However, one can notice from (4.6)–(4.9) and (4.18)–(4.21) that (4.27) is valid only on the gravitational *mass shell* $R_{MN}=0$. The reason is that the d-dimensional background gravitational gauge contains quantum vectors and scalars, being written in terms of m-dimensional fields and therefore the expressions in different sides of (4.27) are computed in different gauges. This is in contrast with (4.26) which holds for any external metric.

4.3. QUANTIZATION OF THE GRAVITINO IN d-DIMENSIONS

The gravitino lagrangian on a d-dimensional metric background is defined by

$$\mathcal{L}=\bar{\psi}_M\gamma^{MNK}\mathcal{D}_N\psi_K, \tag{4.28}$$

where

$$\mathcal{D}_M\psi_N=\partial_M\psi_N-\Gamma^K{}_{NM}\psi_K+\tfrac{1}{4}\gamma_{AB}\omega^{AB}{}_M(e)\psi_N$$

* This does not, however, imply inequivalence of non-anomalous finite parts of effective actions, because they are governed by b_p's only under introduction of a mass, but the latter spoils even classical equivalence.

and the γ-matrix algebra is given in appendix B. The action is invariant under $\delta\psi_M = \mathcal{D}_\mu \varepsilon$ up to the terms vanishing at $R_{MN} = 0$. That is why if we fix a background gauge and calculate the effective action it will be gauge independent only on shell, i.e. $R_{MN} = 0$ (for instance, $\alpha_2, \alpha_3, \alpha_4$ in (4.2) will be gauge dependent, cf. [63]). The main idea in choosing the gauge is to obtain a simple propagator, i.e. to bring a gravitino operator to the general form (2.2). This turns to be somewhat non-trivial for $d \neq 4$. Indeed, let us follow the well-known $d = 4$ recipe [64], taking the gauge $\gamma_M \psi_M = \zeta(x)$ and averaging over ζ with the help of the $\hat{\mathcal{D}} = \gamma_M \mathcal{D}_M$ operator

$$\mathcal{L} + \mathcal{L}_{\text{g.b.}} = \bar{\psi}_M D^{(\xi)}_{MN} \psi_N, \qquad D^{(\xi)}_{MN} = \Sigma^{(\xi)}_{MKN} \mathcal{D}_K, \tag{4.29}$$

$$\Sigma^{(\xi)}_{MKN} = \gamma_{MKN} + \xi \gamma_M \gamma_K \gamma_N = -\tfrac{1}{2}(\gamma_N \gamma_K \gamma_M + \eta \gamma_M \gamma_K \gamma_N), \qquad \eta = -2(\xi + \tfrac{1}{2}), \tag{4.30}$$

where ξ (or η) is a gauge parameter. The corresponding partition function is (assuming ψ_M to be Majorana)

$$Z = \sqrt{\det D^{(\xi)}} \Big/ \sqrt{(\det \hat{\mathcal{D}})^3}\,. \tag{4.31}$$

The usual $d = 4$ gauge choice is $\eta = 0$ ($\xi = -\frac{1}{2}$) [64, 63], because it provides diagonality of the "$\mathcal{D}^2$" term in the "squared" $D^{(\xi)}$ operator. This is based essentially on the identity (B.6) showing the absence of "non-diagonal" terms for $d = 4$. One can easily convince himself that non-diagonal terms, however, are unavoidable for any ξ and $d \neq 4$.

The solution of this problem comes from the observation that if we substitute $D^{(\xi)}$ by the operator $\tilde{D}_{MN} = \Lambda_{MK} D^{(\xi)}_{KL} \Lambda_{LN}$, where $\Lambda = \bar{\Lambda}^T$ is an *algebraic* operator, then we do not change the non-trivial dependence of Z on the metric ($\log\det \Lambda \sim \delta^{(d)}(0)$). Namely, let us take

$$\Lambda_{MN} = g_{MN} + a\gamma_M \gamma_N, \qquad \bar{\Lambda}_{MN} = \Lambda_{NM}(\bar{\gamma}_M = -\gamma_M)\,. \tag{4.32}$$

The idea is to choose constants a and ξ in order to simplify $\tilde{D}$. Direct computation shows that (cf. appendix B)

$$\tilde{D}_{MK} = \left[g_{MK}\gamma_N - \tfrac{1}{2}\left(b\gamma_M \gamma_N \gamma_K + 4c\gamma_{(M} g_{K)N} \right) \right] \mathcal{D}_N,$$

where $b = \eta(1 + ad)^2 - (2a + a^2 d)(d - 4) + 2a^2(d - 2) - 1$, $c = 1 + a(d - 2)$. The condition $b = c = 0$ has only one solution

$$a = -\frac{1}{d-2}, \qquad \eta = \tfrac{1}{2}(d-4), \qquad \text{i.e. } \xi = -\tfrac{1}{4}(d-2)\,. \tag{4.33}$$

Hence we conclude that for all $d>2$ there exists the "*standard gauge*" (4.33), where the gravitino operator has its simplest form*

$$D_{MN} = g_{MN}\hat{\mathcal{D}},$$
$$\Delta_{3/2\,MN} = -(D^2)_{MN} = -g_{MN}\mathcal{D}^2 + \tfrac{1}{4}Rg_{MN} - \tfrac{1}{2}\gamma_{KL}R^{KL}_{MN}. \quad (4.34)$$

As a result, we can use this operator, for example, in off-shell calculations of divergences. It is important to note that this observation simplifies even the $d=4$ calculations, e.g. of the gravitino contribution in anomalies (compare with the approaches of refs. [64, 65]) and in off-shell divergences (cf. [63]).

All that is left is to use (4.34) with (2.2), (2.7)–(2.9) in order to calculate the gravitino contribution in b_ps:

$$\bar{b}_p^{(3/2)} = -\frac{1}{\gamma}\left[\bar{b}_p(\Delta_{3/2}) - 3b_p(\Delta_{1/2})\right], \quad (4.35)$$

where $\Delta_{1/2} = -\hat{\mathcal{D}}^2 = -\mathcal{D}^2 + \frac{1}{4}R$ and $\gamma = 1, 2, 4$ for Dirac, Majorana and Majorana-Weyl cases as in (3.7). The final expressions for $\bar{b}_p^{(3/2)}$ are [cf. (4.1)–(4.3)]

$$N = -\frac{\nu}{\gamma}(d-3), \qquad \rho = -\tfrac{1}{12}N, \qquad \nu = 2^{[d/2]}, \quad (4.36)$$

$$\alpha_1 = \tfrac{1}{180}N + \frac{\nu}{\gamma}\frac{(d-19)}{96}, \qquad \alpha_2 = -\tfrac{1}{180}N, \qquad \alpha_3 = \tfrac{1}{288}N, \qquad \alpha_4 = -\tfrac{1}{120}N, \quad (4.37)$$

$$\sigma_1 = \tfrac{1}{15120}N, \qquad \sigma_2 = \tfrac{1}{3240}N + \frac{\nu}{\gamma}\frac{(d+5)}{1440}, \quad (4.38)$$

while for $\bar{b}_p^{(1/2)} = -(1/\gamma)\bar{b}_p(\Delta_{1/2})$ we get

$$N = -\frac{\nu}{\gamma}, \qquad \alpha_1 = \tfrac{1}{180}N + \frac{\nu}{\gamma}\frac{1}{96}, \qquad \sigma_2 = \tfrac{1}{3240}N + \frac{\nu}{\gamma}\frac{1}{1440}, \quad (4.39)$$

with all other coefficients having the same structure as above. If $d=4$ these values are the same as in [64, 65] (α_1) and in [63] ($\alpha_2, \alpha_3, \alpha_4; \rho$). Finally, let us note that the following "reduction" relation holds: $\bar{b}_p^{(3/2)}(d) = \bar{b}_p^{(3/2)}(m) + (d-m)\bar{b}_p^{(1/2)}(m)$ [cf. (4.26), (4.27)] if we omit the ν/γ factors.

* Thus the usual $d=4$ gauge $\eta=0$ appears to be distinguished by its connection with the "standard" one.

4.4. APPLICATIONS

Now it is possible to compute the leading L^{d-p}, $p \leqslant 6$ one-loop gravitational infinities (i.e. b_p, $p \leqslant 6$) for different systems of fields. Remarkable cancellations are know to occur in $d=4$ supergravities and thus it is natural to start with the maximal possible one in $d>4$, i.e. SG_{11}^{1} [66,2] (maximal SGs in $d<11$ can be obtained by a reduction, all others by a reduction and truncation, of this theory). It contains one gravitation g_{MN}, one Majorana gravitino ψ_M and one antisymmetric gauge tensor A_{MNK}. Making use of (4.6)–(4.9), (4.18)–(4.21) and (4.36)–(4.38) we find for $\bar{b}_p = \bar{b}_p^{(h)} + \bar{b}_p^{(3)} + \bar{b}_p^{(3/2)}$ if $d=11$: $N = 44 + 84 - 128 = 0$, $\rho = -\frac{85}{2}$, $\alpha_1 = \frac{1}{180}(149 + 219 - 368) = 0$, $\alpha_2 = 7$, $\alpha_3 = \frac{421}{24}$, $\alpha_4 = -\frac{85}{12}$, $\sigma_1 = 0$, $\sigma_2 = \frac{1}{180}(3 + 93 - 96) = 0$, and so, *on shell* ($R_{MN} = 0$),

$$\bar{b}_0 = \bar{b}_2 = \bar{b}_4 = \bar{b}_6 = 0. \tag{4.40}$$

Thus there are no $L^{11}, \ldots, L^5$ one-loop divergences in eleven dimensional supergravity*. Using the lemma (2.14) we conclude that (4.40) as the property of $d=11$ theory implies the same for all lower dimensional theories which can be obtained by reduction, i.e. for all *maximal supergravities*. For example, (4.40) is valid for SG_4^8. Then the result $b_4 = 0$ is recognized as the vanishing of anomalies in the version of SG_4^8 directly following from dimensional reduction (i.e. containing 63 A, 7 $A_{\mu\nu}$, 1 $A_{\mu\nu\lambda}$) [31–33]. Thus we understand the absence of anomalies (or topological counterterms) in SG_4^8 as a *consequence* of the absence of L^7 infinities in SG_{11}^1**. The new non-trivial result is $b_6(\mathrm{SG}_4^8) = 0$ [which holds again only for the "reduction" version of SG_4^8 because $\bar{b}_6^{(3)}(4) \neq 0$, $\bar{b}_6^{(2)}(4) - \bar{b}_6^{(0)}(4) \neq 0$ cf. subsect. 4.2]***.

Next we pass to maximal supergravities in $d = 5, 6, 7$. According to (2.6) and (4.40) we conclude that SG_5^8 [68], SG_6^4 and SG_7^4 [69] *are one-loop finite in the corresponding number of dimensions* and hence provide the first examples of $d>4$ dimensional one-loop finite gravitational theories (recall that pure gravity is *not* finite if $d>4$ and so the finiteness of these theories is due to *cancellations* and not merely to the non-existence of non-zero on-shell invariants as in the case of $d=4$ SGs). Therefore, it is this class of theories which may be considered as *a natural candidate for the Kaluza-Klein program*. As for on-shell finiteness of SG_{11}^1 conjectured in [5,6], it seems rather unprobable in view of the property $b_8(\mathrm{SG}_{11}^1) \neq 0$ (which is supported by the analogy with SYM_{10}^1 case and also by the non-zero result for the 4-particle amplitude in SG_4^8 [17,18]).

* We recall that ρ, α_2, α_3, α_4 are gauge dependent; there exists a supergauge where they are zero and so (4.40) is valid off shell.

** Let us note that the $\alpha_2(\mathrm{SG}_4^8) = 0$ off-shell result of ref. [63] does not contradict $\alpha_2(\mathrm{SG}_{11}^1) \neq 0$ because the $d=4$ and $d=11$ gauges are different.

*** To illustrate the meaning of b_6 in the $d=4$ case suppose we give a large mass M to all fields in the theory. Then expanding the effective action in powers of $1/M$ we get $(1/M^2)b_6$ as a first non-trivial term (for applications of this fact see, e.g., [19,67]). Also, the vanishing of b_6 is connected with the vanishing of the 3-particle amplitude, i.e. with the absence of R^3 terms in the expansion of the effective action.

One important clarification is needed concerning the above finiteness conclusion. It is *irreducible* supergravity which is finite if it is finite in the gravitational sector. Given the vanishing of gravitational infinities we are to check that the theory, obtained by reduction from irreducible supergravity, is again an irreducible one. It appears that this property is valid only for *maximal* supergravities (i.e. one can use reduction but not truncation). In other cases irreducibility is lost because reduction breaks the O(d) (Lorentz) symmetry, essential for irreducibility in d dimensions. If we take irreducible but non-maximal SG in $d > 4$ and reduce it to $d = 4$ we obtain a $d = 4$ SG plus some matter multiplets (for instance, $\mathrm{SG}_5^1 \to \mathrm{SG}_4^2 + \mathrm{YM}_4^2$ multiplet [70]), a theory known to be one-loop infinite [71]. However, if we are interested not only in $d = 4$ reduced theory we may question about quantum properties of *non-maximal* supergravities as they are in d dimensions. Then a distinguished candidate is $N = 1$, $d = 10$ supergravity [72–74], connected with the open sector of superstring theory [1, 16]. It contains one graviton, one Majorana-Weyl gravitino, one antisymmetric tensor A_{MN}, one scalar and one Majorana-Weyl spinor. The contributions of all these fields in b_p can be computed using the formulas of sect. 4 (taking $d = 10$ and $\gamma = 4$) with the final conclusion that (4.40) is valid also for SG_{10}^1. This result cannot be considered simply as a consequence of (4.40) for SG_{11}^1, because to relate SG_{11}^1 and SG_{10}^1 we are not only to reduce but also to truncate the former*. Eq. (4.40) is also true for all theories following from SG_{10}^1 by the reduction, for example, for $d = 4$ $\mathrm{SG}_4^4 + 6\mathrm{SYM}_4^4$ – one [72] ($b_4 = 0$ corresponds to the vanishing of anomalies in this theory, first noted in [33]). However, keeping in mind the previous discussion, one should refrain from considering (4.40) as an indication of finiteness of corresponding theories in $d \leqslant 7$, because being obtained from a non-maximal $d = 10$ supergravity they have reducible supersymmetry and thus may be infinite in spite of finiteness in gravitational sector (compare with the opposite belief for $d = 4$ case in [22]**. Thus (4.40) is to be considered mainly as a property of SG_{10}^1 itself, implying the absence of $L^{10}, \dots, L^4$ divergences in this theory***. One can readily check that (4.40) is also valid for SYM_{10}^1 on the gravitational background and hence is also true for the coupled $\mathrm{SG}_{10}^1 + \mathrm{SYM}_{10}^1$ theory, coinciding with the $\alpha' \to 0$ limit of type I superstring theory [16]. One more remark utilizes the independence of b_p, $p \leqslant 6$, on the choice of representation of antisymmetric tensors in $d > 6$. Namely, there exists the version of SG_{10}^1 with $A_{M_1 \dots M_6}$ instead of A_{MN} [73]. Using the $p < d$ equivalence relation in (4.22) we conclude that all b_p, $p < 10$, are *the same* in both theories [i.e. we again have (4.40) and no anomalies in $d = 4$]. An analogous remark is also true for the (probably non-existing [76]) version of SG_{11}^1 with $A_{M_1 \dots M_6}$. The

* Curiously, $\alpha_2(\mathrm{SG}_{10}^1) = 10 - 4 = 6$, while $\alpha_2(\mathrm{SG}_{11}^1) = 11 - 4 = 7$.

** One more counter-example to the claim of possibility to obtain a finite theory by reducing non-maximal supergravity is provided by $\mathrm{SG}_5^4 \to \mathrm{SG}_4^4 + 4\mathrm{SYM}_4^4$ [75], known to be infinite [71].

*** In view of the connection with string theory one may ask about some analogous properties for $d = 26$ gravity; however they are apparently absent, predicting problems in the closed Bose string loop calculations.

moral is that duality transformations in higher dimensions do not influence the infinities of reduced theories.

Finally, we are going to illustrate the $p > d$ case in the quantum (in) equivalence relation (4.22) using the example of b_6 for SG_4^N. As is known [74, 32], it is possible to establish $b_4 = 0$ for all $N = 3, \dots, 8$ by suitable duality transformations of scalars. Then a natural question is whether the spectrum, which gives $b_4 = 0$, also provides $b_6 = 0$. The answer is "yes" for $N = 4, 8$ but "no" for $N = 5, 6$ (suggesting that $SG_4^{5,6}$ with the anomaly free spectrum actually cannot be constructed). Numerically we get: (i) $\alpha_1(N) = -\frac{1}{2}(N-3)$, $\sigma_2(N) = -\frac{1}{6}$ for $N \geqslant 3$ if all spin zero particles are represented by scalars; (ii) $\alpha_1(A_{\mu\nu}) - \alpha_1(A) = \frac{1}{2}$, $\alpha_1(A_{\mu\nu\lambda}) - 0 = -1$, $\sigma_2(A_{\mu\nu}) - \sigma_2(A) = \frac{1}{6}$, $\sigma_2(A_{\mu\nu\lambda}) - 0 = -1$, and thus it is not possible to arrange the spectrum so that $\alpha_1 = \sigma_2 = 0$ except for $N = 4$ *and* 8. The result $b_4(N=3) = 0$ but $b_6(N=3) \neq 0$ can probably be understood from the helicity sum rules: $b_{2k} \sim \Sigma_\lambda (-1)^{2\lambda} d(\lambda) \lambda^k = 0$, $k < N$ [41, 5].

5. Concluding remarks

In this paper we have considered quantum properties of theories obtained by the simplest dimensional reduction (with internal dimensions being $S^1 \times \cdots \times S^1$). The question left open is about properties of differently reduced theories, given those of higher dimensional theory. The general strategy to provide an answer is to study the relation of d-dim and reduced quantum theories separately in each particular case of reduction. This recipe is, of course, rather unconstructive. That is why we give examples of possible more explicit answers starting with the idea that different reductions correspond to different choices of vacuum in d dimensions (and assuming the knowledge of d-dim results for arbitrary backgrounds)*. Let us confine our discussion to $d = 11$ supergravity. Several $d = 11$ internal "vacuum" spaces were already considered in the literature: (a) $N^7 = S^1 \times \cdots \times S^1$ [3]; (b) $N^7 =$ "flat group" space [12]**; (c) $N^7 = (SU(3) \times SU(2) \times U(1))/K$, e.g. $\mathbb{C}P^2 \times S^2 \times S^1$ [4]; (d) $N^7 = S^7$ [77–80]. Different reductions (neglecting all massive modes) correspond to different versions of "$N = 8$, $d = 4$" SG. Only in the first case is it known that reduction preserves supersymmetry. If this is also true for the fourth case (d) the resulting theory may be connected with the SO(8) gauged version of SG_4^8 [81]. To give an idea how one can use $d = 11$ results in analysis of this theory let us present the following heuristic argument for the zero value of its β-function, starting with the $b_4 = 0$ property of SG_{11}^1. To realize the vacuum $M^4 \times S^7$ as a classical solution for SG_{11}^1 we are to assume a non-zero A_{MNK}-background [77, 78]. Suppose we calculated $b_4(SG_{11}^1)$ with $g_{MN} \neq \delta_{MN}$, $A_{MNP} \neq 0$. Then in view of supersymmetry we

* It should be understood that we are speaking about quantum versions of differently reduced theories and *not* about d-dim theory quantized near different vacua.

** This case was not actually studied for $d = 11$.

again must have $b_4 = 0$ on shell. But this implies the absence of Λ-term (on-shell) renormalization in corresponding $d = 4$ theory and thus [82] zero β-function. Our conjecture is that the $b_4 = 0$ property of SG_{11}^1 is, in fact, the origin of not only the vanishing of anomalies in the "reduced" version of SG_4^8 but also of vanishing of the β-function in the gauged version*.

It may be instructive to present another (even less rigorous) variant of the above argument. Naively, one could hope for the following understanding of F_{MN}^2 renormalization in non-abelian Kaluza-Klein theories. Using the standard ansatz (2.11) (φ_{ij} being non-trivial, corresponding to the coset internal space, cf. [10]), on shell we have $\bar{b}_4 = \alpha_1 R_{MNPK}^2, R^2 \ldots \to (R \ldots + FF + \Lambda)^2 \to R^2 \ldots + RFF + F^4 + \Lambda^2 + \Lambda F^2$. It is the last $\sim F_{MN}^2$ term that contributes in the β-function and hence $\alpha_1 = 0$ implies $\beta = 0$. However, a word of caution is needed here: $\alpha_1 \sim \beta$ does not actually hold in Kaluza-Klein theory. Really, as we already observed in sect. 2 for the simple reduction, if there are non-diagonal metric components in (2.11), divergences of the reduced theory are not exhausted by d-dim curvature invariants (i.e. there may be additional F_{MN}^2 contributions). Though this question should be studied separately for the S^7 reduction, we may speculate that in a *supersymmetric* theory (with gravitational and gauge sectors being interrelated) the above argument is after all a correct one.

Our final comment is about supersymmetry breaking reductions. In general, coset reduction (c) changes the number of degrees of freedom (in contrast with soft supersymmetry breaking by the mass terms) and so one cannot hope for some good quantum properties (for instance, the resulting $d = 4$ theories will have L^4, L^2 and at least topological $\log L^2$ divergences). At the same time it was proved [83] that "generalized dimensional reduction" (b) produces one-loop on-shell renormalizable $d = 4$ theory. To understand if it is possible to provide some simple $d = 11$ explanation of this fact one should specially study the impact of this reduction on the quantum theory.

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Note added in proof:

Meanwhile we became aware of several additional references: ref. [85] (b_6 for matter fields); ref. [86] (b_4 for antisymmetric tensors); ref. [87] (soft breaking of SYM_4^4)

* Another hint for this relation is provided by the analogy in sum rules which are connected with these two properties (cf. [5]). Note, however, that it is still unclear, if one can construct a version of SG_4^8 having simultaneously zero anomalies and β-function (the version of ref. [81] employs only scalars).

Appendix A

NOTATION AND USEFUL FORMULAS

Throughout this paper we employed the euclidean metric g_{MN} and the following curvature conventions:

$$R^M{}_{NPQ}=\partial_P\Gamma^M{}_{NQ}-\cdots,\qquad R_{NQ}=R^M{}_{NMQ},\qquad R^{MN}_{PQ}=g^{NK}R^M{}_{KPQ},\quad (A.1)$$

where $M, N, \ldots=1,\ldots,d$. Two other groups of indices are $\mu,\nu=1,\ldots,n$; $i, j=1,2,\ldots,d-n$, $n<d$. Using the matrix notations

$$A^{bc}_\mu=A^a_\mu f^{bac},\qquad f_{acd}f_{bcd}=C_2\delta_{ab},\quad (A.2)$$

and defining the gauge strength invariants

$$\mathcal{G}_1=\mathrm{tr}(\mathcal{D}_M F_{NK})^2,\qquad \mathcal{G}_2=\mathrm{tr}(\mathcal{D}_M F_{NK}\mathcal{D}_N F_{MK}),$$
$$\mathcal{G}_3=\mathrm{tr}(\mathcal{D}_M F_{MN})^2,\qquad \mathcal{G}_4=\mathrm{tr}(F_{MN}F_{NK}F_{KM}),\quad (A.3)$$

one can prove that

$$\mathcal{G}_2=\tfrac{1}{2}\mathcal{G}_1,\qquad \mathcal{G}_3=\tfrac{1}{2}\mathcal{G}_1+2\mathcal{G}_4-\tfrac{1}{2}R_{MNKL}\mathrm{tr}(F_{MN}F_{KL})+\mathrm{div}.\quad (A.4)$$

Assuming the "mass shell" condition $R_{MN}=0$, one can establish analogous relations for the curvature invariants (cf. [57, 59])

$$I_1=R^{MN}_{PQ}R^{PQ}_{KS}R^{KS}_{MN},\qquad I_2=R^{MN}_{PQ}R^{QS}_{NK}R^{KM}_{SP},$$
$$I_3=R^{PN}_{MQ}R^{QS}_{NK}R^{KM}_{SP},\qquad I_4=R_{MNPQ}R^{MP}_{KS}R^{NQ}_{KS},\quad (A.5)$$
$$I_5=R_{MNPQ}\mathcal{D}^2R_{MNPQ},$$
$$I_4=\tfrac{1}{2}I_1,\qquad I_3=I_2-\tfrac{1}{4}I_1,\qquad I_5=-I_1-4I_2.\quad (A.6)$$

Let us also introduce

$$E=I_1-2I_2,\quad (A.7)$$

connected with the Euler number [84] ($d=2m$)

$$\chi_d=\frac{1}{2\cdot 4\cdot\cdots\cdot d}\times\frac{1}{(4\pi)^{d/2}}\int R^{M_1M_2}_{N_1N_2\ldots}R^{M_{d-1}M_d}_{N_{d-1}N_d}\varepsilon^{N_1\ldots N_d}$$

$$\times\varepsilon_{M_1\ldots M_d}\sqrt{g}\,\mathrm{d}^dx\equiv\frac{1}{(4\pi)^{d/2}}\int\mathcal{E}_d\sqrt{g}\,\mathrm{d}^dx. \tag{A.8}$$

Namely, if $d=4$ and $d=6$

$$\mathcal{E}_4=\tfrac{1}{2}R^*R^*,\qquad \mathcal{E}_6=\tfrac{2}{3}E. \tag{A.9}$$

Appendix B

IDENTITIES FOR γ MATRICES IN d DIMENSIONS

Let γ_M be $\nu\times\nu$ ($\nu=2^{[d/2]}$) Dirac matrices. If

$$\gamma_{M_1\ldots M_k}=\gamma_{[M_1\ldots M_k]}=\frac{1}{k!}\left(\gamma_{M_1\ldots}\gamma_{M_k}+\cdots\right) \tag{B.1}$$

(the same "weighted" convention is used throughout also for symmetrization), then

$$\gamma_M\gamma_N=g_{MN}+\gamma_{MN},\qquad \gamma_M\gamma_N\gamma_K=g_{MN}\gamma_K-g_{MK}\gamma_N+g_{NK}\gamma_M+\gamma_{MNK}, \tag{B.2}$$

$$\gamma_M\gamma_N\gamma_K\gamma_S=g_{MN}g_{KS}-g_{MK}g_{NS}+g_{NK}g_{MS}+\gamma_{MN}g_{KS}+g_{MN}\gamma_{KS}$$
$$-\gamma_{MK}g_{NS}-g_{MK}\gamma_{NS}+\gamma_{MS}g_{NK}+g_{MS}\gamma_{NK}+\gamma_{MNKS}, \tag{B.3}$$

$$\gamma_M\gamma_N\gamma_M=(2-d)\gamma_N,\qquad \gamma_M\gamma_N\gamma_K\gamma_M=d\cdot g_{NK}+(d-4)\gamma_{NK}, \tag{B.4}$$

$$\gamma_M\gamma_N\gamma_K\gamma_S\gamma_M=(2-d)(g_{NK}\gamma_S-g_{NS}\gamma_K+g_{KS}\gamma_N)+(6-d)\gamma_{NKS}, \tag{B.5}$$

$$\gamma_M\gamma_N\gamma_K\gamma_S\gamma_P\gamma_M=d(g_{NK}g_{SP}-g_{NS}g_{KP}+g_{NP}g_{KS})$$
$$+(d-4)(g_{NK}\gamma_{SP}+g_{SP}\gamma_{NK}-\gamma_{NS}g_{KP}-\gamma_{KP}g_{NS}+g_{KS}\gamma_{NP}$$
$$+g_{NP}\gamma_{KS})+(d-8)\gamma_{NKSP}, \tag{B.6}$$

$$\mathrm{tr}\,\mathbb{1}=\nu=2^{[d/2]},\qquad \mathrm{tr}\,\gamma_M=0,\qquad \mathrm{tr}(\gamma_M\gamma_N)=\nu g_{MN}, \tag{B.7}$$

$$\mathrm{tr}(\gamma_M\gamma_N\gamma_K\gamma_S)=\nu(g_{MN}g_{KS}-g_{MK}g_{NS}+g_{NK}g_{MS}). \tag{B.8}$$

In particular, if $\gamma \cdot F = \gamma_{MN} F_{MN}$, then

$$\mathrm{tr}(\gamma \cdot F)^2 = -2\nu F_{MN} F_{MN}, \qquad \mathrm{tr}(\gamma \cdot F)^3 = 8\nu F_{MN} F_{NK} F_{KM}. \tag{B.9}$$

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