Nuclear Physics B234 (1984) 472–508


# ONE-LOOP EFFECTIVE POTENTIAL IN GAUGED O(4) SUPERGRAVITY AND THE PROBLEM OF THE Λ TERM

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Received 27 May 1983

We calculate the one-loop, off-shell, effective action in O(4) gauged supergravity assuming an (anti) de Sitter metric and constant scalar fields as a background. The problem of the large induced $\Lambda$ term (present already for free matter fields) is stressed and the possibility of dynamical breakdown of local supersymmetry is pointed out. We illustrate our techniques and qualitative conclusions on a number of examples, including $\phi^4$ theory and QED scalar potentials on a de Sitter background and an effective action in Einstein theory with a cosmological constant. Possible solutions of the $\Lambda$-term problem are also discussed.

## 1. Introduction

If supergravities are to be relevant for unification of interactions then very probably it will be in their gauged form [1–4]. Leaving aside the higher-loop renormalizability question, the first major problem of O($N$) gauged supergravities [1, 2, 4] is the occurrence of an enormous negative cosmological constant $\Lambda_0 \sim -g^2/k^2$ in all known classical ground states of these theories. Thus the background space is anti de Sitter (AdS) with the characteristic scale being of the Planck order. Also, in analogy with the indefiniteness of the gravitational action, tree-level scalar potentials are unbounded from below. Though classically this does not necessarily imply an instability in view of the AdS background [5], unboundedness may lead to some difficulty after a desired but yet imaginary supersymmetry breaking which provides a small final cosmological constant.

The Planck order of the bare cosmological constant ($\Lambda_0 \sim m_{\mathrm{P}}^2$) suggests that quantum effects are to be important in these theories. If so one could hope for a resolution of the above two problems by accounting for quantum corrections which may stabilize the scalar potential and make $\Lambda$, for example, look like $m_{\mathrm{P}}^2\exp(-c/g^2)$. Quantum fluctuations may even be the origin of local supersymmetry breaking (which thus will be *dynamical*), providing an alternative to tree-level supersymmetry breaking mechanisms (the well-known statement of the impossibility of perturbative supersymmetry breaking applies only for the *global* supersymmetry case [6]). With these motivations in mind we begin here the study of *finite* quantum corrections in gauged supergravities*, calculating the one-loop effective potential in O(4) theory [2].

This calculation is of interest from several points of view. In recent studies of gravitational effects on phase transitions in the early universe [9, 10], GUT effective potentials are investigated on a given de Sitter background and effects of back reaction and quantum gravity are completely ignored. This, of course, seems legitimate in a late enough stage of evolution, usually considered. At the same time, the inclusion of quantum-gravitational fluctuations** may explain the nature of the initial de Sitter world and show the way to avoid fine tuning of the cosmological constant. However, the quantum Einstein-plus-GUT theory is known to be one-loop, on-shell, infinite. An attractive feature of (gauged) extended supergravities is that they provide models of one-loop, on-shell, renormalizable (finite) theories [12, 7, 8], unifying gravity with matter. That is why it seems natural to begin the study of quantized gravity effects in the context of supergravities, with the simplest non-trivial candidate being just the O(4) gauged one ($N \leqslant 3$ theories have no scalars).

From the technical point of view we are interested in the effective action $\Gamma[g_{\mu\nu}, \phi]$ as a functional of the background geometry and scalar fields. However, this quantity is very difficult to find explicitly, even at one loop. That is why we make a simplifying assumption that the effective geometry (i.e. the solution of $\delta\Gamma/\delta g_{\mu\nu} = 0$, $\delta\Gamma/\delta\phi = 0$) is (anti) de Sitter space with some cosmological constant $\Lambda$ and that the effective scalar fields are constant. It is the "effective potential" $\Gamma(\Lambda, \phi)$ which we are going to calculate in this paper. Its extrema give us (in principle) information about the effective cosmological constant and the possibility of dynamical supersymmetry breaking. Actual interpretation of the results depends on the resolution of three important issues, namely, gauge dependence, off-shell infinities and the reliability of the one-loop approximation, which are discussed in subsect. 2.1.

In subsects. 2.2, 2.3, 2.4 we develop *techniques for one-loop effective action calculations* on a de Sitter background *for fields of all spins* $s \leqslant 2$ occurring in supergravities. The crucial issue for the present investigation is the new result in establishing *the spectrum of the gravitino operator*. Using the known spectra of all relevant operators we then employ the $\zeta$-function method [13, 14] to find the *finite parts* of the determinants. As a by-product, we streamline and simplify the procedure of the calculation of the effective potentials for spins $s \leqslant 1$ as compared to the previous treatments (cf. [9, 10, 15–17]).

This procedure is illustrated in the examples of $\phi^4$ theory and scalar QED in sect. 3. Taking gravity to be classical, we, however, account for the back reaction of quantized matter by assuming that the effective $\Lambda$ may not coincide with its

* One-loop infinities in these theories were already studied in [7, 8].

** For an earlier attempt to include gravitational corrections in the GUT effective potential see [11].

tree-level value $\Lambda_0$. An acceptable value of $\Lambda$ can always be obtained by fine tuning of $\Lambda_0$. At the same time, if $\Lambda_0 = 0$, then (already in the case of free matter fields) there is a new quantum solution, $\Lambda = \text{const}/\hbar k^2$, representing an *enormous cosmological constant induced* by quantum fluctuations. This finite Planck order $\Lambda$ term is present even for supersymmetric theories (in which only infinite contributions to $\Lambda$ have better chances to cancel). To trust one-loop extrema for $\phi$ one is to consider the analog of the Coleman-Weinberg mechanism [18] in dS space. That is why we present the expression for the scalar QED effective potential, calculated in a rather general class of covariant gauges, generalizing previous results [9, 10]. We stress that if the $\xi R\phi^2$ term is included in the tree lagrangian, the assumption of $\lambda \sim g^4$ [18] *alone* is not sufficient to make the one-loop approximation gauge-independent and thus reliable. If $\Lambda_0 = 0$ then again a Planck-order $\Lambda$ term is generated implying the necessity of including quantum-gravitational effects.

These are dealt with in sect. 4 in the example of Einstein gravity. Here we compute the one-loop effective action $\Gamma(\Lambda, \Lambda_0)$ as a function of the a priori arbitrary parameter $\Lambda$ of effective dS geometry and of the tree-level cosmological constant $\Lambda_0$. This calculation is a remarkable example of an *exactly solvable problem* of establishing the finite part of an off-shell quantum-gravitational effective action. It is worth noting that the de Sitter background plays, in quantum gravity, the role analogous to that of a *constant field strength background* in gauge theories (see e.g. [19]) and thus may be useful for higher-loop studies. In view of the off-shell ($\Lambda \neq \Lambda_0$) gauge dependence of $\Gamma$ the question still remains about the physical significance of the effective $\Lambda$. To extract information stable against the choice of a particular gauge, we work out the expressions for $\Gamma$ in a number of classes of gauges, and show, in particular, that gauges exist in which $\Gamma$ is free from $\Lambda_0$-dependent off-shell infinities. The analysis of the effective equation for $\Lambda$ shows that if $\lambda_0 = \Lambda_0 k^2 < 1$ the one-loop correction is small and $\Lambda \simeq \Lambda_0 + O(\hbar)$. If $\lambda_0 \simeq 1$ one cannot trust the one-loop approximation and the summation of higher-loop series is needed. A special case is $\Lambda_0 = 0$. Here again (as in the case of matter fields) an enormous quantum $\Lambda$ term is generated probably implying an instability of flat space against the creation of a de Sitter universe.

Finally, we are prepared for the study of the effective potential $\Gamma(\Lambda, \phi, g^2/k^2)$ in O(4) supergravity (sect. 5). Putting first $\Lambda$ and $\phi$ equal to their tree-level values we find a gauge-independent one-loop correction to the classical (AdS) vacuum action which appears to be positive for small $g^2$. At the same time this gives a simple proof of the semiclassical stability of the AdS vacuum (cf. [5]). Then we turn to off-shell effective equations, starting with the supersymmetry preserving $\phi = 0$ solution. The corresponding Planck-order value of $\Lambda = \text{const}/k^2 + O(g^2)$ implies the importance of higher-loop corrections. We conclude that the problem of the induced cosmological constant does not find a solution (at one loop) in $O(N)$ gauged supergravities. The set of two effective equations has also a *supersymmetry breaking solution*: $\phi^2 \sim 1 - \text{const}/\Lambda k^2 + O(g^2)$, $\Lambda \sim \text{const}/k^2 + O(g^2)$. Though the $\phi \neq 0$ result cannot be rigorously proved in the one-loop approximation, we still consider it as an indication of the possibility *of dynamical breaking of local supersymmetry*.

In concluding sect. 6. we comment on the probable existence of a *new* class of gauged supergravities (analogous to the $SU_2 \times SU_2$ model of ref. [3]) with a zero "conformal anomaly" coefficient, in which an *induced $\Lambda$ term may not appear*. We also speculate on the possible resolution of the $\Lambda$-term problem after summation of higher-loop contributions.

In the appendix we draw attention to the fact that the off-shell effective action can be calculated *without fixing any gauge*. Though the thus obtained $\Gamma$ does not have a correct on-shell limit it may be advantageous to use it (instead of the standard gauge-dependent effective action) in attempts to find the gauge-independent approximation, trustworthy at one loop.

## 2. General strategy and basic techniques

### 2.1. EFFECTIVE ACTION AND EFFECTIVE EXTREMA

In this paper we will be concerned with the one-loop approximation to the off-shell euclidean effective action, defined in the standard way (see, e.g. [20–22]):

$$\exp\left(-\frac{1}{\hbar}\Gamma[\phi]\right)=\int \mathrm{d}\eta \exp\left\{-\frac{1}{\hbar}\left(I[\phi+\eta]-\frac{\delta\Gamma}{\delta\phi}\eta\right)\right\}, \tag{2.1}$$

namely,

$$\Gamma[\phi]=I[\phi]+\Gamma_1[\phi]+\mathrm{O}(\hbar^2), \qquad \Gamma_1=\tfrac{1}{2}\hbar\ln\det\frac{\delta^2 I}{\delta\phi^2}, \tag{2.2}$$

(we do not indicate explicitly the gauge fixing and ghost terms). If $\phi_0$ is a stable classical solution, the *perturbative* solution $\tilde{\phi}$ of the effective equation

$$\frac{\delta\Gamma}{\delta\phi}=0, \tag{2.3}$$

will be

$$\tilde{\phi}=\phi_0+\hbar\phi_1+\mathrm{O}(\hbar^2), \qquad \phi_1=-\frac{1}{\hbar}\frac{(\delta\Gamma_1/\delta\phi)_{\phi_0}}{(\delta^2 I/\delta\phi^2)_{\phi_0}}. \tag{2.4}$$

Then

$$\Gamma[\tilde{\phi}]=I[\phi_0]+\Gamma_1[\phi_0]+\mathrm{O}(\hbar^2), \tag{2.5}$$

and hence the one-loop correction $\phi_1$ is apparently uninteresting (it becomes relevant only at two loops). The problem thus reduces to the well-known semiclassical expansion around $\phi_0$ (taking account of zero modes, etc.). One can in principle try to solve $\delta(I+\Gamma_1)/\delta\phi=0$ non-perturbatively ("non-analytically" in $\hbar$) but the resulting solution $\bar{\phi}$ will be sensible only if it will belong to a domain of applicability of the one-loop approximation, i.e. if higher loops will give only small corrections to it. This is possible, for example, when $I$ and $\Gamma_1$ happen to be of the same order [18].

Additional complications arise when gauge dependence of $\Gamma$ (and thus the solutions of (2.3)) is taken into consideration. It is $\Gamma$, computed on a solution of (2.3), that is gauge independent. For the perturbative solution $\tilde{\phi}$ this is true to each order in $\hbar$ (e.g. the on-shell one-loop correction $\Gamma_1[\phi_0]$ in (2.5) is gauge independent). As for the non-perturbative solution, one is to rely on the existence of some (loop-wise) gauge-independent approximation for $\Gamma$, providing gauge independence of the effective solution and thus of all physical quantities (like ratios of masses and critical parameters). The validity of this approximation again depends on the smallness of higher-loop corrections. For example, in scalar QED such an approximation is provided by the condition $\lambda \sim g^4$ [18, 23].

Having recalled these general facts let us specialize to the case of interest when $I$ is the gravity or supergravity action. Leaving aside (for a moment) the question of the reliability of the one-loop approximation (which at present is the only practically available one) we confront the problem of *off-shell* non-renormalizability of $\Gamma$. As a result, a naive solution of (2.3) will be gauge and cut-off ($L\to\infty$) dependent. The most satisfactory way around this difficulty would be to find a gauge-independent approximation, restoring the off-shell renormalizability (or finiteness) property known to hold on the classical "mass shell." It is not clear whether such an approximation actually exists (even supposing a higher-loop on-shell renormalizability of supergravities). In this situation we shall simply assume that (non-perturbative) solutions of one-loop effective equations do have some gauge invariant meaning and a priori may reproduce some qualitative features of true effective extrema. To cope with the cut-off and gauge dependence problem the following heuristic recipes may be operative: (i) one may choose an "$R$ gauge," in which $\Gamma$ appears to be renormalizable, employing the following statement: *if a (super) gauge theory is on-shell renormalizable (or finite) then there exists a gauge where its effective action is off-shell renormalizable (or finite)**; the solution in this distinguished gauge may be "near" to that in a gauge-independent approximation; (ii) working in a general as possible class of gauges one can *define the one-loop $\Gamma$ to be finite*, e.g. using the $\zeta$-function prescription [13, 14]. The residual normalization point (and gauge parameters) dependence of $\Gamma$ (and $\bar{\phi}$) is then supposed to be irrelevant for qualitative physical results identified as being stable under variations of gauge parameters. The

* $R$ gauges for pure gravity were found in [24]; when the bare $\Lambda$ term is included the only renormalizations needed are of the topological, gravitational and cosmological coupling constants [25] (see also sect. 4).

point we want to stress here is the *non-triviality* of any off-shell problem (for example, that of dynamical (super) symmetry breaking) in gauge theories renormalizable only *on shell*. A possible resolution of this difficulty could be found in the way of changing the definition of $\Gamma$ (2.1) for some "distinguished," "gauge independent" one like that of ref. [22] or that of the appendix (where $\Gamma$ is formally computed without any gauge fixing). However, the definition of ref. [22] does not provide renormalizable results [26] and thus will not be considered below. We shall mainly use the above recipe (ii), illustrating also (i) in sect. 4.

Now let us write down several general formulas for the one-loop effective action*

$$\Gamma_1 = \sum_i c_i \ln\det\left(\Delta^{(i)}/\mu^2\right),$$

$$\tfrac{1}{2}\ln\det\left(\Delta/\mu^2\right) = -\tfrac{1}{2}\left[\tfrac{1}{2}B_0 L^4 + B_2 L^2 + B_4\left(\ln\left(L^2/\mu^2\right) - \gamma_0\right)\right] + \Gamma_f(\Delta), \quad (2.6)$$

$$\Gamma_f(\Delta) = \tfrac{1}{2}B_4 \ln\left(\rho^2/\mu^2\right) - \tfrac{1}{2}\zeta'(0). \quad (2.7)$$

Eq. (2.6) is the result in the proper-time regularization (with the cut-off $\varepsilon = (L/\mu)^{-2} \to 0$) while eq. (2.7) is its analog in the $\zeta$-function prescription [13, 14]. The operators $\Delta^{(i)}$ are defined on *differentially unconstrained* fields and have the general form $-D^2 + X$. Therefore the $B_p = (1/16\pi^2)\int \bar{b}_p \sqrt{g}\, \mathrm{d}^4x$ coefficients have the well-known structure [27, 28]

$$\bar{b}_0 = \operatorname{tr}\mathbf{1}, \qquad \bar{b}_2 = \operatorname{tr}\left(\tfrac{1}{6}R - X\right), \qquad \bar{b}_4 = \operatorname{tr}\left(\tfrac{1}{2}X^2 + \cdots\right), \quad (2.8)$$

and can be used to compute divergences for an arbitrary background (for such an off-shell computation in gauged supergravities see ref. [8]). The parameter $[\rho] = \mathrm{cm}^{-1}$ is the scale of the eigenvalues of $\Delta$,

$$\Delta\varphi_n = \lambda_n \varphi_n, \qquad \lambda_n = \rho^2 \bar{\lambda}_n, \quad (2.9)$$

while the (generalized) $\zeta$ function is defined by

$$\zeta(p) = \sum_n d_n/\bar{\lambda}_n^p, \qquad \zeta' = \mathrm{d}\zeta/\mathrm{d}p, \quad (2.10)$$

for $\operatorname{Re} p > 2$ and by analytic continuation in all other points ($d_n$ is the multiplicity of $\lambda_n$). We shall assume that the sum in (2.10) goes over *all* modes of $\Delta$ *including negative and zero ones* (zero modes will contribute an infinite constant, which should be properly extracted, while the negative eigenvalues will make $\zeta$ complex, indicating an instability). This unconventional definition is convenient because it gives $\zeta(0) = B_4$

* We shall use the euclidean notation and often put $\hbar = 1$; note that $\gamma_0 = 0.577\ldots$; our curvature conventions are $R^{\lambda}{}_{\mu\lambda\nu} = R_{\mu\nu}$, $R^{\lambda}{}_{\mu\nu\rho} = \partial_\nu \Gamma^{\lambda}_{\mu\rho} - \cdots$.

for a differentially unconstrained operator $\Delta$. If $\Delta$ is split into $\Delta_1 \otimes \cdots \otimes \Delta_k$ by some differential change of variables, then $B_4(\Delta) = \sum_i \zeta_i(0) -$ (number of zero modes of the jacobian), see subsect. 2.4. Note that it is $B_4$ that governs the scale dependence of $\Gamma_1$.

### 2.2. RELEVANT OPERATORS AND THEIR SPECTRA

In this paper we compute effective actions assuming the background geometry to be euclidean de Sitter space ($S^4$)

$$R_{\lambda\mu\nu\rho} = \tfrac{1}{3}\Lambda\left(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu}\right), \qquad R_{\mu\nu} = \Lambda g_{\mu\nu},$$

$$\int d^4x\sqrt{g} = 24\pi^2/\Lambda^2, \qquad \rho^2 = \tfrac{1}{3}\Lambda. \tag{2.11}$$

If background values of scalars are constant then all relevant operators can be written as $-\mathscr{D}^2 + X$ with $X$ being some *constant* matrix. It is possible to find the spectra of all such operators on an $S^4$ background and thus to compute the *finite* $\zeta'(0)$ part of $\Gamma$ in (2.7). In practice, we need first to factorize the operators $\Delta^{(i)}$ into products of "constrained" $\Delta_s$ ones, corresponding to irreducible representations of SO(5). It is the spectra of $\Delta_s$ that can be explicitly found. For spins $s = 1$ and 2 this procedure was already described in [29] (see also [30]). Our treatment of the $s = \frac{3}{2}$ case will follow the same pattern.

Introducing decompositions of vector, tensor and gravitino fields (omitting for simplicity harmonic forms)

$$A_\mu = A_\mu^\perp + \mathscr{D}_\mu\phi, \qquad \mathscr{D}_\mu A_\mu^\perp = 0, \tag{2.12}$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \tfrac{1}{4}g_{\mu\nu}h, \qquad \bar{h}_{\mu\nu}g_{\mu\nu} = 0, \tag{2.13}$$

$$\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu}^\perp + \mathscr{D}_\mu\xi_\nu^\perp + \mathscr{D}_\nu\xi_\mu^\perp + \mathscr{D}_\mu\mathscr{D}_\nu\sigma - \tfrac{1}{4}g_{\mu\nu}\mathscr{D}^2\sigma, \tag{2.14}$$

$$\psi_\mu = \varphi_\mu + \tfrac{1}{4}\gamma_\mu\psi, \qquad \gamma_\mu\varphi_\mu = 0,$$

$$\varphi_\mu = \varphi_\mu^\perp + \left(\mathscr{D}_\mu - \tfrac{1}{4}\gamma_\mu\hat{\mathscr{D}}\right)\zeta, \qquad \mathscr{D}_\mu\varphi_\mu^\perp = 0, \qquad \hat{\mathscr{D}} \equiv \gamma_\mu\mathscr{D}_\mu, \tag{2.15}$$

we define the "constrained" operators $\Delta_s(X)$

$$\Delta_0\phi = \left(-\mathscr{D}^2 + X\right)\phi,$$

$$\Delta_{1\mu\nu}A_\nu^\perp = \left(-\mathscr{D}_{\mu\nu}^2 + g_{\mu\nu}X\right)A_\nu^\perp, \tag{2.16}$$

$$\Delta_{2\alpha\beta}^{\mu\nu}\bar{h}_{\mu\nu}^\perp = \left(-\mathscr{D}_{\alpha\beta}^{2\mu\nu} + \delta_\alpha^\mu\delta_\beta^\nu X\right)\bar{h}_{\mu\nu}^\perp, \tag{2.17}$$

$$\Delta_{1/2}\psi = \left(-\mathscr{D}^2 + \Lambda + X\right)\psi, \tag{2.18}$$

$$\Delta_{3/2\mu\nu}\varphi_\nu^\perp = \left(-\mathscr{D}_{\mu\nu}^2 + \tfrac{4}{3}\Lambda g_{\mu\nu} + g_{\mu\nu}X\right)\varphi_\nu^\perp. \tag{2.19}$$

Explicit $\Lambda$ terms in (2.18), (2.19) are convenient for correspondence with "squared" operators, e.g. $\det \Delta_{1/2} = [\det(\hat{\mathcal{D}} + \sqrt{X})]^2$. "Squaring" $\hat{\mathcal{D}}_{\mu\nu} + \sqrt{X} g_{\mu\nu}$ defined on $\varphi_\mu$ we in general get

$$(\Delta_{3/2})_\varphi = -\mathcal{D}^2_{\mu\nu} + \tfrac{1}{4} R g_{\mu\nu} - \tfrac{1}{2} R_{ab\mu\nu}\gamma^a\gamma^b + X g_{\mu\nu}. \tag{2.20}$$

Being restricted on $\varphi^\perp$, (2.20) reduces for $S^4$ (2.11) to (2.19). One can straightforwardly prove the following relations (which are true up to a "change of variables zero modes" contribution to be discussed in subsect. 2.4):

$$\det(-\mathcal{D}^2 + X)_{A_\mu} = \det \Delta_1(X) \det \Delta_0(X - \Lambda), \tag{2.21}$$

$$\det(-\mathcal{D}^2 + X)_{\bar{h}_{\mu\nu}} = \det \Delta_2(X) \det \Delta_1(X - \tfrac{5}{3}\Lambda) \det \Delta_0(X - \tfrac{8}{3}\Lambda), \tag{2.22}$$

$$\left[\det(\hat{\mathcal{D}} + \sqrt{X})_{\varphi^\perp_\mu}\right]^2 = \det \Delta_{3/2}(X) = \left[\frac{\det(\hat{\mathcal{D}} + \sqrt{X})_{\psi_\mu}}{\det(-\mathcal{D}^2 + X)_\psi}\right]^2. \tag{2.23}$$

Recalling that the irreducible representations of SO(5) are labelled by two positive integers $(n, m)$, $n \geqslant m$, with the dimension of the representation and the value of the second Casimir operator being [31, 32]:

$$d(n, m) = \tfrac{1}{6}(2m+1)(n-m+1)(n+m+2)(2n+3),$$
$$C_2 = \tfrac{1}{2}(L_{ab})^2 = n(n+3) + m(m+1), \tag{2.24}$$

let us present the information about the spectra of the operators $\Delta_s$:

*(a)* $s = 0$; $(n, 0)$ representation of SO(5),

$$\bar{\lambda}_n = n^2 + 3n + \bar{X}, \qquad d_n = \tfrac{1}{6}(n+1)(n+2)(2n+3), \qquad n = 0, 1, \ldots, \tag{2.25}$$

where we used the notation $\lambda_n = \frac{1}{3}\Lambda\bar{\lambda}_n$, $X = \frac{1}{3}\Lambda \bar{X}$ (the minimal value of $n$ is that for which $d_n > 0$);

*(b)* $s = 1$; $(n, 1)$; $A^\perp_\mu$,

$$\bar{\lambda}_n = n^2 + 3n - 1 + \bar{X}, \qquad d_n = \tfrac{3}{6} n(n+3)(2n+3), \qquad n = 1, 2, \ldots. \tag{2.26}$$

*(c)* $s = 2$; $(n, 2)$; $\bar{h}^\perp_{\mu\nu}$

$$\bar{\lambda}_n = n^2 + 3n - 2 + \bar{X}, \qquad d_n = \tfrac{5}{6}(n-1)(n+4)(2n+3), \qquad n = 2, \ldots. \tag{2.27}$$

*(d)* $s = \frac{1}{2}$; $(n \pm \frac{1}{2}, \frac{1}{2})$, both representations have the same spectra. Thus the $(n - \frac{1}{2}, \frac{1}{2})$ contribution

$$\bar{\lambda}_n = (n+1)^2 + \bar{X}, \qquad d_n = \tfrac{2}{3} n(n+1)(n+2), \qquad n = 1, \ldots, \tag{2.28}$$

is to be doubled.

*(e)* $s=\frac{3}{2}$; $(n\pm\frac{1}{2},\frac{3}{2})$; contributions of both signs are again the same; e.g. for $(n-\frac{1}{2},\frac{3}{2})$ we have

$$\bar{\lambda}_n=(n+1)^2+\bar{X},\qquad d_n=\tfrac{4}{3}(n-1)(n+1)(n+3),\qquad n=2,\ldots. \quad (2.29)$$

Let us briefly comment on the derivation of (2.29). The first problem is to identify the parts of $\psi_\mu$ with irreps of SO(5). Starting with a 5-vector spinor $\psi_A$, we split it according to $(1,0)\times(\frac{1}{2},\frac{1}{2})=(\frac{3}{2},\frac{1}{2})+(\frac{1}{2},\frac{1}{2})$, e.g., on $\psi_\mu$ and $\psi_5$. Multiplying by $(n,0)$, carried by the scalar spherical functions, we get: $(\frac{3}{2},\frac{1}{2})\times(n,0)=\Sigma_\pm\{(n\pm\frac{1}{2},\frac{3}{2})+(n\pm\frac{1}{2},\frac{1}{2})+(n\pm\frac{3}{2},\frac{1}{2})\}$, where the three terms correspond to $\varphi_\mu^\perp$, $\zeta$ and $\psi$ in (2.15). Next, we are to connect the known value of $C_2$ (2.24) with the eigenvalue $(\bar{\lambda})^{1/2}$ of $\hat{\mathfrak{D}}_{\mu\nu}$ acting on $\varphi_\mu^\perp$: $C_2=\bar{\lambda}+\frac{3}{2}$. This relation follows either from an explicit $4+1$ analysis of a 5-dimensional gravitino operator or from observation that $\varphi_\mu^\perp$ can be viewed as a spinor with the additional $(1,0)$ internal index, so that $C_2=\bar{\lambda}+1\times3-\frac{3}{2}$ (see, e.g. [33]).

### 2.3. EXPRESSIONS FOR THE $\zeta$ FUNCTIONS

Now we are ready for the computation of the finite part of the effective action (2.6). Introducing the function

$$F(p,k,a,b)=\sum_{\nu=\frac{1}{2}k+1}^{\infty}\frac{\nu(\nu^2-a)}{(\nu^2-b)^p},\qquad \operatorname{Re}p>2, \quad (2.30)$$

(where $k$ is an integer and $\Delta\nu=1$) defined for all $p$ by analytic continuation, we notice that the $\zeta$ functions for the operators $\Delta_s$ (2.16)–(2.19) with the spectra (2.25)–(2.29) can be written as*

$$\zeta_s(p)=\tfrac{1}{3}(2s+1)F\left(p,2s+1,\left(s+\tfrac{1}{2}\right)^2,b_s\right), \quad (2.31)$$

where

$$b_0=\tfrac{9}{4}-\bar{X},\qquad b_1=\tfrac{13}{4}-\bar{X},\qquad b_2=\tfrac{17}{4}-\bar{X},$$

$$b_{1/2}=-\bar{X},\qquad b_{3/2}=-\bar{X}. \quad (2.32)$$

Starting with (2.30) it is easy to prove that (cf. [33, 30])**

$$F(p=0)=\tfrac{1}{4}b(b-2a)+\tfrac{1}{24}a(3k^2+6k+2)-\tfrac{1}{64}k^2(k+2)^2+\tfrac{1}{120}. \quad (2.33)$$

* Let us stress once more that all sums start with the minimal value of $\nu$ for which $d_n>0$ and thus possible negative and zero modes are included.

** Note a misprint in eq. (10) of Chadha et al. [33].

In an analogous way, we found

$$F(p=1)=\tfrac{1}{2}b-\tfrac{1}{12}-\tfrac{1}{8}k(k+2)-\tfrac{1}{2}(b-a)\Psi\Big(\frac{k}{2}+1\pm\sqrt{b}\Big),$$

$$\Psi(x\pm y)\equiv\psi(x+y)+\psi(x-y), \tag{2.34}$$

where $\psi$ is a logarithmic derivative of Euler's $\Gamma$ function. Observing that

$$\frac{\mathrm{d}}{\mathrm{d}b}F'(p=0)=F(p=1),\qquad F'=\frac{\mathrm{d}F}{\mathrm{d}p}, \tag{2.35}$$

we can find $F'(p=0)$ (and thus all the $\zeta'(0)$) integrating (2.35)*

$$F'(p=0)=\tfrac{1}{4}b^2-\tfrac{1}{12}b-\tfrac{1}{8}bk(k+2)-\tfrac{1}{2}\int_0^b\mathrm{d}z\,(z-a)\times\Psi\big(\tfrac{1}{2}k+1\pm\sqrt{z}\big)+C, \tag{2.36}$$

$$C=(F')_{p=b=0}=2\zeta_{\mathrm{R}}'\big(-3,\tfrac{1}{2}k+1\big)-2a\zeta_{\mathrm{R}}'\big(-1,\tfrac{1}{2}k+1\big). \tag{2.37}$$

Eq. (2.37) follows from (2.30) and the definition of the generalized Riemann $\zeta$ function: $\zeta_{\mathrm{R}}(p,q)=\sum_{n=q}^{\infty}n^{-p}$, $\operatorname{Re}p>1$, $\zeta_{\mathrm{R}}'=\mathrm{d}\zeta_{\mathrm{R}}/\mathrm{d}p$. In what follows we shall use the following notation

$$F_s(b_s)\equiv F\big(1,2s+1,(s+\tfrac{1}{2})^2,b_s\big),$$

$$F_s'(b_s)\equiv F'\big(0,2s+1,(s+\tfrac{1}{2})^2,b_s\big). \tag{2.38}$$

To establish the form of the effective equations we actually need to know only $F_s$ (2.34) because all the field dependence of $\zeta'(0)$ in (2.7) is contained in $b_s$ (2.32). That is why we illustrate the derivation of (2.34) in some detail. If one understands the infinite sum (2.30) for $\operatorname{Re}p\leqslant 2$ being regulated by a cut-off, $\nu\leqslant N$, then formally $F_{(N)}'(p=0)=-\sum_{\nu=t}^{N}\nu(\nu^2-a)\ln(\nu^2-b)$ and

$$\frac{\mathrm{d}}{\mathrm{d}b}F_{(N)}'(p=0)=F_{(N)}(p=1)=\sum_{\nu=t}^{N}\nu(\nu^2-a)/(\nu^2-b),\qquad t\equiv\tfrac{1}{2}k+1.$$

Using the formula [34]

$$\sum_{n=0}^{m-1}(n+C)^{-1}=\psi(m+C)-\psi(C), \tag{2.39}$$

* This trick is analogous to that used in [15,16] to obtain the $s=0$ and $s=\frac{1}{2}$ effective actions in terms of integrals over the mass parameter.

and noting that $\psi(N+C) \underset{(N\to\infty)}{\to} \ln(N+C)+O(1/N^2)$, we get

$$F_{(N)}(p=1)=\tfrac{1}{2}N(N-1)-\tfrac{1}{2}t(t-1)$$
$$+\tfrac{1}{2}(b-a)\left[2\ln N-\Psi(t\pm\sqrt{b})\right]+O(1/N). \qquad (2.40)$$

By noting that $\frac{1}{2}N(N-1)=\sum_{n=0}^{N-1}n\to\zeta_R(-1,0)=-\frac{1}{12}$, etc., we find correspondence with the result (2.34) of analytic continuation.

### 2.4. CHANGES OF VARIABLES AND ZERO MODES

To prove (2.21)–(2.23) one is to change the variables according to (2.12)–(2.15). The resulting jacobians are given by

$$\mathrm{d}A_\mu\to\mathrm{d}A_\mu^\perp\,\mathrm{d}\phi\left[\det J_1\right]^{1/2},\qquad J_1=\Delta_0(0),$$
$$\mathrm{d}\bar{h}_{\mu\nu}\to\mathrm{d}\bar{h}_{\mu\nu}^\perp\,\mathrm{d}\xi_\lambda^\perp\,\mathrm{d}\sigma\left[\det J_2\right]^{1/2},\qquad J_2=\Delta_1(-\Lambda)\otimes\Delta_0(-\tfrac{4}{3}\Lambda)\otimes\Delta_0(0),$$
$$\mathrm{d}\varphi_\mu\to\mathrm{d}\varphi_\mu^\perp\,\mathrm{d}\zeta\left[\det J_{3/2}\right]^{-1/2},\qquad J_{3/2}=\Delta_{1/2}(-\tfrac{4}{3}\Lambda). \qquad (2.41)$$

These $J_s$ factors are cancelled by analogous ones arising after explicit substitution of (2.12)–(2.15) in $A_\mu(-\mathscr{D}^2+X)A_\mu,\dots$ and subsequent integration over $A_\mu^\perp,\phi,\dots$. However, this cancellation is true only for *non-zero* modes: the important point is that the decomposition like (2.12)–(2.15) *introduces additional zero modes* not present for an initial unconstrained operator (for example, in $A_\mu(-\mathscr{D}^2+X)A_\mu=A_\mu^\perp\Delta_1(X)A_\mu^\perp+\phi\Delta_0(X-\Lambda)\Delta_0(0)\phi$ the $\phi$ part has a zero mode $\phi=\text{const}$, while $-\mathscr{D}^2+\Lambda$ *does not* have a *non-trivial* zero mode corresponding to it). Thus in general

$$\det\Delta=(\det J)^{-1}\Big(\prod_s\det\Delta_s\det{}'J\Big), \qquad (2.42)$$

where the prime means that zero modes are to be omitted. As a result,

$$B_4(\Delta)=\sum_s\zeta_s(0)-\mathscr{N}(J), \qquad (2.43)$$

with $\mathscr{N}$ being the number of zero modes of the jacobian. For example, for (2.41) we get (using (2.25)–(2.28))*

$$\mathscr{N}(J_1)=1,\qquad\mathscr{N}(J_2)=10+5+1=16,\qquad\mathscr{N}(J_{3/2})=4+4=8^{**}. \qquad (2.44)$$

* These numbers are related to those of harmonic forms, Killing tensors and spinors on $S^4$.

** For example, $B_4(\hat{\mathscr{D}}_{\psi_\mu})=\frac{1}{2}B_4(\hat{\mathscr{D}}^2_{\psi_\mu})=\frac{1}{2}\sum_s\zeta_s(0)-4$.

A useful check of the consistency of the previous discussion is to compute $\Sigma_s \zeta_s(0)$ for (2.21)–(2.23) using (2.31)–(2.33) and compare the right-hand side of (2.43) with $B_4$ calculated for unconstrained operators with the help of the standard algorithm (2.8). In view of the fact that it is the $\Delta_s$ representation which is well-suited for the calculation of finite parts, we shall write (for notational simplicity) effective actions only in terms of $\Sigma_s \ln\det\Delta_s$, keeping, however, in mind that $B_4$ in (2.7) must be calculated using "unconstrained" $\Delta$ representations or equivalently, employing (2.43).

Finally, we want to note that the important difference between $B_4$ and $\Sigma_s \zeta_s(0)$ was first pointed out in [30]. However, the true origin of this difference, namely, $\mathcal{N}(J_s) \neq 0$ in (2.43), was not clearly revealed there, because the authors used the standard definition of $\zeta$ functions when only *positive* modes are included in the sum. As a consequence, $B_4 \neq \zeta(0)$ already for a scalar (or any unconstrained) operator, if it has zero or negative modes, which makes comparison with $B_4$ rather indirect. In view of the above discussion we also do not completely agree with the criticism of the constrained operator representation of the effective action, expressed in [30].

## 3. Examples of the calculation of effective potentials in de Sitter space

### 3.1. SELF-INTERACTING SCALAR FIELD THEORY

To illustrate the general strategy of establishing effective equations we are going to consider here two simple examples, interesting in their own right. The first is provided by

$$\mathcal{L} = -\frac{1}{k^2}(R - 2\Lambda_0) + \tfrac{1}{2}(\partial_\mu\phi)^2 + \frac{\lambda}{4!}\phi^4 + \tfrac{1}{2}\xi R\phi^2, \qquad k^2 = 16\pi G. \tag{3.1}$$

Supposing gravity to be classical we get the following one-loop effective action for gravitational de Sitter space (2.11) and constant scalar backgrounds

$$\Gamma_1 = \tfrac{1}{2}\ln\det\frac{\Delta_0(X)}{\mu^2}, \qquad X = \tfrac{1}{2}\lambda\phi^2 + 4\xi\Lambda. \tag{3.2}$$

$\Lambda$ and $\phi$ are to be determined from the effective equations. Such procedure is based on the assumption that the full effective gravitational equations $\delta\Gamma/\delta g_{\mu\nu} = 0$ admit a de Sitter solution and so one has simply to find a correct value of $\Lambda$. This assumption is known to be true for the $\lambda = 0$ and $\xi = \frac{1}{6}$ case when $\Gamma_1$ can be explicitly evaluated for a conformally flat metric, integrating the known expression for the conformal anomaly (see, e.g. [35, 36]). As for the more general "massive" case there are also good reasons to hope for a de Sitter-like solution (cf. [36, 37]). In any case, the above assumption provides us with a simple possibility to account for the back reaction of quantized matter in background gravity (more exact treatment, including the representation of (3.2) in terms of an eigenvalue sum for $\Delta_0$ on an arbitrary conformally flat background, is rather cumbersome and is difficult to carry out for $s > 0$ spin fields).

Using the results of the previous paragraph (2.6)–(2.8), (2.31), (2.36) we get

$$B_0 = \tfrac{3}{2}y^2, \qquad B_2 = y(1-6\xi) - \tfrac{3}{4}\lambda xy,$$
$$B_4 = \zeta_0(0) = \tfrac{29}{90} + 12\xi^2 - 4\xi + \tfrac{3}{16}\lambda^2 x^2 + \tfrac{1}{2}(6\xi - 1)\lambda x,$$
$$y \equiv 1/\Lambda, \qquad x \equiv \phi^2/\Lambda. \tag{3.3}$$

All infinities can be absorbed by renormalization; working in the $\zeta$-function prescription (2.7) we have

$$\Gamma = I + \Gamma_1 = 24\pi^2\left[-\frac{2}{k^2}(2y - \Lambda_0 y^2) + \tfrac{1}{24}\lambda x^2 + 2\xi x\right]$$
$$+ \tfrac{1}{2}B_4 \ln\frac{1}{3\mu^2 y} - \tfrac{1}{6}\left\{\tfrac{1}{4}b_0^2 - \tfrac{11}{24}b_0 - \tfrac{1}{2}\int_0^{b_0} \mathrm{d}z\,(z - \tfrac{1}{4})\Psi\left(\tfrac{3}{2} \pm \sqrt{z}\right)\right\} + \text{const}, \tag{3.4}$$

where $b_0 = \frac{9}{4} - 12\xi - \frac{3}{2}\lambda x$. Eq. (3.4) is in clear correspondence with the results of refs. [15, 16] for a free massive scalar field, obtained by essentially more complicated methods. The effective equations

$$\partial\Gamma/\partial\Lambda = \left(\frac{\partial\Gamma}{\partial y} + \phi^2\frac{\partial\Gamma}{\partial x}\right)\left(-\frac{1}{\Lambda^2}\right) = 0, \qquad \frac{\partial\Gamma}{\partial\phi} = \frac{2\phi}{\Lambda}\frac{\partial\Gamma}{\partial x} = 0,$$

are equivalent to $\partial\Gamma/\partial x = \partial\Gamma/\partial y = 0$ if $\phi \neq 0$. If $\phi = 0$ we get only one equation for $\Lambda$

$$\frac{96\pi^2}{\Lambda k^2}(\Lambda_0 - \Lambda) - \tfrac{1}{2}\hbar B_4(x=0) = 0, \qquad B_4(x=0) = \tfrac{29}{90} + 12\xi^2 - 4\xi. \tag{3.5}$$

If $\Lambda_0 \neq 0$ and $\Lambda_0 k^2 \ll 1$ we get a trustworthy perturbative solution: $\Lambda = \Lambda_0 + O(\hbar\Lambda_0 k^2)$. If $\Lambda_0 k^2 \sim 1$ the quantum correction in (3.5) has the same order as the classical term and one cannot neglect quantum gravity. Essentially the "quantum" solution appears also if $\Lambda_0 = 0$; for $\xi = \frac{1}{6}$ we find

$$\Lambda = 17280\pi^2/\hbar k^2. \tag{3.6}$$

The existence of such a solution was first noted in [13] (see also [36]). Though the result (3.6) lies beyond the range of applicability of the approximation, neglecting quantum-gravitational effects, its analog will be shown to exist also when gravity is quantized (sects. 4, 5). Formally, (one-loop) contributions of free fields of any spin are all of the same order and thus a solution with $\Lambda k^2$ of the order of some natural number is always supposed to exist. As a consequence, an enormous cosmological constant is induced by quantum corrections in all reasonable theories, *including supersymmetric ones*. Global supersymmetry helps to cancel quartic ($B_0$) and quadratic ($B_2$) divergences [38], but not *finite* contributions to $\Lambda$ governed by $B_4$ (cf. (3.5)). For example, $B_4 \neq 0$ even in $N=4$ super-Yang-Mills theory. It is only in theories with antisymmetric tensors with a zero (curved space) conformal anomaly ($B_4 = 0$) [39], where the situation might be improved (see also sects. 5, 6). If $\phi \neq 0$ we have

$$0 = \frac{\partial \Gamma}{\partial y} = \frac{96\pi^2}{k^2}(\Lambda_0 y - 1) - \frac{\hbar}{2y}B_4, \tag{3.7}$$

$$\begin{aligned} 0 = \frac{\partial \Gamma}{\partial x} &= 24\pi^2\left(2\xi + \tfrac{1}{12}\lambda x\right) \\ &+ \tfrac{1}{4}\hbar\lambda\left[\left(6\xi - 1 + \tfrac{3}{4}\lambda x\right)\ln\frac{1}{3\mu^2 y} + \tfrac{2}{3} - 6\xi - \tfrac{3}{4}\lambda x \right. \\ &\left. + \left(6\xi - 1 + \tfrac{3}{4}\lambda x\right)\Psi\left(\tfrac{3}{2} \pm \sqrt{\tfrac{9}{4} - 12\xi - \tfrac{3}{2}\lambda x}\right)\right], \end{aligned} \tag{3.8}$$

(the latter easily follows from (2.34), (2.35)). If $\xi < 0$ then quantum effects give small corrections to the classical solution $\Lambda = \Lambda_0$, $x = -24\lambda^{-1}\xi$. If $\xi = 0$ and $\Lambda \to 0$ then (3.8) reduces to the Coleman-Weinberg-type equation with the non-perturbative solution lying away from the domain of applicability of the one-loop approximation [18]. If $\xi > 0$ no perturbative solution exists while the non-perturbative one is again untrustworthy. Let us formally ignore this difficulty in order to illustrate an interesting mechanism of *natural* suppression of the $\Lambda$ term, which may be operative in a more realistic context. Namely, let us take $\xi = \frac{1}{6}$ and assume that $\lambda < 1$, $\sigma \equiv \lambda x < 1$. Then (3.8) reduces to

$$\Lambda \simeq 3\mu^2 \exp\left(-128\pi^2/3\lambda^2 x\right), \tag{3.9}$$

providing a natural hierarchy between the scale of $\Lambda$ and the Planck (or GUT) scale*. Still this is not a final solution of the $\Lambda$-term problem, because to satisfy (3.7) we need fine tuning of $\Lambda_0$.

For the application of the above theory to the early Universe, one is to take $\mu \sim m_X$ and to fine tune $\Lambda_0$ so that we get $\Lambda \simeq 0$ for the $\phi \neq 0$ solution. Then

* The known experimental bound $|\Lambda k^2| < 10^{-122}$ suggests such an $\exp(-c/g^2)$ mechanism.

quantum corrections in the $\partial\Gamma/\partial\Lambda = 0$ equation (i.e. the back reaction of quantized matter) can be neglected and one simply has to study the minima of the effective potential for a given $\Lambda$ (cf. [9,10]). The only problem is whether the one-loop approximation is believable. This problem can in principle be solved as in flat space [18] by introducing gauge fields.

### 3.2. SCALAR ELECTRODYNAMICS

Our next example is the following theory ($a = 1,2$)

$$\mathcal{L} = -\frac{1}{k^2}(R - 2\Lambda_0) + \frac{1}{4g^2}F^2_{\mu\nu} + \tfrac{1}{2}(\mathcal{D}_\mu\phi^a)^2 + \frac{\lambda}{4!}(\phi^a\phi_a)^2 + \tfrac{1}{2}\xi R(\phi^a\phi_a). \quad (3.10)$$

Using the general class of gauges ($\phi$ is the background and $\varphi$ is the quantum scalar fields)

$$\mathcal{L}_g = \frac{1}{2\alpha}\left(\partial_\mu A_\mu + \beta g \varepsilon_{ab}\phi_a\varphi_b\right)^2, \quad (3.11)$$

it is straightforward to find the corresponding one-loop effective action on the de Sitter background (see (2.16) for our notation)

$$\Gamma = I + \tfrac{1}{2}\{\ln\det\Delta_1(\Lambda + g^2\phi^2) + \ln\det\Delta_0(\tfrac{1}{2}\lambda\phi^2 + 4\Lambda\xi)$$
$$+ \ln\det B + \ln\det E - 2\ln\det\Delta_0(\beta g^2\phi^2)\}, \quad (3.12)$$

$$B = -\mathcal{D}^2 + (\tfrac{1}{6}\lambda + \beta^2\alpha^{-1}g^2)\phi^2 + 4\Lambda\xi,$$
$$E = -\mathcal{D}^2 + \alpha g^2\phi^2 + \alpha B^{-1}g^2\phi^2(1-\beta/\alpha)^2\mathcal{D}^2. \quad (3.13)$$

Here we made use of the decomposition (2.12) and thus are to take into account (2.43), (2.44). To illustrate the gauge dependence of $\Gamma$ it is sufficient to put $\alpha = \beta$. Then

$$B_4 = \sum_i \zeta_i(0) - 1 = -\tfrac{2}{45} + 24\xi^2 - 8\xi + \tfrac{1}{4}x^2(9g^4 + \tfrac{5}{6}\lambda^2 + 2\alpha\lambda g^2)$$
$$+ x(3g^2 + 6\alpha\xi g^2 + 4(\xi - \tfrac{1}{6})\lambda), \quad (3.14)$$

where $x = \phi^2/\Lambda$ as in (3.3). Instead of writing down the explicit expression for the effective potential (following from (2.31), (2.32), (2.36))* we directly pass to the effective equations for $\phi \neq 0$. They easily follow from (2.34), (2.35)

$$\frac{\partial \Gamma}{\partial y} = \frac{96\pi^2}{k^2}(\Lambda_0 y - 1) - \frac{1}{2y} B_4 = 0, \tag{3.15}$$

$$\begin{aligned} \frac{\partial \Gamma}{\partial x} = {} & 24\pi^2\left(2\xi + \tfrac{1}{12}\lambda x\right) + \frac{1}{2}\frac{\partial B_4}{\partial x} \ln \frac{1}{3\mu^2 y} \\ & - \tfrac{1}{6}\Big\{ -9g^2\Big[-\tfrac{11}{6} - \tfrac{3}{2}g^2 x + \left(\tfrac{3}{2}g^2 x + 1\right)\Psi\Big(\tfrac{5}{2} \pm \sqrt{\tfrac{1}{4} - 3g^2 x}\Big)\Big] \\ & \quad + \tfrac{3}{2}\lambda\Big[-\tfrac{11}{24} + 6\xi + \tfrac{3}{4}\lambda x - \left(\tfrac{3}{4}\lambda x + 6\xi - \tfrac{1}{8}\right)\Psi\Big(\tfrac{3}{2} \pm \sqrt{\tfrac{3}{2}\lambda x + 12\xi}\Big)\Big] \\ & \quad + \left(\tfrac{1}{2}\lambda + 3\alpha g^2\right)\Big[-\tfrac{11}{24} + 6\xi + \tfrac{1}{4}\lambda x + \tfrac{3}{2}\alpha g^2 x - \left(\tfrac{1}{4}\lambda x + \tfrac{3}{2}\alpha g^2 x + 6\xi - \tfrac{1}{8}\right) \\ & \qquad \times \Psi\Big(\tfrac{3}{2} \pm \sqrt{\tfrac{1}{2}\lambda x + 3\alpha g^2 x + 12\xi}\Big)\Big] \\ & \quad - 3\alpha g^2\Big[-\tfrac{11}{24} + \tfrac{3}{2}\alpha g^2 x - \left(\tfrac{3}{2}\alpha g^2 x - \tfrac{1}{8}\right)\Psi\Big(\tfrac{3}{2} \pm \sqrt{3\alpha g^2 x}\Big)\Big]\Big\} = 0. \end{aligned} \tag{3.16}$$

To obtain a reasonable one-loop solution for $\phi$ which survives in higher orders in flat space one is to take $\lambda \sim g^4$ [18]. Such an assumption makes it possible to omit all gauge-dependent contributions in $\Gamma$ and thus to prove the gauge independence of the relevant physical quantities. Here we want to stress, that *in curved space* this condition alone is *not sufficient* for a gauge-independent approximation *if* $\xi \neq 0$. In fact, $\alpha$-dependent terms are present in (3.15), (3.16) even for $\lambda \sim g^4$. This implies that an additional assumption like $\xi \sim O(g^2)$ is needed in order to make the Coleman-Weinberg mechanism work in curved space**. This fact seems not to be well understood in recent studies of the early universe (cf. [9, 10]).

To obtain an appropriate solution for $\Lambda$ on a GUT scale fine tuning of $\Lambda_0$ is again needed, because (3.15) *naturally* predicts the $\Lambda$ of Planck order (cf. (3.6)). On smaller scales the back reaction of quantized matter (already accounted for in (3.15), (3.16)) and quantum-gravitational corrections are to be taken into consideration.

## 4. Einstein gravity with the cosmological constant

As a next example on the way to $N = 4$ supergravity we shall calculate here the one-loop effective action for quantized gravity on a de Sitter background, assuming that $S^4$ is the solution of full effective equations. To give an idea of our program

* Note that our approach is much simpler than that of Shore [9], who used the $\alpha = 1$, $\beta = 0$ gauge.

** Note that $\xi = \frac{1}{6}$ is thus excluded.

consider the classical action on the $S^4$ background

$$I = -\frac{1}{k^2}\int d^4x\sqrt{g}\,(R - 2\Lambda_0) = \frac{48\pi^2}{\Lambda_0 k^2}(y^2 - 2y), \qquad y = \Lambda_0/\Lambda. \qquad (4.1)$$

Varying $y$ (or $\Lambda$) we get the classical solution $y = 1$ (or $\Lambda = \Lambda_0$). In the same manner $\partial\Gamma/\partial\Lambda = 0$ will give the value of the effective parameter $\Lambda$. Prior to any physical interpretation of $\Lambda$ one is to solve problems of gauge dependence, off-shell infinities and the reliability of the one-loop approximation*. Still we consider the above program worth following, because there seems to be practically no off-shell (i.e. non-semiclassical) calculations of the *finite* part of the *gravitational* effective action (see, however, [40]) while the de Sitter background provides (a physically interesting) *exactly solvable example*. Also, it illustrates important theoretical issues concerning the gauge dependence of off-shell problems.

Let us start with the bilinear part of the Einstein lagrangian on $S^4$ background ([29, 30]) using the decomposition (2.13), (2.14)

$$\begin{aligned}\mathcal{L}_2 &= \frac{1}{2k^2}\Big[\tfrac{1}{2}\bar{h}_{\mu\nu}\big(-\mathcal{D}^2 + \tfrac{8}{3}\Lambda - 2\Lambda_0\big)\bar{h}_{\mu\nu} \\ &\qquad - \tfrac{1}{8}h\big(-\mathcal{D}^2 - 2\Lambda_0\big)h - \big(\mathcal{D}_\mu\bar{h}_{\mu\nu} - \tfrac{1}{4}\mathcal{D}_\nu h\big)^2\Big] \\ &= \frac{1}{2k^2}\Big\{\tfrac{1}{2}\bar{h}^\perp\Delta_2\big(\tfrac{8}{3}\Lambda - 2\Lambda_0\big)\bar{h}^\perp + 2(\Lambda - \Lambda_0)\xi^\perp\Delta_1(-\Lambda)\xi^\perp \\ &\qquad - \tfrac{3}{16}\big[\sigma\Delta_0(0)\Delta_0\big(-\tfrac{4}{3}\Lambda\big)\Delta_0(4\Lambda_0 - 4\Lambda)\sigma \\ &\qquad + 2\sigma\Delta_0(0)\Delta_0\big(-\tfrac{4}{3}\Lambda\big)h + h\Delta_0\big(-\tfrac{4}{3}\Lambda_0\big)h\big]\Big\}. \end{aligned} \qquad (4.2)$$

The simplicity of the de Sitter background gives as a possibility to calculate $\Gamma$ explicitly for a wide class of gauges; this is important in attempting to extract gauge-independent information. Let us start with the standard covariant background gauges

$$\mathcal{L}_g = \frac{\gamma}{2k^2}\big(\mathcal{D}_\mu\bar{h}_{\mu\nu} - \tfrac{1}{4}\beta\mathcal{D}_\nu h\big)^2, \qquad (4.3)$$

($\gamma = \beta = 1$ corresponds to the de Donder gauge). Changing the variables according

* In fact, even the assumption that $S^4$ can be an effective geometry may be criticized in view of the conformal non-invariance of Einstein theory.

to (2.41) and integrating $\exp(-I_2 - I_g)$ over $\bar{h}^{\perp}$, $\xi^{\perp}$ and $\sigma$ and ghosts we finally get

$$Z(\gamma,\beta) = e^{-\Gamma_1} = \det \Delta_1(-\Lambda)\left[\det \Delta_1\left(\Lambda(2\gamma^{-1}-1) - 2\gamma^{-1}\Lambda_0\right)\right]^{-1/2}$$
$$\times\left[\det \Delta_2\left(\tfrac{8}{3}\Lambda - 2\Lambda_0\right)\right]^{-1/2} \det \Delta_0\left(4\Lambda_0(\beta-3)^{-1}\right)$$
$$\times\left[\det\left(\mathfrak{D}^4 + (a\Lambda + b\Lambda_0)\mathfrak{D}^2 + c\Lambda_0^2 + d\Lambda\Lambda_0\right)\right]^{-1/2}, \quad (4.4)$$

$$a = \frac{4}{(\beta-3)^2}\left(\beta^2 + 3 - 2\beta - 2\gamma^{-1}\right), \qquad b = \frac{4}{(\beta-3)^2}\left(-\beta^2 + 3 + 2\gamma^{-1}\right),$$

$$c = \frac{16}{\gamma(\beta-3)^2}, \qquad d = c(\gamma - 1). \quad (4.5)$$

It is straightforward to calculate the infinite part of $\Gamma_1$ either by converting all operators into unconstrained ones (using (2.21), (2.22) and employing the $B_4$ algorithm) or computing $\Sigma_i \zeta_i(0) - \mathfrak{N}$, $\mathfrak{N} = 16 - 2 = 14$ according to (2.43), (2.44) (ghosts *are also decomposed* in (4.4)). On mass shell $\Lambda = \Lambda_0$ (4.4) reduces to the *gauge-independent* expression (first found in [29]; see also [30, 25])

$$Z_{\text{on shell}} = \left[\frac{\det \Delta_1(-\Lambda)}{\det \Delta_2\left(\tfrac{2}{3}\Lambda\right)}\right]^{1/2}, \quad (4.6)$$

(we omit zero-mode contributions, recalling subsect 2.4). It might be helpful to rewrite (44) explicitly for some simple choices of $\beta$ and $\gamma$*

$$Z(1,1) = Z_0\left[\frac{\det \Delta_1(-\Lambda)}{\det \Delta_1(\Lambda - 2\Lambda_0)}\right]^{1/2} \frac{\det \Delta_0(-2\Lambda)}{\det \Delta_0(-2\Lambda_0)}, \quad (4.7)$$

$$Z(\infty,\infty) = Z_0\left[\frac{\det \Delta_0(0)}{\det \Delta_0(4\Lambda_0 - 4\Lambda)}\right]^{1/2}, \quad (4.8)$$

$$Z(\infty,1) = Z_0\left[\frac{\det \Delta_0(-2\Lambda)}{\det \Delta_0(-2\Lambda_0)}\right]^{1/2}, \quad (4.9)$$

$$Z_0 \equiv \left[\frac{\det \Delta_1(-\Lambda)}{\det \Delta_2\left(\tfrac{8}{3}\Lambda - 2\Lambda_0\right)}\right]^{1/2}. \quad (4.10)$$

Now let us consider another class of gauges more suited for the calculation of the

* The "Landau" gauge $\gamma = \infty$, $\beta = 1$ is distinguished by the fact that, as was noted in [26], the corresponding $\Gamma$ coincides (for the $S^4$ background) with the off-shell effective action defined according to Vilkovisky [22].

finite part of $\Gamma_1$. Observing that the gauge transformation $\delta h_{\mu\nu} = \mathcal{D}_\mu \varepsilon_\nu + \mathcal{D}_\nu \varepsilon_\mu$ implies $\delta \bar{h}^{\perp}_{\mu\nu} = 0$, $\delta \xi^{\perp}_\mu = \varepsilon^{\perp}_\mu$, $\delta\sigma = 2\varepsilon$, $\delta h = 2\mathcal{D}^2\varepsilon$, $\varepsilon_\mu = \varepsilon^{\perp}_\mu + \mathcal{D}_\mu \varepsilon$, we can choose the following gauge

$$h = 0, \qquad \xi^{\perp}_\mu = a^{\perp}_\mu(x). \tag{4.11}$$

Averaging over $a^{\perp}_\mu$ with the help of the operator $H = (1/2r)\Delta_1(p\Lambda + q\Lambda_0)\Delta_1(-\Lambda)$ ($p, q, r$ are gauge parameters), i.e. using the following gauge breaking term

$$\mathcal{L}_g = \frac{1}{k^2 r}\xi^{\perp}\Delta_1(p\Lambda + q\Lambda_0)\Delta_1(-\Lambda)\xi^{\perp}, \tag{4.12}$$

we find (after proper inclusion of ghost and $H$ determinants and the jacobian in (2.41))*

$$Z(p,q,r) = \left[\frac{\det \Delta_1(-\Lambda)}{\det \Delta_2(\frac{8}{3}\Lambda - 2\Lambda_0)}\right]^{1/2}\left[\frac{\det \Delta_1(p\Lambda + q\Lambda_0)}{\det \Delta_1((p+r)\Lambda + (q-r)\Lambda_0)}\right]^{1/2}$$
$$\times\left[\frac{\det \Delta_0(0)}{\det \Delta_0(4\Lambda_0 - 4\Lambda)}\right]^{1/2}, \tag{4.13}$$

which again coincides with (4.6) on shell. The simplest gauge is $r = 0$, where (4.13) is equivalent to (4.8). To study the effective action (2.6), (2.7) for (4.13) we need first to know the $B_p$ coefficients; a long calculation gives

$$B_0 = \frac{3}{\Lambda^2}, \qquad B_2 = \frac{3}{2\Lambda^2}(\rho_1\Lambda + \rho_2\Lambda_0), \tag{4.14}$$

$$\rho_1 = -3r - \tfrac{50}{3}, \qquad \rho_2 = 3r + 6,$$

$$B_4 = \sum_i \zeta_i(0) - 14 \equiv \gamma_1 + \gamma_2 y + \gamma_3 y^2,$$

$$\gamma_1 = \tfrac{9}{4}r^2 + \tfrac{9}{2}pr - \tfrac{3}{2}r + \tfrac{1724}{45},$$

$$\gamma_2 = -\tfrac{9}{2}r^2 + \tfrac{9}{2}qr - \tfrac{9}{2}pr + \tfrac{3}{2}r - 78,$$

$$\gamma_3 = \tfrac{9}{4}r^2 - \tfrac{9}{2}qr + 27, \qquad y = \Lambda_0/\Lambda, \tag{4.15}$$

(for comparison, in the de Donder gauge $\rho_1 = -\frac{92}{3}$, $\rho_2 = 20$, $\gamma_1 = \frac{419}{45}$, $\gamma_2 = -52$, $\gamma_3 = 30$ [25]). On shell $y = 1$ and $B_4 = -\frac{571}{45}$ in agreement with [30]. Even neglecting power divergences, we confront the problem of off-shell non-renormalizability

* It should be noted that the well-known indefiniteness of the Einstein action [41] manifests itself in the $h = 0$ gauge in the necessity of rotating the contour of integration over $\sigma$ ($\sigma \to i\sigma$) (cf. (4.2)).

(compare (4.1) and (4.15)). It is thus interesting to note that there exists a one-parameter class of gauges ($r \neq 0$)

$$q = \frac{r^2+12}{2r}, \qquad p = \frac{-3r^2+2r-68}{6r}, \tag{4.16}$$

in which the theory is free from $y$-dependent logarithmic infinities (i.e. $\gamma_2 = \gamma_3 = 0$). The constant $B_4 \ln L^2$ infinity ($B_4 = -\frac{571}{45}$) can be absorbed in renormalization of the topological coupling constant, if we add

$$I_\chi = \alpha\chi, \qquad \chi = \frac{1}{32\pi^2}\int R^*R^*\sqrt{g}\,\mathrm{d}^4x \tag{4.17}$$

($\chi = 2$ for $S^4$). Starting with (4.13) and using the $\zeta$-function prescription (2.7), (2.31), (2.32), (2.38) we get

$$\begin{aligned}\Gamma = I &+ \tfrac{1}{2}B_4 \ln\frac{\Lambda_0}{3\mu^2 y} - \tfrac{1}{6}\{5F_2'(-\tfrac{15}{4}+6y)\\ &-3F_1'(\tfrac{25}{4}) + 3F_1'(\tfrac{13}{4} - 3(p+r) - 3(q-r)y)\\ &-3F_1'(\tfrac{13}{4} - 3p - 3qy) + F_0'(\tfrac{57}{4} - 12y) - F_0'(\tfrac{9}{4})\}.\end{aligned} \tag{4.18}$$

On shell ($y = 1$) it reduces to (cf. (4.1), (4.6))

$$\Gamma_{\text{on shell}} = -\frac{48\pi^2}{\lambda_0} + \tfrac{1}{2}\hbar(-\tfrac{571}{45})\ln\frac{\lambda_0}{3\mu^2k^2} + \text{const}, \qquad \lambda_0 \equiv \Lambda_0 k^2, \tag{4.19}$$

giving us the one-loop corrected de Sitter vacuum action (a complete semiclassical analysis should include treatment of zero modes etc., cf. [29]). We remark that (4.19) is real because the operators in (4.6) have no negative modes according to (2.26), (2.27)*. Raising an analogous question for (4.13) we get the following conditions for the absence of negative modes (assuming, e.g. $r = 0$): $1 \leqslant y \leqslant \frac{8}{3}$. Hence any extremum of (4.18) lying out of this interval will be complex. Varying (4.18) with respect to $y$ we get the effective equation for $\Lambda$ (supposing $\Lambda_0 \neq 0$)

$$\begin{aligned}&\frac{96\pi^2}{\lambda_0}(y-1) - \tfrac{1}{6}\{30F_2(-\tfrac{15}{4}+6y) - 9(q-r)F_1(\tfrac{13}{4} - 3(p+r) - 3(q-r)y)\\ &\qquad + 9qF_1(\tfrac{13}{4} - 3p - 3qy) - 12F_0(\tfrac{57}{4} - 12y)\}\\ &\quad + \tfrac{1}{2}(\gamma_2 + 2\gamma_3 y)\ln\frac{\lambda_0}{3\mu^2k^2y} - \frac{1}{2y}B_4 = 0.\end{aligned} \tag{4.20}$$

* This is a simple proof of the semiclassical stability of the de Sitter background (see also [42] for the general classical proof of stability).

In the "$R$ gauges" (4.16) $\mu$-dependent terms vanish and we get, using (2.34)

$$\frac{96\pi^2}{\lambda_0\hbar}(y-1)-\frac{1}{2y}B_4+A(y)=0, \qquad B_4=-\tfrac{571}{45}, \tag{4.21}$$

$$\begin{aligned}A(y)\equiv{}&135+\tfrac{3}{2}r+\tfrac{27}{4}(r^2+pr-qr)\\&-y\left[81+\tfrac{27}{4}(r^2-2qr)\right]-15(5-3y)\Psi\left(\tfrac{7}{2}\pm\sqrt{6y-\tfrac{15}{4}}\right)+\tfrac{9}{4}(q-r)\\&\times\left[-1+3(p+r)+3(q-2)y\right]\Psi\left(\tfrac{5}{2}\pm\sqrt{\tfrac{13}{4}-3(p+r)-3(q-r)y}\right)\\&-\tfrac{9}{4}q(-1+3p+3qy)\Psi\left(\tfrac{5}{2}\pm\sqrt{\tfrac{13}{4}-3p-3qy}\right)\\&+6(6y-7)\Psi\left(\tfrac{3}{2}\pm\sqrt{\tfrac{57}{4}-12y}\right).\end{aligned}$$

If the natural coupling constant $\lambda_0<1$ the classical term is the leading one and we get the perturbative solution*

$$y\simeq 1+\frac{\hbar\lambda_0}{96\pi^2}\left(\tfrac{1}{2}B_4-\hat{A}(1)\right)+O\left(\lambda_0^2\right). \tag{4.22}$$

If, however, $\lambda_0\sim 1$ (i.e. $\Lambda_0$ is of the Planck order) higher-loop corrections are important and perturbation theory breaks down. Looking formally for non-perturbative solutions of (4.21) we again see that no real solution exists unless $1\leqslant y\leqslant\frac{8}{3}$ (in the opposite case, $\psi$ functions of negative arguments appear which are complex). This "quantum stability" of the classical vacuum $y=1$ seems rather remarkable.

Now let us discuss the special case of $\Lambda_0=0$. Suppose, we are interested in spontaneous creation of de Sitter space from flat space due to quantum-gravitational fluctuations. Taking $\Lambda_0=0$ ($y=0$) in (4.18) we get

$$\Gamma=-\frac{96\pi^2}{\lambda}+\tfrac{1}{2}B_4\ln\frac{\lambda}{3\mu^2\kappa^2}+c, \tag{4.23}$$

where $\lambda=\Lambda k^2$, $B_4=\text{const}$ (e.g. $\gamma_1$ in (4.15)) and $c$ is a complex gauge-dependent constant**. The formal extremum of (4.23) is given by

$$\Lambda=-\frac{192\pi^2}{\hbar B_4 k^2}. \tag{4.24}$$

This result is the gravitational analog of the phenomenon of quantum generation of

* One is to extract from $A(1)$ an infinite constant $\sim\psi(0)$ corresponding to the zero mode of $[\Delta_0(4\Lambda_0-4\Lambda)]$, which drops out from (4.13).

** Note that $\Gamma$ is *always* complex due to the indefiniteness of the gravitational action.

the Planck-order cosmological constant already observed in sect. 3 (cf. (3.6)). Though $B_4$ is gauge dependent, in any reasonable gauge it is equal to some "natural" number and thus $\Lambda k^2$ is not unnaturally small but rather of the order unity. This implies that higher-loop corrections are important and one has to sum the perturbation series to give a definite answer concerning the value of the induced $\Lambda$ (cf. sect. 6). Still it seems possible that the origin of de Sitter space needed in the "inflationary universe" scenarios (see e.g. [10] and references therein) might be due to quantum gravitational effects (for another suggestion see [43]).

## 5. O(4) gauged supergravity

The lagrangian of this theory is given by [2, 4]

$$\mathcal{L} = -\frac{1}{k^2}R + \frac{2}{k^2}\frac{\partial_\mu\Phi\partial_\mu\Phi^*}{\left(1-|\Phi|^2\right)^2} - \frac{8g^2}{k^4}\left(1+\frac{2}{1-|\Phi|^2}\right) + \tfrac{1}{8}\left(F^{ij}_{\mu\nu}\right)^2$$

$$+\left\{\tfrac{1}{8}\frac{\Phi}{1-\Phi^2}\left(\Phi\delta_{ik}\delta_{je} - \tfrac{1}{2}\varepsilon_{ijkl}\right)\left(F^{ij}_{\mu\nu}F^{kl}_{\mu\nu} + iF^{ij}_{\mu\nu}\tilde{F}^{kl}_{\mu\nu}\right) + \text{h.c.}\right\}$$

$$+\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}^i_\mu\gamma_5\gamma_\nu D_\rho\psi^i_\sigma + \frac{2g}{k\sqrt{1-|\Phi|^2}}\bar{\psi}^i_\mu\sigma_{\mu\nu}\psi^i_\nu$$

$$+\tfrac{1}{2}\bar{\chi}^i\hat{D}\chi^i + \frac{g\sqrt{2}}{k\sqrt{1-|\Phi|^2}}\bar{\psi}^i_\mu\gamma_\mu\left(\Phi_1 + i\gamma_5\Phi_2\right)\chi^i$$

$$+\text{quartic fermionic terms}, \tag{5.1}$$

where $k^2 = 16\pi G$, $\gamma_{(\mu}\gamma_{\nu)} = g_{\mu\nu}$, $\gamma_5^2 = 1$, $\sigma_{\mu\nu} = \frac{1}{2}\gamma_{[\mu}\gamma_{\nu]}$, $\hat{D} = \gamma_\mu D_\mu$, $D_\mu = \mathcal{D}_\mu + gA_\mu$, $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$. It contains a graviton; an O(4) gauge field ($i, j = 1,\ldots,4$); a complex scalar field $\Phi = \Phi_1 + i\Phi_2$ parametrizing SU(1,1)/U(1) space and thus satisfying $|\Phi| < 1$; four gravitinos; and four spinors (all Majorana). The theory is invariant under gauged $N = 4$ supersymmetry

$$\delta\psi^i_\mu = \left(D_\mu + \frac{g}{k\sqrt{1-|\Phi|^2}}\gamma_\mu\right)\varepsilon^i,$$

$$\delta\chi^i = \frac{\sqrt{2}\,g}{k\sqrt{1-|\Phi|^2}}\left(\Phi_1 + i\gamma_5\Phi_2\right)\varepsilon^i, \tag{5.2}$$

and possesses a unique stable supersymmetric classical "vacuum" [5]

$$g_{\mu\nu} = \text{AdS}, \qquad \text{with } \Lambda_0 = -\frac{12g^2}{k^2}, \qquad \Phi = 0, \qquad \psi, A, \chi = 0, \tag{5.3}$$

with $\Phi = 0$ being the only extremum (maximum) of the classical potential*

$$V_0 = -\frac{8g^2}{k^4}\left(1 + \frac{2}{1 - |\Phi|^2}\right). \tag{5.4}$$

An apparent problem in the way of calculating the effective action is the *negative* sign of the tree-level cosmological constant (recall that the spectra of the operators in subsect. 2.1 were given for the $S^4$ background with $\Lambda > 0$). Though formally the effective $\Lambda$ may well turn out to be positive, it seems safer (at the one-loop level) to keep the same sign of the tree and the effective $\Lambda$. To overcome this difficulty one may use the following recipe: consider an unphysical theory with $g^2 = -\tilde{g}^2 < 0$ (and thus with $\Lambda_0 > 0$), calculate $\Gamma_1$ for $\Lambda > 0$ and finally make an analytic continuation $\Lambda \to -\Lambda$, $g^2 \to -g^2$. This procedure probably corresponds to an appropriate choice of boundary conditions on eigenfunctions in a possible direct AdS-background calculation (cf. [45, 5]). We shall call the unphysical $g^2 < 0$, $\Lambda > 0$ theory an "$S^4$ version" in contrast to the "AdS version" with $g^2 > 0$, $\Lambda < 0$. The calculation of $\Gamma$ will be done for the $S^4$ version, assuming that the background is given by (2.11) and

$$\Phi_1 = \phi = \text{const}, \qquad \phi^2 < 1, \qquad \Phi_2 = 0, \qquad \psi, \chi, A = 0, \tag{5.5}$$

(the dependence on the pseudoscalar $\Phi_2$ follows from U(1) invariance). In view of the off-shell non-renormalizability of gauged supergravities we shall use the $\zeta$-function prescription to obtain the finite expression for $\Gamma(\Lambda, \phi, g^2/k^2)$ (see the discussion in subsect. 2.1). Also, we shall carry out calculations in a wide class of gauges with the idea of understanding which properties of $\Gamma$ are gauge independent.

Let us start with the scalar fields contribution in the effective action, using the following conventions:

$$k^2 = 2, \qquad g^2 = -\tilde{g}^2 = -1, \qquad \alpha = \frac{1}{1-\phi^2} > 1, \qquad V_0 = 2(1 + 2\alpha),$$

$$\text{on shell:} \qquad \alpha = 1, \qquad \Lambda = V_0 = 6, \tag{5.6}$$

(actual dependence on $k^2$ and $g^2$ is easy to restore in the final answer). Shifting the fields, $\Phi_1 \to \Phi_1 + \phi$, $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ we get for the scalar terms in (5.1)

$$\mathcal{L}_\Phi = \alpha^2(\partial\Phi_1)^2 + \alpha^2(\partial\Phi_2)^2 + \left[V_0 + 8\alpha^2\phi\Phi_1 + 4\alpha^2(4\alpha - 3)\Phi_1^2 + 4\alpha^2\Phi_2^2\right]$$
$$\times\left(1 + \tfrac{1}{2}h + \tfrac{1}{16}h^2 - \tfrac{1}{4}\bar{h}_{\mu\nu}^2\right) + \cdots. \tag{5.7}$$

* Thus there is no tree-level supersymmetry breaking in this theory, which is possible, however, for $N \geqslant 5$ gauged supergravities [5, 44].

Assuming (as always, in one-loop calculations) that it is the background field that is constrained ($\phi^2 < 1$) while quantum fields are arbitrary $-\infty < \Phi_{1,2} < \infty$ (in a similar fashion, $g_{\mu\nu}$ has a euclidean signature while $h_{\mu\nu}$ is unconstrained) we can integrate over $\Phi$ with the result given by

$$Z_\Phi = \left[\det \Delta_0(16\alpha - 12)\det \Delta_0(4)\right]^{-1/2}, \tag{5.8}$$

and also by additional terms in the quantum gravitational lagrangian

$$\Delta\mathcal{L}_h = -4\alpha(\alpha-1)h\Delta_0^{-1}(16\alpha-12)h + V_0\left(\tfrac{1}{16}h^2 - \tfrac{1}{4}\bar{h}_{\mu\nu}^2\right). \tag{5.9}$$

In the derivation of (5.8) we made the rescaling $\Phi \to \alpha^{-1}\Phi$ and assumed that the corresponding jacobian is cancelled by the proper local ($\sigma$ model type) measure $[\mathrm{d}\Phi] = \prod_x \mathrm{d}\Phi\,\mathrm{d}\Phi^*/(1-|\Phi|^2)^2$. This choice is necessary for consistency of the theory (e.g. for SU(1,1) invariance of the measure) and also for cancellation of quartic divergences*. It is worth stressing that it is only gravitational ($\sim\sqrt{g}$) quartic infinities which automatically cancel in supergravities**. In view of the non-polynomiality of scalar and vector kinetic terms in scalars (cf. (5.1)), scalar-field dependent quartic infinities will be present in the naively defined functional integral. To eliminate them (and also to restore the global symmetry of the measure) one is to include a proper local scalar-dependent factor in the measure. As for finite parts, there is a possible source of ambiguity here: if $c = \text{const}$, one may define $\det(c\Delta)$ either as $\det[c\delta(x,x')]\det\Delta$ or, in the $\zeta$-function prescription, as $c^{B_4}\det\Delta$. It is only in the first definition that scalar-dependent factors in kinetic terms are completely cancelled by the measure. With the latter definition in addition to (5.8) we get

$$\Delta\Gamma_{1\Phi} = \tfrac{1}{2}\left[B_4\left(\Delta_0(16\alpha-12)\right) + B_4\left(\Delta_0(4)\right)\right]\ln\alpha^2, \tag{5.10}$$

(let us note in passing that the contribution of scalar factors of kinetic terms in the finite part of the effective action for ungauged $N=4$ supergravity was also discussed in [48]).

Assuming (5.5), one can rewrite the gauge vector piece of (5.1) as

$$\mathcal{L}_A = \tfrac{1}{8}\Sigma_{ijkl}(\phi)F_{\mu\nu}^{ij}F_{\mu\nu}^{kl},$$

$$\Sigma_{ijkl} = \frac{1+\phi^2}{1-\phi^2}\delta_{i[k}\delta_{l]j} - \frac{\phi}{1-\phi^2}\varepsilon_{ijkl}. \tag{5.11}$$

Changing the variables $A_\mu = A_\mu^\perp + \mathcal{D}_\mu a$ (as in (2.12)) we finally get

$$Z_A = \left[\frac{\det\Delta_0(0)}{\det\Delta_1(+\Lambda)}\right]^{1/2\times 6}, \tag{5.12}$$

* In general for $\mathcal{L} = g_{ij}(\varphi)\partial_\mu\varphi^i\partial_\mu\varphi^j$ the proper measure is $[\mathrm{d}\varphi^i] = \prod_{x,i}[\det g_{ij}(\varphi)]^{1/2}\mathrm{d}\varphi^i$ (see ref. [46]).
** For a discussion of power divergences in supergravities see [25, 8, 47].

if the $\det \Sigma$ factor is completely cancelled by the measure, or in the $\zeta$-function prescription, also (cf. (5.10))

$$\Delta \Gamma_{1A} = \tfrac{1}{2} B_4(\Delta_1(+\Lambda)) \ln \det \Sigma, \qquad \det \Sigma_{[ij][kl]} = \left(\frac{1-\phi}{1+\phi}\right)^{12}. \tag{5.13}$$

Next we are to include the gravitational contribution. Comparing (5.9) and (4.2) we see that $V_0$ plays the role of the bare cosmological constant $\Lambda_0$. Thus the only complication is due to the scalar field-scalar graviton mixing in (5.9). It can be avoided by choosing $h=0$ as part of the gravitational gauge. For example, in the gauge (4.11), (4.12), we again get (4.13) *but now with* $\Lambda_0 \to V_0$. Working in the standard class of gauges (4.3) one has to add (5.9) to (4.2) and then to integrate over $\bar{h}^{\perp}$, $\xi^{\perp}$, $\sigma$, $h$ with the result analogous to (4.4). For example, in the de Donder gauge ($\gamma = \beta = 1$) we find

$$\begin{aligned} Z_g = {} & \det \Delta_1(-\Lambda) \det \Delta_0(-2\Lambda) \left[\det \Delta_0(16\alpha - 12)\right]^{1/2} \\ & \times \left[\det \Delta_2(\tfrac{8}{3}\Lambda - 2V_0) \det \Delta_1(\Lambda - 2V_0) \right. \\ & \left. \times \det \Delta_0(-2V_0) \det \Delta_0(-12) \det \Delta_0(8\alpha - 4)\right]^{-1/2}. \end{aligned} \tag{5.14}$$

On the classical mass shell (5.6) this expression coincides, of course, with (4.13) or (4.6).

Let us now discuss the fermionic sector contribution in $\Gamma$. Integrating over $\chi$ the spinor part of (5.1)

$$\mathcal{L}_\chi = \tfrac{1}{2}\bar{\chi}\hat{\mathcal{D}}\chi - i\sqrt{\alpha}\,\phi\bar{\psi}\chi, \qquad \psi = \gamma_\mu \psi_\mu, \tag{5.15}$$

we get

$$Z_\chi = [\det \hat{\mathcal{D}}]^{1/2\times 4} = \det \Delta_{1/2}(0), \tag{5.16}$$

and also the additional term

$$\Delta \mathcal{L}_\psi = \tfrac{1}{2}(\alpha - 1)\bar{\psi}\hat{\mathcal{D}}^{-1}\psi, \tag{5.17}$$

in the gravitino lagrangian. Introducing the decomposition (2.15), the latter can be rewritten as

$$\begin{aligned} \mathcal{L}_\psi = {} & \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \gamma_5 \gamma_\nu \mathcal{D}_\rho \psi_\sigma + m\bar{\psi}_\mu \sigma_{\mu\nu}\psi_\nu \\ = {} & \tfrac{1}{2}\bar{\varphi}^{\perp}(\hat{\mathcal{D}} - m)\varphi^{\perp} + \tfrac{3}{16}\left[\bar{\zeta}(\hat{\mathcal{D}} + 2m)\Delta_{1/2}(-\tfrac{4}{3}\Lambda)\zeta \right. \\ & \left. - 2\bar{\zeta}\Delta_{1/2}(-\tfrac{4}{3}\Lambda)\psi - \bar{\psi}(\hat{\mathcal{D}} - 2m)\psi\right], \qquad m = i\sqrt{2\alpha}, \end{aligned} \tag{5.18}$$

(let us recall that $-\hat{\mathfrak{D}}^2 = -\mathfrak{D}^2 + \frac{1}{4}R = \Delta_{1/2}(0)$, so $\mathfrak{D}^2 - \frac{1}{4}\hat{\mathfrak{D}}^2 = -\frac{3}{4}\Delta_{1/2}(-\frac{4}{3}\Lambda)$). Observing that under (5.2) $\delta\varphi^\perp = 0$, $\delta\zeta = \varepsilon$, $\delta\psi = (\hat{\mathfrak{D}} + 2m)\varepsilon$, we recognize $\psi = 0$ as the simplest gauge in which there is no $\psi - \chi$ mixing (5.17). Integrating over $\varphi^\perp$, $\zeta$, $\psi$ and including the ghost and jacobian (2.41) factors we are left with

$$Z_\psi = \left[\frac{\det(\hat{\mathfrak{D}} - m)_{\varphi^\perp}}{\det(\hat{\mathfrak{D}} + 2m)}\right]^{1/2\times 4} = \frac{\det \Delta_{3/2}(m^2)}{\det \Delta_{1/2}(4m^2)}, \tag{5.19}$$

$$m^2 = -2\alpha .$$

(In the final form we used (2.18), (2.19) and (2.23).) It is possible also to find a simple one-gauge parameter extension of (5.19). Integrating first over $\varphi^\perp$ and $\zeta$ and adding (5.17) one obtains the residual $\psi$ term,

$$\tilde{\mathcal{L}}_\psi = \tfrac{1}{4}\bar{\psi}\left[(\Lambda + 3m^2)(\hat{\mathfrak{D}} + 2m)^{-1} + 2(\alpha - 1)\hat{\mathfrak{D}}^{-1}\right]\psi, \tag{5.20}$$

which vanishes on shell (5.6) in agreement with the on-shell supersymmetry of (5.18). Though off shell it is possible to integrate over $\psi$ without any gauge fixing (see appendix) here we follow the standard route, adding a gauge fixing term

$$\mathcal{L}_g = \tfrac{1}{4}\bar{\psi}H\psi, \qquad H = \gamma_0(\hat{\mathfrak{D}} + 2m)^{-1},$$

(any other choices of $H$ are also admissible but they complicate the result). Trading $\gamma_0(\Lambda, \alpha, \gamma)$ for a new gauge parameter $\gamma$, we get the expression

$$Z_\psi^{(\gamma)} = \frac{\det \Delta_{3/2}(m^2)\det \Delta_{1/2}(\gamma(\alpha - 1))}{\det \Delta_{1/2}(4m^2)\det \Delta_{1/2}(0)}, \tag{5.21}$$

coinciding with (5.19) for $\gamma = 0$.

We conclude that the total four-gauge parameter dependent off-shell effective action is given by the product of (5.8), (5.12), (4.13) (with $\Lambda_0 = V_0$), (5.17) and (5.21) (for simplicity we shall omit contributions (5.10), (5.13), which vanish on shell (5.6), from the following discussion)*. It is straightforward though tedious to compute the infinities of this action**. Recalling eqs. (2.43), (2.44), (4.14)–(4.16) we

* An analogous $\gamma$-dependent factor was also emitted in the derivation of (5.21). The neglect of such factors can be considered as part of the definition of $\Gamma_1$.

** For a previous discussion of off-shell infinities of gauged supergravities, in the mixed gravitational-gauge field background, see ref. [8].

get

$$B_0 = 0, \qquad B_2 = \frac{3}{2\Lambda^2}\{-3(r+6)(\Lambda - V_0) + 8(\gamma-2)(\alpha-1)\},$$

$$B_4 = \sum_i \zeta_i(0) - \mathfrak{N}$$

$$= \frac{3}{2\Lambda^2}\{(\Lambda - V_0)[(\tfrac{3}{2}r^2 + 3pr - r + \tfrac{58}{3})\Lambda$$

$$-(\tfrac{3}{2}r^2 - 3qr - 25)V_0 + \tfrac{8}{3}(\gamma-4)(\alpha-1)]$$

$$+4(\alpha-1)[(\tfrac{56}{3} - \gamma^2)(\alpha-1) + \tfrac{4}{3}(1+2\alpha)(\gamma-4) + \tfrac{4}{3}(5+49\alpha)]\},$$

$$\mathfrak{N} = 14 + 6 - \tfrac{1}{2}\times 4\times 8 = 4. \qquad (5.22)$$

Thus on shell (5.6) the theory is free from quartic and quadratic infinities [8] and is renormalizable. Only the last term in (5.22) survives and corresponds to the renormalization of the topological coupling constant (4.17) and the gauge coupling constant for general backgrounds [7, 8]

$$(B_4)_{\text{on shell}} = -4 = \frac{3}{2\Lambda^2}(-\tfrac{1}{2}R^*R^* - 4\Lambda^2)_{S^4}. \qquad (5.23)$$

Comparing (5.22) and (5.23) *we notice that* $\Sigma_i\zeta_i(0) = 0$ *on shell* and thus $B_4 = -4$ is completely due to the zero modes of the jacobians (2.41), a rather unexpected result in view of the non-zero $\beta$ function in the $N = 4$ model (and the non-self-dual nature of $S^4$, cf. [49])*.

As follows from (5.22), our four-parameter gauge freedom is not sufficient to make the theory renormalizable also off shell. That is why we rely upon the $\zeta$-function prescription (2.7) to define finite $\Gamma$. Let us first write down the functional form of the effective action in the simplest possible gauge $r = 0$, $\gamma = 0$ (i.e. $h = 0$,

* This interesting property is also true for $N = 5$ and 8 gauged supergravities: in view of the zero $\beta$ function for $N \geqslant 5$, $B_4 = \frac{3}{2}\Lambda^{-2}[\frac{1}{2}(3-N)\times R^*R^*] = -2(N-3)$, while the number of zero modes of all jacobians (2.41) is $\mathfrak{N} = 14 + \frac{1}{2}N(N-1) - \frac{1}{2}N\times 8$. Thus from (2.43) $\Sigma_i\zeta_i(0) = \frac{1}{2}(N-5)(N-8)$, if $N \geqslant 5$. $\Sigma_i\zeta_i(0) = 0$ holds also for $N = 4$ super-Yang-Mills theory on the $S^4$ background, having $B_4 = -1$ and $\mathfrak{N} = 1$.

$\xi^{\perp}=0$, $\psi=0$), restoring the dependence on $g^2$ and $k^2$:

$$\begin{aligned}\Gamma_{1\,\text{simpl}} = {} & \tfrac{1}{2}\ln\det\Delta_2\left(\tfrac{8}{3}\Lambda - k^2V_0\right) - \tfrac{1}{2}\ln\det\Delta_1(-\Lambda) \\ & + 3\ln\det\Delta_1(\Lambda) + \tfrac{1}{2}\ln\det\Delta_0\left(2k^2V_0 - 4\Lambda\right) \\ & - \tfrac{7}{2}\ln\det\Delta_0(0) + \tfrac{1}{2}\ln\det\Delta_0\left(\frac{8g^2}{k^2}(3-4\alpha)\right) \\ & + \tfrac{1}{2}\ln\det\Delta_0\left(-\frac{8g^2}{k^2}\right) - \ln\det\Delta_{3/2}(m^2) \\ & + \ln\det\Delta_{1/2}(4m^2) - \ln\det\Delta_{1/2}(0)\,, \end{aligned} \tag{5.24}$$

where now $V_0 = -(8g^2/k^4)(1+2\alpha)$, $m^2 = (4g^2/k^2)\alpha$. Using the results of sect. 2 we get (cf. (4.18))

$$\begin{aligned}\Gamma_{\text{simpl}} = I + \tfrac{1}{2}B_4\ln\frac{\Lambda}{3\mu^2} - \tfrac{1}{6}\{ & 5F_2'\left(-\tfrac{15}{4} - 2y - 4x\right) \\ & - 3F_1'\left(\tfrac{25}{4}\right) + 18F_1'\left(\tfrac{1}{4}\right) + F_0'\left(\tfrac{57}{4} + 4y + 8x\right) \\ & - 7F_0'\left(\tfrac{9}{4}\right) + F_0'\left(\tfrac{9}{4} - 6y + 8x\right) + F_0'\left(\tfrac{9}{4} + 2y\right) \\ & - 16F_{3/2}'(-x) + 8F_{1/2}'(-4x) - 8F_{1/2}'(0)\}\,, \end{aligned} \tag{5.25}$$

where

$$y = \frac{12g^2}{k^2\Lambda}\,, \qquad x \equiv \alpha y = \frac{y}{1-\phi^2}\,, \tag{5.26}$$

$$B_4 = 31 + \tfrac{1}{3}\left(19y^2 + 110x^2 + 12xy + 74y + 172x\right), \tag{5.27}$$

$$I = -\frac{4\pi^2}{3g^2}\left(y^2 + 2xy + 6y\right). \tag{5.28}$$

If we now substitute $\Lambda \to -\Lambda$, $g^2 \to -g^2$ in the *quantum* part $\Gamma_1$, $x$ and $y$ remain invariant and we simply get (5.25) with $\ln\Lambda \to \ln(-\Lambda)$. This gives the *final result* for the effective action in the physical AdS version of the theory. Let us first discuss the gauge-independent, on-shell limit of this action, i.e. the one-loop-corrected classical vacuum action, taking $\Lambda = -12g^2/k^2$, $x = y = -1$ (cf. (4.19)):

$$\Gamma_{\text{on shell}} = \frac{4\pi^2}{g^2} - 2\hbar\ln\frac{4g^2}{\mu^2k^2} + \text{const}\,. \tag{5.29}$$

This quantity is real and finite (after proper subtraction of the zero modes contribution); note also that the quantum correction is *positive* for small enough $g^2$. One can show that these properties hold also for $N > 4$ gauged O($N$) supergravities if one considers the one-loop correction for the supersymmetric vacuum state (then $B_4 = -2(N-3)$). The reality of (5.29) follows from the absence of negative modes for all the operators in the on-shell limit of (5.24) in the $S^4$ version. This observation provides *a simple proof of the stability of the AdS vacuum* (5.3) (for a different proof see [5]). A natural question then is about the off-shell negative modes in (5.24). Employing (2.25)–(2.29) we get the following restrictions from the Bose, $\frac{8}{3}\Lambda \geqslant -(4g^2/k^2)(1+2\alpha) \geqslant \Lambda$, and Fermi, $\Lambda \geqslant -(12g^2/k^2)\alpha$, sectors. Recalling that $\alpha > 1$ in (5.6) we conclude that (5.25) is real *only if* $\Lambda = -12g^2/k^2$ and $\alpha = 1$, i.e. *on shell*. This rather startling conclusion is due to the opposite bounds in Bose and Fermi sectors and thus is probably due to supersymmetry. It seems to be gauge independent (cf. (5.21), (4.13)) and thus *amplifies* the statement of the classical stability of the AdS vacuum.

In what follows we shall formally ignore the complex nature of any non-classical extrema of (5.25) anticipating that this may be an artifact of the one-loop approximation. Turning to the effective equations, following from (5.25)

$$\frac{\partial \Gamma}{\partial \Lambda} = -y\left(\frac{\partial \Gamma}{\partial y} + \alpha \frac{\partial \Gamma}{\partial x}\right) = 0, \qquad \frac{\partial \Gamma}{\partial \phi} = 2x\phi \frac{\partial \Gamma}{\partial x} = 0, \tag{5.30}$$

we see that the supersymmetry preserving solution $\phi = 0$ is always present. The corresponding $\Lambda$ satisfies the equation, analogous to (4.20) (we use the AdS version)

$$\begin{aligned} &-\frac{8\pi^2}{g^2}(1+y) - \frac{1}{2y}(31 + 82y + 47y^2) + (41 + 47y)\ln\left(\frac{-4g^2}{\mu^2 k^2 y}\right) \\ &\quad -\tfrac{1}{3}\left[-15F_2\left(-\tfrac{15}{4} - 6y\right) + 6F_0\left(\tfrac{57}{4} + 12y\right) + 2F_0\left(\tfrac{9}{4} + 2y\right)\right. \\ &\quad \left. + 8F_{3/2}(-y) - 16F_{1/2}(-4y)\right] = 0. \end{aligned} \tag{5.31}$$

Supposing $g^2$ to be small we get

$$\Lambda \simeq -\frac{192\pi^2}{B_4^{(0)} k^2 \hbar} + O(g^2), \tag{5.32}$$

where $B_4^{(0)} = B_4(g=0) = 31$ in the gauge we used (cf. (5.27)). Thus *gauging gives only small corrections to a Planck-order quantum cosmological constant induced already for ungauged $N = 4$ supergravity*. As a consequence, O(4) supergravity does not improve the situation as compared to the pure gravity (4.24) and matter field (3.6) cases.

Now let us discuss eqs. (5.30) for $\phi \neq 0$:

$$\frac{\partial \Gamma}{\partial y} = -\frac{8\pi^2}{3g^2}(3+x+y) - \frac{1}{6y}(93+19y^2+110x^2+12xy+74y+172x)$$

$$+\tfrac{1}{3}(37+6x+19y)\ln\left(-\frac{4g^2}{\mu^2 k^2 y}\right) - A_1 = 0,$$

$$\frac{\partial \Gamma}{\partial x} = -\frac{8\pi^2}{3g^2}y + \tfrac{1}{3}(86+6y+110x)\ln\left(-\frac{4g^2}{\mu^2 k^2 y}\right) - A_2 = 0,$$

$$A_1(x,y) = \tfrac{1}{3}\Big\{-5(5+y+2x)\Psi\Big(\tfrac{7}{2} \pm i\sqrt{\tfrac{15}{4}+2y+4x}\Big)$$

$$-2(7+2y+4x)\Psi\Big(\tfrac{3}{2} \pm \sqrt{\tfrac{57}{4}+4y+8x}\Big)$$

$$+3(1-3y+4x)\Psi\Big(\tfrac{3}{2} \pm \sqrt{\tfrac{9}{4}-6y+8x}\Big)$$

$$-(1+y)\Psi\Big(\tfrac{3}{2} \pm \sqrt{\tfrac{9}{4}+2y}\Big) + \tfrac{131}{3} + 6x + 19y\Big\},$$

$$A_2(x,y) = \tfrac{1}{3}\Big\{-10(5+y+2x)\Psi\Big(\tfrac{7}{2} \pm i\sqrt{\tfrac{15}{4}+2y+4x}\Big)$$

$$-4(7+2y+4x)\Psi\Big(\tfrac{3}{2} \pm \sqrt{\tfrac{57}{4}+4y+8x}\Big)$$

$$-4(1-3y+4x)\Psi\Big(\tfrac{3}{2} \pm \sqrt{\tfrac{9}{4}-6y+8x}\Big)$$

$$+4(4+x)\Psi(3 \pm i\sqrt{x}) - 8(1+4x)\Psi(2 \pm i\sqrt{4x})$$

$$+\tfrac{254}{3} + 6y + 80x\Big\}. \tag{5.33}$$

Expanding in $\hbar$ we get the perturbative solution $\Lambda = -12g^2/k^2 + O(\hbar)$, $\alpha = 1 + O(\hbar)$. Expanding instead in $g^2$ and noting that according to (5.26) $x = g^2\bar{x}$, $y = g^2\bar{y}$, we find

$$\bar{y} = C_1 + O(g^2), \qquad \bar{x} = C_2 + O(g^2), \tag{5.34}$$

where $C_1$ and $C_2$ are calculable constants. Thus up to all problems of interpretation, raised above, we find indications for *dynamical local supersymmetry breaking*, i.e. $\phi \neq 0$. Contrary to some expectations this breaking does not, however, help to solve the problem of an enormous $\Lambda$ term which is thus relegated to higher-loop summation approaches (cf. sect. 6).

Still it seems possible that this induced $\Lambda$-term problem may find a perturbative solution in yet unknown* *gauged* supergravity with $B_4^{(0)} \equiv B_{4 \text{ on shell}}(g=0)=0$ (cf. sect. 6). Suppose first that such a theory will have zero tree-level cosmological term in the supersymmetric ground state like $\phi=0$. Then *quantum corrections will respect* this remarkable property, because the analog of (5.31) will look like

$$\frac{8\pi^2}{g^2} + a_1 \ln\frac{\lambda}{\tilde{\mu}^2} \simeq 0\,,$$

$$\lambda = \Lambda k^2\,, \qquad \tilde{\mu}^2 \equiv a_2 \mu^2 k^2 + \mathrm{O}(g^2)\,, \qquad a_1, a_2 = \text{const}\,. \tag{5.35}$$

Hence

$$\Lambda = m_{\mathrm{P}}^2 \left(1+\mathrm{O}(g^2)\right) \exp\left(-\frac{\text{const}}{g^2}\right), \tag{5.36}$$

an extremely desired result. If at the tree level $\Lambda_0 \sim \mathrm{O}(g^2)$ will turn to be non-zero, then an additional $\Lambda_0/\Lambda$ term appears in (5.35) (cf. (5.31)) and the solution (5.36) is excluded. But here a rescue may come from *dynamical supersymmetry breaking*: *$\Lambda_0$ can be compensated by non-zero scalar field contributions*. We stress that *this is possible in principle only if no $\Lambda$ term is induced in the $g^2 \to 0$ limit* (*i.e. if $B_4^{(0)}=0$*). Though some conditions for the realization of this possibility can be guessed from (5.33), a quantitative analysis obviously depends on details of the theory and thus awaits for its actual construction.

## 6. Concluding remarks

In this paper we considered the one-loop effective potential in O(4) gauged supergravity in an attempt to understand whether quantum corrections can resolve the problem of the tree-level cosmological constant in this theory. Though a one-loop calculation is obviously inconclusive it seems improbable that taking into account any finite number of loops may be of help. Our method can be straightforwardly applied to $N \geqslant 5$ gauged O($N$) supergravities [4]. We anticipate no qualitative differences in the results in spite of a zero $\beta$ function in the latter theories [7]. A peculiar quantum behaviour might be in principle expected in theories with a zero (on-shell) "conformal anomaly" (or topological infinity) coefficient $B_4^{(0)}$ because if $B_4^{(0)} \neq 0$ a Planck-order cosmological term is induced by quantum fluctuations of fields of any spin on a gravitational background. However, $B_4^{(0)} \neq 0$ for all O($N$)

* The only exception is the $SU_2 \times SU_2$ theory of ref. [3] with the pseudoscalar traded for the antisymmetric tensor as discussed in sect. 6.

gauged supergravities. Trying to make $B_4^{(0)}$ equal to zero, one is to introduce antisymmetric tensors [39] but this seems to be impossible without breaking O($N$) *gauged* supersymmetry (though probably possible for ungauged U($N$) supergravities following the idea of Nicolai and Townsend [39]). One is thus led to the question of whether it is possible to construct a realistic (i.e. *gauged*) supergravity with zero $B_4^{(0)}$. It seems to us that the answer may be "yes" if one gives up the idea of a simple total gauge group. To provide an example of such a theory, consider the $SU_2 \times SU_2$ gauged $N = 4$ supergravity of Freedman and Schwarz [3]:

$$\mathcal{L} = -\frac{1}{k^2}R + \tfrac{1}{2}(\partial_\mu\phi)^2 + \tfrac{1}{2}e^{2k\phi}(\partial_\mu B)^2$$

$$+ \tfrac{1}{4}e^{-k\phi}\left[\left(F^i_{1\mu\nu}\right)^2 + \left(F^i_{2\mu\nu}\right)^2\right] - 2(g_1^2 + g_2^2)e^{k\phi} + \partial_\mu B J_\mu + \cdots, \qquad (6.1)$$

where $i = 1, 2, 3$, $g_1$ and $g_2$ are gauge couplings,

$$J_\mu = -\tfrac{1}{2}k\varepsilon_{\mu\alpha\beta\gamma}\left(A^i_{1\alpha}\partial_\beta A^i_{1\gamma} + \tfrac{1}{3}g_1\varepsilon_{ijk}A^i_{1\alpha}A^j_{1\beta}A^k_{1\gamma}\right) + (1 \to 2) + \text{fermionic currents},$$

and we omitted a number of terms irrelevant to our discussion. The crucial fact is that irrespective of gauging, the pseudoscalar $B$ contributes in the lagrangian as explicitly shown in (6.1), i.e. only through $\partial_\mu B$. That is why it is possible to convert $B$ into an antisymmetric tensor $A_{\mu\nu}$ just in the same way as was done by Nicolai and Townsend [39] for ungauged $SU_4$ theory with the resulting $e^{-2k\phi}(\partial_\mu \tilde{A}_{\mu\nu} + J_\nu)^2$ term instead of the $B$ terms in (6.1). *Hence we found gauged supergravity with* $B_4^{(0)} = 0$. At the same time, the quantum status of this theory remains unclear because of the absence of an obvious $\phi = \text{const}$ "vacuum" in this theory (as compared with O($N$) gauged theories). The difficulty is due to the $e^{k\phi}$ scalar potential that is reminiscent of the 2-dim Liouville field theory (see e.g. [50]). It is disturbing to find that *no classical solutions exist* if we admit $\phi \neq \text{const}$, but still hope for $g_{\mu\nu} =$ (anti) de Sitter. Thus one has probably to look for soliton black hole-type solutions, like those already discussed for ungauged $N = 4$ theory in [51] (cf. [50]). Then it should be possible to study quantum properties of this theory (including on-shell $\beta$ functions for $g_1$ and $g_2$).

Even admitting that this particular $N = 4$ $SU_2 \times SU_2$ theory might be pathological at the quantum level, we see no reasons for the non-existence of its more "fortunate" higher-$N$ generalizations with the total *gauge* group G being, e.g. for $N = 8$, $G_2 \times G_2$ or $SU_5 \times SU_3 \times U_1$ (we have to employ all 28 vectors of ungauged $SO_8$ supergravity). To establish zero $B_4^{(0)}$ we have to assume that 7 out of a total of 70 scalars are singlets of G and thus can be converted into 7 antisymmetric tensors $A_{\mu\nu}$ (plus one $A_{\mu\nu\lambda}$). Another suggestion is to use $N = 3$ multiplets as building blocks (cf. [39]). It remains to be seen whether such theories can actually be constructed and whether they have brighter perspectives for phenomenological applications than O($N$) gauged supergravities. An insight into the question of their existence might be provided by further study of the solutions of $d=11$ supergravity* (cf. [52]).

Our next remark concerns the possible resolution of a large induced $\Lambda$-term problem due to summation of higher-loop contributions. Consider first the effective action in a pure Einstein gravity for a de Sitter background (2.11),

$$\Gamma \sim \int \mathrm{d}^4x \sqrt{g} \left( -\frac{1}{k^2} R + R^2 \ln\left(\frac{R}{\mu^2}\right) + \cdots \right).$$

Introducing the dimensionless parameter $\lambda = \Lambda k^2$ we symbolically have (for $\mu^2 k^2 \sim 1$)

$$\Gamma \sim \frac{1}{\lambda} + \ln\lambda + \lambda(\ln\lambda + \ln^2\lambda) + \cdots . \tag{6.2}$$

This implies that the one-loop effective extremum $\lambda \sim 1$ is untrustworthy because for $\lambda \sim 1$ higher-loop corrections are all of the same order. To give an idea of what might happen after summation of the series, let us turn to higher-derivative renormalizable theory $\mathcal{L} = -(1/k^2)(R - 2\Lambda_0) + (1/g^2)(R_{\mu\nu}^2 - \frac{1}{3}R^2) + \cdots$, which is known to be asymptotically free in $g^2$ [25]. The latter property implies that the corresponding ($S^4$ background) effective action can be approximated as ($\lambda_0 = \Lambda_0 k^2$)**

$$\Gamma \sim \frac{1}{\lambda} - \frac{\lambda_0}{2\lambda^2} + \frac{b}{a + g^2 \ln\lambda}, \qquad a, b = \text{const}. \tag{6.3}$$

If $b > 0$ this function *is extremal near* $\lambda \sim \exp(-a/g^2)$. As a result, we get a very small effective $\Lambda$ term *irrespective* of the *value of the tree-level cosmological constant*. This observation shows the way to solving the $\Lambda$-term problem of gauged $O(N)$ supergravities either by summing higher-loop corrections or *by directly coupling them to (higher-derivative) conformal supergravities*, known to be asymptotically free or finite [54].

## Note added

After the completion of this work we became aware of ref. [55] where a similar $\zeta$-function approach to calculations in de Sitter space is discussed in the example of scalar QED and also of ref. [56] where one-loop quadratic divergences of the

* Note, however, that it was proved to be impossible to obtain $SU_2 \times SU_2$ gauged theory by $M^4 \times S^3 \times S^3$ reduction of $d=10$ supergravity [53].

** This expression as well as the $R + R^2$ theory itself may be considered simply as a model of a resummed Einstein theory.

effective potential for matter scalars in $N=1$ supergravity are computed. New $N=4$ gauged supergravities are constructed in [57]. We were also informed that "phenomenological" models of the small effective $\Lambda$ term were recently discussed in [58]. Finally we mention ref. [59] quoted in [55, 56] but yet unavailable to us.

## Appendix

### GAUGE NON-FIXING PROCEDURE FOR THE CALCULATION OF THE OFF-SHELL EFFECTIVE ACTION

The notion of the off-shell effective action is ambiguous in gauge theories. One can either use the standard definition (2.1) with the resulting $\Gamma$ being dependent on the particular gauge used or resort to some "aesthetically distinguished" new recipe like that of Vilkovisky [22] (see also [26]), which is gauge independent by construction. Here we want to point out that there exists yet another possible definition of $\Gamma$. The principal observation is that the naive functional integral in (2.1) is already non-degenerate *if* we assume an off-shell background (e.g. $\delta^2 I/\delta\phi^2$ has no gauge zero modes if $\phi$ does not satisfy $\delta I/\delta\phi = 0$). Thus no gauge fixing is formally needed at all. An apparent objection to this proposal is that the resulting $\Gamma$ will be singular in the on-shell limit $\delta I/\delta\phi \to 0$ (for simplicity we consider only the one-loop approximation). This, however, seems not to be a serious defect if singular terms can be separated in some natural way from the non-trivial part of $\Gamma$. The final quantity $\tilde{\Gamma}$ may be helpful in an attempt to find a gauge-independent approximation and may possess improved ultraviolet properties (especially in supersymmetric theories) as compared to the standard $\Gamma$ in a general gauge.

Now let us illustrate the suggested procedure in a number of examples starting with flat-space scalar QED (see also [26]). The standard effective potential in the $(\alpha, \beta)$ class of gauges (3.11) can be written as (cf. (3.12))

$$V = -\tfrac{1}{2}\int \mathrm{d}k\,\Big\{3\ln\big(k^2+g^2\phi^2\big)+\ln\big(k^2+\tfrac{1}{2}\lambda\phi^2\big)$$
$$+\ln\Big[\big(\alpha^{-1}k^2+g^2\phi^2\big)\big(k^2+\tfrac{1}{6}\lambda\phi^2+\beta^2\alpha^{-1}g^2\phi^2\big)-g^2\phi^2k^2\big(1-\beta\alpha^{-1}\big)^2\Big]$$
$$-2\ln\big(\alpha^{-1/2}\big)\big(k^2+\beta g^2\phi^2\big)\Big\},\qquad \mathrm{d}k=\frac{\mathrm{d}^4k}{(2\pi)^4}\,. \tag{A.1}$$

At the same time, if $\phi \neq 0$ one can calculate it without gauge fixing

$$\tilde{V} = -\tfrac{1}{2}\int \mathrm{d}k\,\big\{3\ln\big(k^2+g^2\phi^2\big)+\ln\big(k^2+\tfrac{1}{2}\lambda\phi^2\big)+\ln\big(\tfrac{1}{6}\lambda g^2\phi^2\big)\big\}\,. \tag{A.2}$$

The last ($\phi \to 0$ singular) term is quartically divergent and thus can be cancelled by a local measure (or is zero in dimensional regularization), while the first two terms coincide with the result in the unitary gauge.

Next consider the Einstein gravity with the $\Lambda$ term (sect. 4). Integrating over $\bar{h}^{\perp}$ and $\sigma$, we find from eqs. (4.2), (2.41)

$$Z=\left[\frac{\det\Delta_1(-\Lambda)}{\det\Delta_2\left(\frac{8}{3}\Lambda-2\Lambda_0\right)\det\Delta_0(4\Lambda_0-4\Lambda)}\right]^{1/2}\int \mathrm{d}h\,\mathrm{d}\xi^{\perp}\exp(-I'),\quad (\mathrm{A.3})$$

$$I'=\frac{12\pi^2}{\Lambda^2k^2}\left\{2(\Lambda-\Lambda_0)\xi^{\perp}\Delta_1(-\Lambda)\xi^{\perp}+\tfrac{1}{2}(\Lambda-\Lambda_0)\right.$$
$$\left.\times h\Delta_0(-2\Lambda_0)\Delta_0^{-1}(4\Lambda_0-4\Lambda)h\right\}. \quad (\mathrm{A.4})$$

If $\Lambda\neq\Lambda_0$ we can formally integrate over $\xi^{\perp}$ and $h$ with the result (after omitting $\sim\ln(\Lambda-\Lambda_0)$ terms)

$$\tilde{Z}=\left[\det\Delta_2\left(\tfrac{8}{3}\Lambda-2\Lambda_0\right)\det\Delta_0(-2\Lambda_0)\right]^{-1/2}. \quad (\mathrm{A.5})$$

This expression is to be compared with that in the $\xi^{\perp}=0$, $h=0$ gauge (4.8) which follows after the "brutal" insertion of $\delta(h)\delta(\xi^{\perp})\Omega_{\text{ghost}}$, $\Omega_{\text{ghost}}=[\det\Delta_0(0)]^{1/2}$, in (A.3). We observe that (A.5) does not have a correct on-shell limit. For example it describes *six* instead of *two* degrees of freedom. This is a manifestation of "massiveness" of the off-shell theory in the approach we use.

Further evidence for this is provided by the example of the gravitino contribution in the effective action (see sect. 5). Integration over $\varphi^{\perp}$ and $\zeta$ in (5.18) gives

$$Z=\left[\det(\hat{\mathcal{D}}-m)_{\varphi^{\perp}}\det(\hat{\mathcal{D}}+2m)\right]^{1/2}\int \mathrm{d}\psi\,\mathrm{e}^{-I'},$$

$$I'=\frac{24\pi^2}{\Lambda^2}\left[\tfrac{1}{4}(\Lambda+3m^2)\bar{\psi}\,(\hat{\mathcal{D}}+2m)^{-1}\psi\right]. \quad (\mathrm{A.6})$$

Again $\Lambda+3m^2\neq0$ off shell (cf. (5.6)) and we find (for one gravitino)

$$\tilde{Z}=\left[\det(\hat{\mathcal{D}}-m)_{\varphi^{\perp}}\right]^{1/2}, \quad (\mathrm{A.7})$$

while the definition of (A.6) with $\psi=0$ as a gauge brings us back to eq. (5.19). Finally we note that to use the analogs of (A.5) and (A.7) in the context of the O(4) supergravity calculation of sect. 5, one has to take into account the scalar-graviton (5.9) and spinor-gravitino (5.17) mixing terms before integrating over $h$ and $\psi$ in (A.3) and (A.6). The resulting $\tilde{\Gamma}$ does not however describe a zero total number of degrees of freedom $(5+1+2+6\times2-4\times4-8\neq0)$ and thus we refrain from further discussion of it here.

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