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ONE-LOOP β-FUNCTION IN CONFORMAL SUPERGRAVITIES

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We present a detailed calculation of the one-loop β-function in $N=1$ conformal supergravity. The conformal gravitino is found to give a negative contribution of the gravitational infinities while the "Weyl graviton" and the ordinary matter fields are known to give positive ones. β-functions are also obtained for the U(N)-extended conformal supergravities. As a result we prove that the $N=1$, 2, 3 theories are asymptotically free in the Weyl coupling constant (just like the Weyl theory) while the $N=4$ theory is (one-loop) finite and thus anomaly free. The sequence of U(N) β-functions ($N \leqslant 4$) is found to be in remarkable correspondence with the analogous sequence for gauged O(n) Poincaré supergravities.

1. Introduction

Conformal supergravities [1–7] are presently interesting, mainly in view of the fact that they provide the framework for a deeper and more systematic understanding of the structure of ordinary Poincaré (or De Sitter) supergravity theories (see e.g. [8]). However, one must also keep in mind the possibility that superconformal theories may appear to be more directly related to the description of fundamental interactions. For example, an appropriate supersymmetric extension of the $(R\phi^2 + 6(\partial\phi)^2 + \alpha^{-2}W)$ renormalizable gravitational lagrangian (W is the square of Weyl tensor) may lead to a viable quantum gravity plus unified matter theory where the problem of ghosts will be solved due to a high type of the symmetry (there is equal number of Fermi and Bose ghosts) and/or accounting for radiative corrections (see e.g. [9, 10] and refs. there).

Apart from these general considerations, conformal supergravities are interesting field theoretical models with the highest degree of symmetry presently known. The fact that the global superconformal symmetry (known to improve quantum behaviour, for example, in $N=4$ super Yang–Mills theory, see e.g. [11]) is localized here may provide a deeper insight, e.g., into the problem of anomalies. Therefore, the study of the quantum properties of conformal supergravities seems to be an important avenue for research.

Here we make the first step in this direction by calculating the one-loop β-functions. This work can be considered as an extension of our paper [9] where a systematic approach to one-loop renormalizations in higher derivative gauge theories was worked out. Hence the technique we use is mostly the same as in [9], including the correctly accounting for the contribution of averaging over gauge operators (cf. [12–14]), the background field method (see e.g. [12, 15]) and the algorithms for the divergences of the determinants of various differential operators [12, 16, 17, 9].

In sect. 2 we first discuss the question of counting the degrees of freedom in $N=1$ conformal supergravity, clarifying and correcting earlier statements [1–5] and stressing peculiarities of this counting in higher derivative gauge theories. Then we justify the idea that for a calculation of the one-loop $N=1$ β-function it is sufficient to consider either "electromagnetic" or "gravitational" sectors in the infinite part of the one-loop effective action where only the axial vector field and metrical background, respectively, are non-trivial.

In view of the fact that the value of the $N=1$ β-function is basic for establishing the β-functions in N-extended conformal supergravities, we present (in sects. 3 and 4) independent calculations of it in both sectors and find agreement (which also gives an independent check on the value of the β-function in the Weyl theory found in [9]). The essential technical simplification is achieved here due to the possibility of obtaining the algorithm for the infinities of $\log\det\Delta_3$ for the third-order gravitino operator using the corresponding algorithm for the fourth-order operators found in [9].

In sect. 5 we briefly describe the analogous calculation (in the "gravitational" sector) in $N=2, 3, 4$ extended conformal supergravities [6, 7]. The total one-loop β-function is obtained here by summing the graviton (positive), N gravitinos (negative) and the gauge and matter field contributions. The final result is that $\beta_0>\beta_{\rm I}>\beta_{\rm II}>\beta_{\rm III}>\beta_{\rm IV}=0$, thus indicating that the $N=1, 2, 3$ conformal supergravities are asymptotically free (just like the $N=0$ Weyl theory [18, 9]), while the maximally extended $N=4$ theory is one-loop finite (providing one more candidate for a completely finite theory, see [11] for a review) and thus is free from anomalies.

In sect. 6 we present a discussion of the results comparing them with the analogous ones in O(N) supergravities. Appendix A contains our notations and some useful formulae. In appendix B we summarize, for convenience, the known expressions for the algorithms for one-loop infinities. Appendix C deals with quantization of the ordinary (first derivative) gravitino lagrangian on a curved background in a non-standard gauge $\mathscr{D}_\mu(\psi_\mu-\frac{1}{4}\gamma_\mu\gamma\cdot\psi)=0$ as compared to the known results in the $\gamma_\mu\psi_\mu=0$ gauge and the analogous quantization of the third derivative gravitino lagrangian in conformal supergravity.

2. Counting degrees of freedom and the general recipe for the calculation of the $N=1$ one-loop β-function

The lagrangian of simple $N=1$ conformal supergravity has the following form in terms of the physical fields $g_{\mu\nu}$, A_μ, ψ_μ [1, 3]:

$$\mathscr{L}_1 = W = R^2_{\mu\nu} - \tfrac{1}{3}R^2\,, \qquad \mathscr{L}_2 = -\tfrac{3}{4}F^2_{\mu\nu}\,, \tag{2.2}$$

$$\mathscr{L}_3 = 4\epsilon^{\mu\nu\rho\sigma}\bar{\phi}_\rho\gamma_5\gamma_\sigma D^+_\nu\phi_\mu\,, \tag{2.3}$$

$$\mathscr{L}_4 = \tfrac{1}{4}i\bar{\psi}_\rho\gamma_\rho\mathscr{F}_{\mu\nu}S^{\mu\nu} + \tfrac{1}{2}i(\bar{\psi}_\lambda\gamma_\rho + 2\bar{\psi}_\rho\gamma_\lambda)\mathscr{F}_{\rho\mu}S^{\mu\lambda}\,, \tag{2.4}$$

$$\begin{aligned}\mathscr{L}_5 = -R_{\mu\nu}[2(\bar{\psi}_\lambda\sigma_{\lambda\nu}\phi_\mu - \bar{\psi}_\mu\sigma_{\lambda\nu}\phi_\lambda) + \bar{\psi}_\lambda\gamma_\nu \\ \times(\mathscr{D}_\mu\psi_\lambda - \mathscr{D}_\lambda\psi_\mu - \gamma_\mu\phi_\lambda + \gamma_\lambda\phi_\mu)] + \tfrac{4}{3}R\bar{\psi}_\lambda\sigma_{\lambda\nu}\phi_\nu\,,\end{aligned} \tag{2.5}$$

where $\mathscr{L}_6$ includes terms like $\mathscr{D}R\bar{\psi}\psi$, $(\bar{\psi}\psi)^4$, $R(\bar{\psi}\psi)^2$, etc., which will prove to be irrelevant for our purposes, and

$$R_{\mu\nu} = R_{\mu\nu}(g)\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,,$$

$$\phi_\mu = \tfrac{1}{3}\gamma^\nu(S_{\mu\nu} + \tfrac{1}{4}\gamma_5\tilde{S}_{\mu\nu})\,, \qquad S_{\mu\nu} = D_\nu\psi_\mu - D_\mu\psi_\nu\,, \tag{2.6}$$

$$\tilde{S}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}S^{\rho\sigma}\,, \qquad \mathscr{F}_{\mu\nu} = \gamma_5 F_{\mu\nu} - \tfrac{1}{2}\tilde{F}_{\mu\nu}\,, \tag{2.7}$$

$$(D_\mu)^\rho_\sigma = \delta^\rho_\sigma(\partial_\mu + \tfrac{1}{2}\sigma^{ab}\omega_{ab\mu} - \tfrac{3}{4}i\gamma_5 A_\mu) + \Gamma^\rho_{\sigma\mu}\,. \tag{2.8}$$

Note that $D^+_\mu = \cdots + \tfrac{3}{4}i\gamma_5 A_\mu + \cdots$ and ω and Γ are torsionless; for other notations see appendix A. It is convenient to use instead of A_μ the following variable:

$$B_\mu = \tfrac{3}{4}iA_\mu\,, \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,. \tag{2.9}$$

The action for (2.1) is invariant under the general coordinate and local Lorentz transformations, and also under the "ordinary" Q and "conformal" S supersymmetries (with the parameters ε and λ), the scale (Λ) and chiral (α) transformations

$$\begin{aligned}\delta e^a_\mu &= \bar{\varepsilon}\gamma^a\psi_\mu + \Lambda e^a_\mu\,,\\ \delta\psi_\mu &= 2D_\mu\varepsilon + \gamma_\mu\lambda - \tfrac{1}{2}\Lambda\psi_\mu + i\alpha\gamma_5\psi_\mu\,,\\ \delta B_\mu &= \tfrac{3}{4}\bar{\varepsilon}\gamma_5\phi_\mu + \tfrac{3}{4}\bar{\lambda}\gamma_5\psi_\mu + i\partial_\mu\alpha\,.\end{aligned} \tag{2.10}$$

Let us first consider the quantization of (2.1) in the linear approximation where

$$\mathscr{L}_1 \simeq \tfrac{1}{4}h_{\mu\nu}\Box^2 h_{\mu\nu} - \tfrac{1}{2}\chi_\mu H_{\mu\nu}\chi_\nu - \tfrac{1}{12}\varphi'\Box^2\varphi'\,, \tag{2.11}$$

$$\begin{aligned}&g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}\,, \qquad \varphi = h^\mu_\mu\,, \qquad \chi_\mu = \partial_\rho h_{\rho\mu}\,,\\ &\varphi' = \varphi - \Box^{-1}\partial_\mu\chi_\mu\,, \qquad H_{\mu\nu} = -g_{\mu\nu}\Box + \tfrac{1}{2}\partial_\mu\partial_\nu\,,\\ &\mathscr{L}_3 \simeq -\bar{\psi}_\mu\hat{\partial}^3\psi_\mu - \tfrac{2}{3}\bar{\chi}\hat{\partial}\chi + \tfrac{1}{2}\bar{\psi}\hat{\partial}^3\psi\,,\\ &\quad\hat{\partial} = \gamma_\mu\partial_\mu\,, \qquad \psi = \gamma_\mu\psi_\mu\,, \qquad \chi = \partial_\mu\psi_\mu + \tfrac{1}{2}\hat{\partial}\psi\,.\end{aligned} \tag{2.12}$$

Choosing the following gauges for the coordinate, Λ, ε, λ and α-groups respectively,

$$\begin{aligned}&\text{(a)}\ \ \partial_\mu h_{\mu\nu} = \xi_\nu(x)\,, \qquad \varphi = \eta(x)\,,\\ &\text{(b)}\ \ \partial_\mu\psi_\mu = \zeta(x)\,, \qquad \psi = \rho(x)\,,\\ &\text{(c)}\ \ \partial_\mu A_\mu = \gamma(x)\,,\end{aligned} \tag{2.13}$$

and averaging over them with the help of suitable weight operators [see (A.8)] we get the following result for the free partition function*:

$$Z^{(0)}=Z_h Z_A Z_\psi\,,\qquad Z_A=\frac{\det\Delta_g^{(c)}}{\sqrt{\det(-\Box)_A}}\,,\tag{2.14}$$

$$\begin{aligned}Z_h&=\frac{\det\Delta_g^{(a)}\sqrt{\det\Box^2}\sqrt{\det H_{\mu\nu}}}{\sqrt{\det(\Box^2)_h}}\,,\\ Z_\psi&=\frac{\sqrt{\det(\hat\partial^3)_\psi}}{\det\Delta_g^{(b)}\sqrt{\det\hat\partial}\sqrt{\det\hat\partial^3}}\,,\end{aligned}\tag{2.15}$$

where the ghost operators are $\Delta_{g_{\mu\nu}}^{(a)}=-\delta_{\mu\nu}\Box$, $\Delta_g^{(b)}=-\hat\partial^2$, $\Delta_g^{(c)}=-\Box$. Taking Z_0 to be the contribution of one Bose degree of freedom (A.7) and noting that $\det H_{\mu\nu}=Z_0^{-4}$, we have

$$\begin{aligned}Z_h&=Z_0^{-8-2-4}/Z_0^{-20}=Z_0^6\,,\qquad Z_A=Z_0^2\,,\\ Z_\psi&=Z_0^{-24}/Z_0^{-8-2-6}=Z_0^{-8}\,,\end{aligned}\tag{2.16}$$

and thus $Z^{(0)}=1$, i.e. the total number of degrees of freedom $(6+2-8)$ is zero as it is to be in a supersymmetric theory.

We conclude that the effective number of gravitational degrees of freedom corresponding to the Weyl lagrangian (2.2) is 6 [9] and not 4 ("two gravitons") as one can naively suppose in view of the representation $\mathcal{L}_1\simeq\frac{1}{4}hP^{2^+}\Box^2h$ (P^{2^+} is the transverse traceless spin 2 projector). In an analogous way the $\hat\partial^3$ gravitino lagrangian (2.3) *describes* 8 *and not* 6 ("three gravitinos") Fermi degrees of freedom (cf. [1–5]). This peculiarity in counting degrees of freedom in higher derivative gauge theories is due to the fact that the increase of the number of derivatives in the action in transition, e.g., from Einstein to Weyl theory, is not followed by a corresponding growth (doubling) of the dimension of the invariance group.

The naive counting of degrees of freedom in $N=1$ conformal supergravity (2 gravitons + 1 vector field − 3 gravitinos = 0, see e.g. [4]) is based on the known [5] spectrum of the supersymmetric extension of the lagrangian $\mathcal{L}'=-(1/k^2)R+\alpha^{-2}W$, consisting of one $(2,\frac{3}{2})$ massless and one $(2,\frac{3}{2},\frac{3}{2},1)$ massive $(m=\alpha/k)$ multiplet. Taking $k^2\to\infty$ we apparently get the above naive counting. However, the limit $k^2\to\infty$ is not a regular one [9]. Really (let us consider for simplicity only the gravitational part), for a finite k^2 we schematically have $\mathcal{L}'=-(1/k^2)(\bar h\Box\bar h-\varphi\Box\varphi)+(1/4\alpha^2)\bar h\Box^2\bar h$ (remember that φ drops from W) and so the contribution of the $\bar h_{\mu\nu}$ path integral $Z_0Z_m^5$ is supplemented by the Z_0 contribution of φ. The resulting number of degrees of freedom is $6+1=7=2+5$ corresponding

* Note that if one uses the variables $(\bar h_{\mu\nu},\varphi)$ and (φ_μ,ψ) [see (A.9)], φ and ψ drop from (2.11) and (2.12) due to the scale invariance and S supersymmetry and so there is no need for averaging over gauges $\varphi=\eta$, $\psi=\rho$.

to the spectrum* $2_0^+ + 2_m^+$. Hence the singularity of the limit $k^2 \to \infty$ is connected (i) with the absence (due to conformal invariance) of the φ-contribution in the limiting theory and (ii) with the difference in the number of degrees of freedom for the massive and massless spin-2 field.

Let us now consider the quantization of the theory (2.1) on the non-trivial background of the fields $\hat{\Phi} = \{\hat{g}, \hat{A}, \hat{\psi}\}$. The (one-loop) effective action is defined as follows:

$$Z[\hat{\Phi}] = e^{-I_{\text{eff}}[\hat{\Phi}]} = \int [d\Phi]\, e^{-\tilde{I}[\Phi, \hat{\Phi}]} ,$$
$$\tilde{I} = I[\Phi + \hat{\Phi}] - I[\hat{\Phi}] - \frac{\delta I}{\delta \Phi} \Phi . \quad (2.17)$$

Here, as always, one must choose a gauge which breaks the "quantum" gauge group but respects the background gauge invariance. This yields the formal background gauge invariance of the effective action. However, this formal invariance may be broken by a regularization which manifests itself in the anomalies or non-invariant terms in the *finite* $I_f = I_{\text{eff}} - I_\infty$ part of the effective action. As is well-known, the conformal, S superconformal and chiral anomalies often arise when one uses the regularizations preserving general covariance and Q supersymmetry (for example, ordinary dimensional regularization or its modification by dimensional reduction, see e.g. [19]). Let us remark in passing that the recipe for maintaining the background conformal invariance of the total effective action in the Weyl theory by substituting the background metric by the one belonging to the same conformal equivalence class but having zero curvature scalar [20, 9] possibly allows a supersymmetric generalization and thus gives a way to escape the anomalies also in conformal supergravity (and provides the possibility of the explicit construction of the background covariant gauges).

Here we shall consider only the infinite part I_∞ of the effective action which is invariant if the gauges are background covariant. However, it is also possible to use background non-covariant gauges taking into account only the invariant part $I_\infty^{(\text{inv})}$ of I_∞ when calculating the β-function. Keeping this in mind we can choose the following gauges for (2.1) [cf. (2.13)]

$$\text{(a)}\quad \mathcal{D}_\mu h_{\mu\nu} = \xi_\nu(x) , \qquad \varphi = \eta(x) ,$$
$$\text{(b)}\quad \mathcal{D}_\mu \psi_\mu = \zeta(x) , \qquad \psi = \rho(x) , \quad (2.18)$$
$$\text{(c)}\quad \mathcal{D}_\mu A_\mu = \gamma(x)$$

(covariant derivatives depend on the background fields). The part of $\tilde{I}$ in (2.17)

* We remark in passing that if an R^2 term is added to $\mathcal{L}$, we get one more ($R^2 \sim \varphi \Box^2 \varphi$) degree of freedom: $7+1 = 2+5+1$ in accordance with the spectrum $2_0^+ + 2_m^+ + 0_{m'}^-$.

which is bilinear in the quantum fields (or the one-loop part) has the following structure:

$$\tilde{I} = (h\Delta_4 h + A\Delta_2 A + AM_1 h) + (\bar{\psi}\Delta_3\psi) + (\bar{\psi}M_2 A + \bar{\psi}M_3 h + \text{h.c.}) \,, \tag{2.19}$$

where the operators Δ and M are background dependent. Averaging over gauges (2.18) we can establish the diagonality of the highest derivative terms in Δ. The ghost operator (its part essential in the one-loop approximation) does not in general factorize on the ghost operators for (a), (b) and (c) gauges in (2.18). This fact along with the mixings in the third bracket of (2.19) makes the direct evaluation of the one-loop infinities a hard problem.

The essential simplification stems from the remark that in view of the formal off-shell renormalizability of the theory (2.1), the invariant part of the divergences is proportional to the action itself [see (B.7)]:

$$I_\infty^{(\text{inv})} = \tilde{\beta} I_0 \,, \qquad \tilde{\beta} = -\frac{\beta}{2(4\pi)^2}\log\frac{L^2}{\mu^2} \,. \tag{2.20}$$

As a consequence, in order to find the β-function it is sufficient to obtain the coefficient before any of the terms in the action which cannot contribute in the non-invariant part of I_∞. It is easy to see that $\mathcal{L}_2$ in (2.1) possesses the desired property. Suppose we have made the gauges (2.18) background covariant. From the structure of the transformation laws (2.10) it follows that this covariantization procedure cannot give new terms in (2.18) which are independent of $\hat{\psi}_\mu$ and thus does not lead to additional $\hat{F}^2_{\mu\nu}$ (or $\hat{\psi}$-independent) terms in I_∞. The conclusion is that all $\hat{F}^2_{\mu\nu}$ terms in I_∞ belong to* $I_\infty^{(\text{inv})}$. The same statement is valid in the case of $\mathcal{L}_1^{(\text{inv})}$ in (2.1).

Let us now suppose that the background fields in (2.17)–(2.19) are chosen in one of the following two ways:

$$\text{E:}\ \hat{g}_{\mu\nu} = \delta_{\mu\nu} \,, \qquad \hat{\psi}_\mu = 0 \,, \qquad \hat{A}_\mu = \text{arbitrary} \,, \tag{2.21}$$

$$\text{G:}\ \hat{g}_{\mu\nu} = \text{arbitrary with } R(\hat{g}) = 0 \,, \qquad \hat{\psi}_\mu = 0 \,, \qquad \hat{A}_\mu = 0 \,. \tag{2.22}$$

We shall speak about (2.21) and (2.22) as "electromagnetic" (E) and "gravitational" (G) "sectors", respectively. The condition $R(\hat{g}) = 0$ in (2.22) establishes the background conformal invariance of (2.18)) and thus of I_∞ and is a matter of convenience only.

It is obvious that the above choices are sufficient for the evaluation of $\tilde{\beta}$ in (2.20) as a coefficient of the $\mathcal{L}_2$ or $\mathcal{L}_1$ terms in I_∞, resepctively. In both cases $M_2 = M_3 = 0$ in (2.19) and the ghost operator can be factorized on the three independent parts.

* Let us recall once more that the non-invariant terms in I_∞ arise only due to the absence of complete background covariance of the gauges (2.18) (they lack background scale invariance and Q and S supersymmetries).

The resulting expressions for the β-function in the $N=1$ theory in the E and G sectors are the following:

$$\beta_I^{\mathrm{E}} = \beta_h^{\mathrm{E}} + \beta_\psi^{\mathrm{E}}, \qquad \beta_I^{\mathrm{G}} = \beta_h^{\mathrm{G}} + \beta_\psi^{\mathrm{G}} + \beta_{\mathrm{A}}^{\mathrm{G}}, \tag{2.23}$$

where β_h^{E} is the contribution (in the E-sector) of the first bracket in (2.19) while β_I^{G} is the sum of the contributions of Δ_p in (2.19) in the G-sector (where $M_1=0$).

3. Calculation of the $N=1$ β-function in the "electromagnetic" sector

As was found in [9],

$$\beta_h^{\mathrm{E}} = 0, \tag{3.1}$$

i.e. the interaction of higher derivative gravity with a gauge field does not give additional $\hat{F}^2_{\mu\nu}$ infinities. Let us briefly recall the derivation of this result. The terms in the first bracket in (2.19) arise from the expansion of $\mathscr{L}_1$ and $\mathscr{L}_2$ in (2.1) in the E-sector (2.21) ($\mathscr{L}_2\sqrt{g}$ gives $h\hat{F}\hat{F}h$ and $h\hat{F}\partial A$ terms). From now on we shall omit $\hat{}$ on background fields. In view of the conformal invariance of $\mathscr{L}_1$ and $\mathscr{L}_2$ only $\bar{h}_{\mu\nu}$ (A.9) contribute and we get, after averaging over gauges (a) and (c) in (2.13),

$$a\mathscr{L}_1 - \frac{1}{3e}\mathscr{L}_2 \simeq \frac{1}{2}\Big\{\tfrac{1}{2}a\bar{h}_{\mu\nu}\Box^2\bar{h}_{\mu\nu} + \frac{1}{e}[A_\mu(-\Box)A_\mu + \bar{h}_{\mu\nu}U^{\mu\nu}_{\alpha\beta}\bar{h}^{\alpha\beta} + 2\bar{h}^{\alpha\beta}\bar{Y}^{\mu\rho}_{\alpha\beta}\partial_\mu A_\rho]\Big\}, \tag{3.2}$$

$$\bar{Y}^{\mu\rho}_{\alpha\beta} = 2\delta^{[\mu}_{\underline{\beta}}F^{\rho]}_{\;\cdot\underline{\alpha}}, \qquad U^{\mu\nu}_{\alpha\beta} = F_{\rho\alpha}F_{\rho\underline{\mu}}g_{\underline{\nu}\beta} + \tfrac{1}{2}F_{\underline{\mu}\alpha}F_{\underline{\nu}\beta} - \tfrac{1}{8}F^2_{\lambda\rho}g_{\underline{\mu}\alpha}g_{\underline{\nu}\beta}, \tag{3.3}$$

where a and e are constants and $P_{\underline{\mu\nu}}$ is a traceless tensor. The contribution of (3.2) in infinities can be easily found (e.g. by the diagram method) with the result (for notations see appendix B)

$$\Delta\bar{b}_4 = \frac{2}{a\cdot e}\,\mathrm{tr}\,(-U^{\alpha\beta}_{\alpha\beta} + \tfrac{1}{4}\bar{Y}^{\mu\rho}_{\alpha\beta}\bar{Y}^{\mu\rho}_{\alpha\beta}). \tag{3.4}$$

Substituting (3.3) in (3.4) (and noting that the corresponding (a) and (c) ghost operators and averaging operators are A_μ independent) we are left with the non-trivial result (3.1).

Now we turn to the evaluation of the gravitino contribution β_ψ^{E} in (2.23). First of all, let us write down the expressions for the relevant ($\mathscr{L}_3+\mathscr{L}_4$) part of (2.1) in the E-sector (2.21) [cf. (2.12)]

$$\mathscr{L}_3 = -\bar{\psi}_\mu\hat{\mathscr{D}}\hat{\mathscr{D}}^+\hat{\mathscr{D}}\psi_\mu - \tfrac{2}{3}\bar{\chi}\hat{\mathscr{D}}\chi + \tfrac{1}{2}\bar{\psi}\hat{\mathscr{D}}^+\hat{\mathscr{D}}\hat{\mathscr{D}}^+\psi + \mathscr{L}_1^V, \qquad \mathscr{L}_4 = \mathscr{L}_2^V, \tag{3.5}$$

where $\mathscr{D}_\mu(\mathscr{D}^+_\mu) = \partial_\mu \mp B_\mu$ [see (2.9)], $\chi = \mathscr{D}_\mu\psi_\mu + \tfrac{1}{2}\hat{\mathscr{D}}\psi$ and

$$\mathscr{L}^V = \bar{\psi}_\mu V^\alpha_{\mu\nu}\mathscr{D}_\alpha\psi_\nu, \tag{3.6}$$

$$V^{\alpha}_{\mu\nu}=a_1B_{\mu\nu}\gamma_\alpha\gamma_5+a_2\gamma_{[\nu}B_{\mu]\alpha}\gamma_5+a_3\gamma_\rho\gamma_5B_{\rho[\mu}g_{\nu]\alpha}$$
$$+b_1\tilde{B}_{\alpha(\mu}\gamma_{\nu)}+b_2\gamma_\rho\tilde{B}_{\rho(\mu}g_{\nu)\alpha}+b_3\gamma_\rho\tilde{B}_{\rho\alpha}g_{\mu\nu}\,, \quad (3.7)$$

$$\mathcal{L}^V_1:\quad \begin{matrix} a_1=1\,, & a_2=\frac{4}{3}\,, & a_3=-\frac{2}{3}\,, \\ b_1=\frac{1}{3}\,, & b_2=0\,, & b_3=-\frac{1}{2}\,, \end{matrix} \quad (3.8)$$

$$\mathcal{L}^V_1+\mathcal{L}^V_2:\quad \begin{matrix} a_1=\frac{7}{3}\,, & a_2=-\frac{2}{3}\,, & a_3=-\frac{4}{3}\,, \\ b_1=0\,, & b_2=-\frac{1}{3}\,, & b_3=-\frac{1}{6}\,. \end{matrix} \quad (3.9)$$

In the derivation of these expressions we used the fact that in the calculation of the β-function one can formally put $\partial_\mu B_{\lambda\rho}=0$, thus obtaining $V^\alpha_{\mu\nu}=-\bar{V}^\alpha_{\nu\mu}$ in (3.6) [the bar is Majorana conjugation (A.1)].

In view of the Q and S invariance of the complete action* we have the possibility to cancel the "non-diagonal" second and third terms in (3.5) using the gauges (b) in (2.18). As a result, the gravitino contribution to the (E-sector) one-loop path integral has the form [cf. (2.15)]

$$Z^{(1)}_\psi[A]=\frac{\sqrt{\det\hat{\Delta}_3}}{\det\Delta^{(b)}_g\sqrt{\det\hat{\mathscr{D}}}\sqrt{\det\hat{\mathscr{D}}^+\hat{\mathscr{D}}\hat{\mathscr{D}}^+}}\,, \quad (3.10)$$

where

$$\hat{\Delta}_{3\mu\nu}=-g_{\mu\nu}\hat{\mathscr{D}}\hat{\mathscr{D}}^+\hat{\mathscr{D}}+V^\alpha_{\mu\nu}\mathscr{D}_\alpha\,, \quad (3.11)$$

$$\Delta^{(b)}_g=-\tfrac{4}{3}\mathscr{D}_\mu\bar{g}_{\mu\nu}\mathscr{D}_\nu=-\mathscr{D}^2-\tfrac{1}{3}\sigma\cdot B\gamma_5\,. \quad (3.12)$$

Here $\bar{g}_{\mu\nu}=g_{\mu\nu}-\frac{1}{4}\gamma_\mu\gamma_\nu$, $\sigma\cdot B=\sigma_{\mu\nu}B_{\mu\nu}$ and we used that $\hat{\mathscr{D}}^+\hat{\mathscr{D}}=\mathscr{D}^2-\sigma\cdot B\gamma_5$. The expression for $V^\alpha_{\mu\nu}$ in (3.11) is given by (3.6) and (3.9). Thus the ψ_μ contribution to I_∞ is given by (B.1) where

$$\bar{b}_4=-\bar{b}_4(\hat{\Delta}_3)+2\bar{b}_4(\Delta^{(b)}_g)+2\bar{b}_4(\hat{\Delta}_2)\,, \quad (3.13)$$

$$\hat{\Delta}_2=-\hat{\mathscr{D}}\hat{\mathscr{D}}^+=-(\mathscr{D}^+)^2-\sigma\cdot B\gamma_5\,. \quad (3.14)$$

With the help of the Δ_2 algorithm (B.5) we easily obtain that

$$\bar{b}_4(\hat{\Delta}_2)=-\tfrac{2}{3}B^2\,,\qquad \bar{b}_4(\Delta^{(b)}_g)=+\tfrac{2}{9}B^2\,,\qquad B^2=B_{\mu\nu}B_{\mu\nu}\,, \quad (3.15)$$

and so the main problem is the calculation of the $\bar{b}_4$ for $\hat{\Delta}_3$ (3.11). One can avoid a tedious diagram evaluation of this coefficient by noting that the multiplication of $\hat{\Delta}_3$ and the first-order operator $\hat{\Delta}_1$ gives a fourth-order operator of the (B.4) type:

$$\hat{\Delta}_{4\mu\nu}=\hat{\Delta}_{3\mu\lambda}\hat{\Delta}_{1\lambda\nu}\,,\qquad \hat{\Delta}_{1\lambda\nu}=-g_{\lambda\nu}\hat{\mathscr{D}}^+\,, \quad (3.16)$$

$$\hat{\Delta}_{4\mu\nu}=g_{\mu\nu}(\mathscr{D}^+)^4+V^{(\alpha\beta)}_{\mu\nu}\mathscr{D}^+_\alpha\mathscr{D}^+_\beta+U_{\mu\nu}\,, \quad (3.17)$$

* Observe that the E-sector one-loop truncated action ($\mathscr{L}_3+\mathscr{L}_4$) is invariant only under $\delta\psi_\mu=\gamma_\mu\lambda$ transformations.

$$V^{(\alpha\beta)}_{\mu\nu} = 2g_{\mu\nu}g^{\alpha\beta}\sigma\cdot B\gamma_5 - V^{(\alpha}_{\mu\nu}\gamma^{\beta)}\,,$$
$$U_{\mu\nu} = g_{\mu\nu}(\sigma\cdot B)^2 - \tfrac{1}{2}V^{\alpha}_{\mu\nu}\gamma^{\beta}B_{\alpha\beta}\gamma_5\,, \tag{3.18}$$

with the b_4 coefficient, calculable according to (B.6),

$$\bar{b}_4(\hat{\Delta}_4) = \bar{b}_4^{(0)} + \mathrm{tr}\,\{-\tfrac{1}{2}V^{\alpha}\gamma_{\alpha}\sigma\cdot B\gamma_5 + \tfrac{1}{2}V^{\alpha}\gamma^{\beta}B_{\alpha\beta}\gamma_5 + \tfrac{1}{24}V^{\alpha}_{\nu\mu}\gamma^{\beta}V^{(\alpha}_{\mu\nu}\gamma^{\beta)} + \tfrac{1}{48}V^{\alpha}_{\mu\nu}\gamma^{\alpha}V_{\nu\mu}{}^{\beta}\gamma^{\beta}\}\,, \qquad V^{\alpha} = V^{\alpha}_{\mu\nu}g^{\mu\nu} \tag{3.19}$$

($\bar{b}_4^{(0)} = 8\bar{b}_4(\hat{\Delta}_2)$ is the value for $V^{\alpha}_{\mu\nu} = 0$). Using the relations

$$b_4(\hat{\Delta}_3) = b_4(\hat{\Delta}_4) - b_4(\hat{\Delta}_1)\,,$$
$$b_4(\hat{\Delta}_1) = 4\cdot\tfrac{1}{2}b_4(\hat{\Delta}_2)\,, \tag{3.20}$$

and substituting (3.7) in (3.19) we finally obtain the following expression for b_4 in (3.13):

$$\bar{b}_4 = [\tfrac{28}{9} - 4(b_1 - b_2 - 4b_3) + 3b_1^2 + 3b_2^2 + 8b_3^2 - 2b_1b_2 + 4b_2b_3 - 4b_1b_3 + 2a_1^2 - a_1a_3 + a_1a_2 - \tfrac{1}{2}a_2a_3 - \tfrac{1}{4}a_3^2 - \tfrac{1}{4}a_2^2]B^2\,, \tag{3.21}$$

or, in view of (3.9),

$$\bar{b}_4 = \tfrac{34}{3}B^2_{\mu\nu}\,. \tag{3.22}$$

Taking into account that in terms of B_μ the lagrangian $\mathcal{L}_2$ is $+\frac{4}{3}B^2_{\mu\nu}$, we conclude that the $N=1$ β-function (2.23) is given by

$$\beta^{\mathrm{E}}_I = \beta^{\mathrm{E}}_\psi = \tfrac{17}{2} \tag{3.23}$$

[we supposed that in (B.7) $\mathcal{L}_0$ stands for $\mathcal{L}_I$ (2.1)].

4. $N=1$ β-function in the gravitational sector: check and summary of the results

Taking into consideration that the β-function in the Weyl theory [$\mathcal{L}_1$ in (2.2)] obtained in [9] coincides with β^{G}_h in (2.23) while β^{G}_A can be extracted from (B.9),

$$\beta^{\mathrm{G}}_h = \tfrac{199}{15}\,, \qquad \beta^{\mathrm{G}}_A = \tfrac{1}{5}\,, \tag{4.1}$$

we are again left with the problem of establishing the gravitino contribution in the G-sector β-function. The calculation of β^{G}_ψ is step by step analogous to that of β^{E}_ψ. The relevant part of the lagrangian (2.1) is ($\mathcal{L}_3 + \mathcal{L}_5$), simplified by taking account of (2.22) (throughout this section

$$(\mathcal{D}_\mu)^{\lambda}_{\ \rho} = \delta^{\lambda}_{\rho}(\partial_\mu + \tfrac{1}{2}\sigma^{ab}\omega_{ab\mu}) + \Gamma^{\lambda}{}_{\rho\mu}$$

plays the role of $\mathscr{D}_\mu$ and $\mathscr{D}^+_\mu$ of sect. 3). It is straightforward to establish the validity in the G-sector of expressions like (3.5), (3.6), (3.10)–(3.18), where now

$$V^\alpha_{\mu\nu} = a_1 R_{\rho[\mu}\epsilon_{\nu]\rho\alpha\sigma}\gamma_5\gamma_6 + a_2 R_{\rho\alpha}\epsilon_{\mu\rho\nu\sigma}\gamma_5\gamma_6 + b_1 R_{\mu\nu}\gamma_\alpha + b_2 R_{\alpha(\mu}\gamma_{\nu)} + b_3\gamma_\rho R_{\rho\alpha} g_{\mu\nu} + b_4 \gamma_\rho R_{\rho(\mu} g_{\nu)\alpha} , \tag{4.2}$$

$$\mathscr{L}^V_2 = \mathscr{L}_5: \quad \begin{matrix} a_1 = \frac{1}{3}, & a_2 = 0, & b_1 = \frac{1}{3}, \\ b_2 = -\frac{1}{3}, & b_3 = 1, & b_4 = -1, \end{matrix} \tag{4.3}$$

$$\mathscr{L}^V_1 + \mathscr{L}^V_2: \quad \begin{matrix} a_1 = -1, & a_2 = -\frac{1}{2}, & b_1 = 1, \\ b_2 = 0, & b_3 = -\frac{1}{2}, & b_4 = 1, \end{matrix} \tag{4.4}$$

$$\Delta^{(b)}_g = -\tfrac{4}{3}\mathscr{D}_\mu \bar{g}_{\mu\nu}\mathscr{D}_\nu = -\mathscr{D}^2 - \tfrac{1}{12}R , \tag{4.5}$$

$$\hat{\Delta}_2 = -\hat{\mathscr{D}}\hat{\mathscr{D}} = -\mathscr{D}^2 + \tfrac{1}{4}R ,$$

$$\hat{\Delta}_{3\mu\nu} = -(\hat{\mathscr{D}}^3)_{\mu\nu} + V^\alpha_{\mu\rho}(\mathscr{D}_\alpha)_{\rho\nu} , \tag{4.6}$$

$$\hat{\Delta}_{1\mu\nu} = -(\hat{\mathscr{D}})_{\mu\nu} ,$$

$$\tilde{\Delta}_{2\mu\nu} = -(\hat{\mathscr{D}}^2)_{\mu\nu} = -\mathscr{D}^2_{\mu\nu} + \tfrac{1}{4}R g_{\mu\nu} - \sigma\cdot R_{\mu\nu} ,$$

$$\sigma\cdot R_{\mu\nu} = \sigma_{ab} R^{ab}_{\mu\nu} , \tag{4.7}$$

$$V^{\alpha\beta}_{\mu\rho} = 2\sigma\cdot R_{\mu\rho} g^{\alpha\beta} - V^{(\alpha}_{\mu\rho}\gamma^{\beta)} ,$$

$$U_{\mu\nu} = \sigma\cdot R_{\mu\rho}\sigma\cdot R_{\rho\nu} - \tfrac{1}{2}V^\alpha_{\mu\rho}\gamma^\beta (R_{\rho\nu\alpha\beta} + \tfrac{1}{2}\sigma\cdot R_{\alpha\beta} g_{\rho\nu}) . \tag{4.8}$$

Making use of the algorithms (B.5) and (B.6) we obtain [notice that $R = 0$ according to (2.22)]

$$\begin{gathered} \bar{b}_4(\hat{\Delta}_1) = \tfrac{1}{2}\bar{b}_4(\tilde{\Delta}_2) = \tfrac{17}{15}W , \\ \bar{b}_4(\Delta^{(b)}_g) = \bar{b}_4(\hat{\Delta}_2) = -\tfrac{1}{10}W , \end{gathered} \tag{4.9}$$

$$\bar{b}_4 = \bar{b}^{(0)}_4 - [4(a_1 - a_2) - \tfrac{1}{3}(b_2 + 4b_3 + b_4) + a_1^2 - 2a_1a_2 + \tfrac{5}{3}a_2^2 + \tfrac{1}{6}(12b_1^2 - \tfrac{7}{2}b_2^2 - 4b_3^2 - \tfrac{7}{2}b_4^2 + 6b_1b_2 + 6b_1b_4 + 5b_2b_4 - 2b_2b_3 - 2b_3b_4)]W , \tag{4.10}$$

$$\bar{b}^{(0)}_4 = -\tfrac{3}{2}\bar{b}_4(\tilde{\Delta}_2) + 4\bar{b}_4(\hat{\Delta}_2) = -\tfrac{19}{5}W , \tag{4.11}$$

or after the substitution of (4.4),

$$\bar{b}_4 = -\tfrac{149}{30}W , \qquad \beta^G_\psi = -\tfrac{149}{30} . \tag{4.12}$$

Summing (4.1) and (4.12) according to (2.23), we conclude agreement with the previous result (3.23), i.e.

$$\beta^G_I = \beta^E_I = \beta_I .$$

Let us briefly comment on the $N = 1$ results obtained in sects. 3 and 4. First of all the coincidence of β^G_I and β^E_I is a useful check of the consistency of the whole scheme of calculation (e.g. the counterterms appear in a supersymmetric combination, etc.). Second, $\beta_I > 0$ implies that $N=1$ conformal supergravity is asymptotically free [see (B.8)] like the Weyl theory [18, 9]. Third, the remarkable fact is the *negative* sign of the "conformal gravitino" contribution (4.12) in the gravitational (W) infinities, while the "conformal graviton" and other matter fields [see (4.1), (B.9)] give *positive* contributions. Hence one may suppose that β_N functions will decrease with the growth of the number of "gravitinos". We shall prove in the next section that this is really the case.

5. Generalization of the case of extended conformal supergravities

Here we shall consider the β-function in $N=2, 3, 4$ U(N)-extended conformal supergravities [6, 7], giving mainly the results (some non-trivial details and proofs will be discussed elsewhere). The basic fact is that in order to calculate the one-loop infinities *in the gravitational sector* one must simply sum the contributions of the "kinetic" terms for all the fields present in the theory. The G-sector one-loop relevant part of the lagrangian can be written in the form

$$\mathscr{L} = \sum_s \varphi_s \Delta_s \varphi_s\,, \tag{5.1}$$

where φ_s are the fields in the theory (including the Weyl graviton $h_{\mu\nu}$, N gravitinos, etc.) and Δ_s are the corresponding background metric dependent differential operators of the appropriate order. To establish the form of Δ_s one simply observes *that they (i.e. $\varphi_s\Delta_s\varphi_s$) must be background conformal invariant* (note that in the graviton, gravitino and the gauge fields cases this is valid before gauge fixing). As a result, practically the only information we need is the spectrum (i.e. the representation and linearized action) of the N-extended theory and not a complete lagrangian known only in the $N=2$ theory [7] (see, however, the discussion of the $N=4$ case below).

Let us begin with the $N=2$ theory where the following fields are present [6, 7]: 1 graviton $h_{\mu\nu}$; 2 gravitinos ψ_μ; 1 axial vector A_μ; 3 SU(2) gauge fields $V^i_{j\mu}$; 2 spinors χ^i; 1 antisymmetric tensor field $T^{ij}_{\mu\nu} = T^{[ij]}_{[\mu\nu]}$. Note that for the SU($N$) case $i, j = 1, \dots, N$ and in our euclidean notations $T^{ij}_{\mu\nu}$ is real while all spinors are chiral (or Majorana). The main problem is to establish the contribution of $T_{\mu\nu}$ to the infinities. As one can readily show (and this is confirmed by the comparison with the $N=2$ lagrangian in [7]) the following operator on $T_{\mu\nu}$ is conformal invariant under $g'_{\mu\nu} = \lambda^2 g_{\mu\nu}$, $T'_{\mu\nu} = \lambda T_{\mu\nu}$:

$$\begin{aligned}\mathscr{L}_T &= -4\mathscr{D}_\mu T^-_{\mu\nu}\mathscr{D}_\rho T^+_{\rho\nu} + 2R_{\mu\rho}T^-_{\mu\nu}T^+_{\rho\nu}\,,\\ T^\pm_{\mu\nu} &= T_{\mu\nu} \pm \epsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}\,.\end{aligned} \tag{5.2}$$

If $g_{\mu\nu}=\delta_{\mu\nu}$, (5.2) can be written in the form

$$\mathscr{L}_T^{(0)} = -2(\partial_\mu T_{\mu\nu})^2 + \tfrac{1}{2}(\partial_\rho T_{\mu\nu})^2 = -(\partial_\mu T^{\parallel}_{\mu\nu})^2 + \tfrac{1}{2}(\partial_\rho T^{\perp}_{\mu\nu})^2 , \tag{5.3}$$

and thus describes *six* degrees of freedom (three physical and three ghost). As a consequence, we have the correct counting of degrees of freedom in the $N=2$ theory

$$N_{\rm II} = 6-2\times 8+2+3\times 2-2\times 2+6=0 . \tag{5.4}$$

By the change of variables ($T\to\xi,\eta$)

$$T_{\mu\nu} = \mathscr{D}_\mu \xi_\nu - \mathscr{D}_\nu \xi_\mu + \epsilon_{\mu\nu\lambda\rho}\mathscr{D}_\lambda \eta_\rho , \tag{5.5}$$

one can rewrite (5.2) in fourth derivative form: $\zeta^+_\mu \Delta_{4\mu\nu}\zeta^-_\nu$, $\zeta^\pm_\mu = \xi_\mu \pm i\eta_\mu$, and hence use the Δ_4 algorithm (B.6) to obtain the contribution of $T_{\mu\nu}$ to the infinities [one should also take into account the measure arising due to the change (5.5) and fix the gauges for ζ^+_μ and ζ^-_μ]. The final result appears (somewhat surprisingly) to be equal to that for *six* scalar fields [cf. (B.9)]

$$\beta_T = \tfrac{1}{10} . \tag{5.6}$$

Using (5.6), (B.9) along with the values for the graviton (4.1) and one gravitino (4.12) contributions we are able to write down the following expression for the $N=2$ one-loop β-function

$$\beta_{\rm II} = \beta^{\rm G}_h + 2\beta^{\rm G}_\psi + \beta_1 + 3\beta_1 + 2\beta_{1/2} + \beta_T = \tfrac{13}{3} . \tag{5.7}$$

The spectrum of the $N=3$ theory consists of: 1 $h_{\mu\nu}$; 3 ψ^i_μ; 1 A_μ; 8 $V^i_{j\mu}$; 1 ($\hat\partial^3$)-spinor Λ; 3 complex scalars E_i; 9 spinors χ^{ij}; 3 $T^{ij}_{\mu\nu}$. The corresponding conformal invariant terms in (5.1) are $\bar\chi\hat{\mathscr{D}}\chi$, etc., while the operator on Λ must be constructed as the conformal invariant (under $g'_{\mu\nu}=\lambda^2 g_{\mu\nu}$, $\Lambda'=\lambda^{-1/2}\Lambda$) generalization of $\hat\partial^3$. Direct computation leads to

$$\mathscr{L}_\Lambda = \bar\Lambda[\hat{\mathscr{D}}^3 + (R_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}R)\gamma_\mu \mathscr{D}_\nu]\Lambda , \tag{5.8}$$

whose contribution to the infinities can be calculated in direct analogy with the case of the Δ_3 gravitino operator (4.6) [one multiplies on $\hat{\mathscr{D}}$ and uses (B.6)]. This yields [cf. (B.9)]

$$\beta_\Lambda = -\tfrac{1}{60} . \tag{5.9}$$

Thus the number of states and the β-function in the $N=3$ theory are given by

$$N_{\rm III} = 6-3\times 8+2+8\times 2-3\times 2+3\times 2-9\times 2+3\times 6=0 , \tag{5.10}$$

$$\beta_{\rm III} = \beta^{\rm G}_h + 3\beta^{\rm G}_\psi + \beta_1 + 8\beta_1 + \beta_\Lambda + 3\times 2\beta_0 + 9\beta_{1/2} + 3\beta_T = 1 . \tag{5.11}$$

In the $N=4$ theory (1 $h_{\mu\nu}$; 4 ψ^i_μ; 15 $V^i_{j\mu}$; 1 complex ($\Box^2$)-scalar C; 4 ($\hat\partial^3$)-spinors Λ^i; 10 complex scalars $E_{(ij)}$; 20 spinors $\chi^{ij}{}_k$; 6 $T^{ij}_{\mu\nu}$) the only new moment is connected with the appearance of the conformal invariant extension of the $\Box^2$ operator, which does not, of course, coincide with $(-\mathscr{D}^2+\frac{1}{6}R)^2$ due to the fact that the dimensionless field C is invariant under the Weyl transformations. One can show that the most general form of such an operator is

$$\mathscr{L}_C=\mathscr{D}^2C^*\mathscr{D}^2C-2(R_{\mu\nu}-\tfrac{1}{3}g_{\mu\nu}R)\mathscr{D}_\mu C^*\mathscr{D}_\nu C+\tfrac{1}{2}\gamma(C_{\lambda\mu\nu\rho})^2C^*C, \qquad (5.12)$$

where $C_{\lambda\mu\nu\rho}$ is the Weyl tensor (A.3) and γ is an *arbitrary constant*, which is to be fixed by the condition that (5.12) is the part of the complete superinvariant $N=4$ lagrangian or by some considerations additional to conformal invariance. At present we are only able to make a hypothesis that the true value of γ is

$$\gamma=-1\,, \qquad (5.13)$$

which seems to be the most natural non-zero value in view of the fact that it implies the presence of the $W(1-|C|^2)$ term in the $N=4$ lagrangian which is reminiscent of the version of $N=4$ conformal supergravity with a manifest rigid SU(1, 1) and an extra local chiral U(1) invariance [7].

The choice (5.13) is also *a posteriori* justified by the fact that it leads to a *zero β-function* in the $N=4$ theory, the result which one could expect [due to the tendency of β_N functions to decrease, cf. (4.1), (4.12), (5.7), (5.11)] from the fact that the $N=4$ superconformal theory is a maximally extended one (cf. the discussion of other probably finite theories in [11]). The zero value for $\beta_{\rm IV}$ is also strongly prompt by the correspondence with the O(N) supergravity theories obvious in the $N=1$, 2, 3 cases (see sect. 6). Collecting all needed results (4.1), (4.12), (B.9), (5.6), (5.9) and the contribution of the field C (5.12) [easily established by the use of the Δ_4 algorithm (B.6)]

$$\beta_C=-\tfrac{4}{15}-2\gamma=\tfrac{26}{15}\,, \qquad (5.14)$$

we finally have

$$N_{\rm IV}=6-4\times8+15\times2+2\times2-4\times3\times2+2\times10-20\times2+6\times6=0\,, \qquad (5.15)$$

$$\beta_{\rm IV}=\beta_h^{\rm G}+4\beta_\psi^{\rm G}+15\beta_1+\beta_C+4\beta_\Lambda+20\beta_0+20\beta_{1/2}+6\beta_T=0\,. \qquad (5.16)$$

6. Discussion

From the results of the previous sections we conclude that β-functions in U(N) extended conformal supergravities are decreasing with N growing from 0 to 4:

$$\tfrac{199}{15}>\tfrac{17}{2}>\tfrac{13}{3}>1>0\,. \qquad (6.1)$$

It is instructive to compare this sequence with the analogous one for gauged O(N) Poincaré supergravities [22, 23]. But let us first make an important clarification.

Conformal supergravities are theories with "explicit" (described by terms in the lagrangian with "wrong" signs) as well as "implicit" (contained in the higher derivative, i.e. Weyl, gravitino, Λ, C terms and in the $T_{\mu\nu}$ field lagrangian (5.3)) *ghosts*. The classification on "ghosts" and "physical particles" thus depends on the overall sign of the superinvariant lagrangian, which should be chosen in accordance with an interpretation of the theory. Throughout this paper we used *euclidean* notations and assumed that the Weyl invariant (2.2) contributes to the total lagrangian with *positive* sign (cf. [6, 7])

$$\mathscr{L}_N = \alpha^{-2}\left[W - \frac{4-N}{4N} F_{\mu\nu}^2 - (V^i_{j\mu\nu})^2 + \cdots \right]. \tag{6.2}$$

As a consequence, the axial field A_μ (note that it is real in (2.1) as follows from the reality of the action) and the SU(N) gauge field $V^i_{j\mu}$ *are ghost-like*. The above choice can be justified as follows. One should not worry about the ghost nature of A_μ and V_μ because a physically meaningful theory should be constructed by coupling (6.2) with the extended Einstein super (conformal) gravity, which contains its own physical gauge fields. The essence of the matter is that α^2 in (6.2) must be taken to be positive in order to establish the absence of tachyons (and hence instability) in the theory (see e.g. [9]). Really, the spectrum of the total theory contains [5], e.g. for $N=1$, one massless $(2, \frac{3}{2})$ and one massive $(2, \frac{3}{2}, \frac{3}{2}, 1)$ multiplet which is ghost-like for $\alpha^2>0$ and tachyonic for $\alpha^2<0$ (one easily convinces oneself of the correctness of this statement by noting that the auxiliary field A_μ contributes as $-A_\mu^2$ in the $N=1$ Einstein supergravity lagrangian)*. It is interesting to note that apart from superinvariance considerations the difference in signs of the W and $F_{\mu\nu}^2$ terms in (6.2) can be readily deduced either from the remarkable fact that the number of Bose ghosts should be equal to that of the Fermi ghosts (i.e. in Einstein plus Weyl supergravity *ghosts fill a supermultiplet*, cf. [5]) or from the fact that matter fields contribute with *different* signs in W and $F_{\mu\nu}^2$ infinities.

In order to facilitate the comparison with O(N) supergravities (where gauge fields are physical) it is useful to consider the formal case of $\alpha^2=-g^2<0$ (gauge fields are now physical) and to introduce the gauge coupling β-function $\beta(g)=-\beta(\alpha)$. It was the Weyl coupling $(\alpha^2>0)\beta(\alpha)$ function which we called "β-function" throughout this paper and hence our results (6.1) indicate the lack of asymptotic freedom if $g^2>0$ in $N=1, 2, 3$ theories.

Now we are in position to compare the behaviour of the $\beta(g)$ functions with N in the U(N) Weyl and gauged O(N) Einstein supergravities (gauge fields are thus assumed to be physical in both theories). We conclude a remarkable and rather unexpected (in view of the fact that the β-functions for O(N) theories are defined

* Observe that the transition from the Minkowski formulation (with $\delta_{\mu\nu} = ++++$) of refs. [1, 5–7] to the euclidean one we use is obtained by changing the sign of the lagrangian and taking the time-like components of the fields to be real. Hence, in the above references vector gauge fields were assumed to be physical while the Weyl invariant was taken with the "wrong" sign.

only on-shell) correspondence. Collecting the results of ref. [22] and the present paper we find the following contributions of fields of different spin (spin 2 field stands for the Einstein or the Weyl graviton, etc.) in the $\beta(g)$ function in the *gravitational* sector (remember that our normalization is $\mathscr{L} = +g^{-2}(V^i_{j\mu\nu})^2 + \cdots$):

	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$T_{\mu\nu}$	Λ	C
O(N):	$-\frac{87}{10}$	$+\frac{137}{60}$	$-\frac{1}{5}$	$-\frac{1}{20}$	$-\frac{1}{60}$	—	—	—
U(N):	$-\frac{199}{15}$	$+\frac{149}{30}$	$-\frac{1}{5}$	$-\frac{1}{20}$	$-\frac{1}{60}$	$-\frac{1}{10}$	$+\frac{1}{60}$	$-\frac{26}{15}$

(6.3)

In the "electromagnetic" or gauge sector we analogously have (see [23] for the O(N) case)

	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$T_{\mu\nu}$	Λ	C
O(N):	0	$-\frac{13}{3}$	$0_{\text{or}}\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{12}$	—	—	—
U(N):	0	$-\frac{17}{2}$	$0_{\text{or}}\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{12}$	>0	<0	<0

(6.4)

(all the numbers here should be multiplied by appropriate group invariants). Finally, let us write down the sequences for the $\beta(g)$ functions for $N = 0, \ldots, 4$ (the values for $N = 0$, 1 in the O(N) and $N = 0$ in the U(N) cases are meaningful only in the gravitational sector)

	0	1	2	3	4	5
O(N):	$-\frac{87}{10}$	$-\frac{77}{12}$	$\nearrow -\frac{13}{3}$	$\nearrow -\frac{5}{2}$	$\nearrow -1$	$\nearrow 0$
U(N):	$-\frac{199}{15}$	$-\frac{17}{2}$	$-\frac{13}{3}$	-1	0	

(6.5)

The correspondence here is shifted ($\Delta N = 1$) probably due to the fact that $U_1 \simeq SO_2$, $SU_2 \simeq SO_3$, $SU_3 \supset SO_4$, $SU_4 \supset SO_5$. Using (6.3) and (6.4) we get a transparent understanding of the growth of $\beta(g)$ with N: it is due to the positive contribution of gravitino (and Λ) in the gravitational sector, while in the gauge sector it is caused by positive contributions of the gauge and $T_{\mu\nu}$ fields*.

Next we want to point out that the results of ref. [9] and the present paper provide the possibility to establish the β-functions for α (as well as for k^2) in the superconformal extensions of the theory

$$\mathscr{L}_{\text{EW}} = -\frac{1}{k^2}(R\phi^2 + 6\partial_\mu\phi\partial_\mu\phi) + \alpha^{-2}W. \tag{6.6}$$

For example, considering the $N = 1$ superconformal extension of the Einstein lagrangian [24], one easily finds that the part contributing to the $F^2_{\mu\nu}$ and W infinities is given by

$$\mathscr{L}' = -6D_\mu\phi^* D_\mu\phi - \tfrac{1}{2}\bar{\chi}\hat{D}_1\chi\,, \tag{6.7}$$

where $D_\mu = \mathscr{D}_\mu - \frac{1}{2}iA_\mu$, $D_{1\mu} = \mathscr{D}_\mu + \frac{1}{4}iA_\mu$ and ϕ and χ are the complex scalar and Majorana spinor fields of the $N = 1$ scalar multiplet. The resulting contribution to

* Remarkably enough $T_{\mu\nu}$ *supports* asymptotic freedom in contrast with the gravitino and matter fields.

the infinities is obtained with the help of (B.5) and (B.9):

$$\Delta\bar{b}_4 = \Delta\beta(W - \tfrac{3}{4}F^2_{\mu\nu}), \qquad \Delta\beta = \tfrac{1}{12} \tag{6.8}$$

(note that the $R\phi\phi^*$ term contribution to the divergences is suppressed due to the presence of W in (6.6), cf. [9]). As a consequence, the total $\beta(\alpha)$ function is obtained by summing the pure $N=1$ part (3.23) and (6.8) ($\tilde{\beta}_I = \beta_I + \Delta\beta$) and therefore is increasing. Thus one cannot obtain a zero one-loop β-function in any $N \leq 4$ superconformal extension of (6.6) (cf. [10]). However, there remains an interesting possibility that the β-function will be zero in a desired but yet unknown $N > 4$ superconformal extension of (6.3) which will probably have *negative* $\beta(\alpha)$ in the absence of the Einstein part [cf. (6.1)].

Let us now consider the renormalization of the O(N) Poincaré supergravity SO(N) gauge field coupling constant e^2 in the case of a hybrid theory like (6.6):

$$\begin{aligned}\mathcal{L} = \mathcal{L}_{O(N)}(g, \psi, B; \Lambda, C; \phi, \chi; V, A, t, \ldots) \\ + \mathcal{L}_{U(N)}(g, \psi, V, A; \Lambda, C, T, \ldots; \mathcal{D}, \ldots),\end{aligned} \tag{6.9}$$

where B^{ij}_μ, Λ^i and C are the SO(N) gauge, spinor and scalar fields of O(N) supergravity, ϕ and χ belong to a compensating scalar multiplet [cf. (6.7)], V^{ij}_μ and A_μ are the auxiliary fields of the Poincaré supergravity and at the same time the U(N) gauge fields of the U(N) conformal supergravity, $\mathcal{D}$ (for $N > 1$) is the auxiliary field for the U(N) theory [7], while for the auxiliary field $t^{ij}_{\mu\nu}$ of Poincaré supergravity one has (neglecting, for simplicity, the ϕ-dependence, see [7, 8])

$$t^{ij}_{\mu\nu} = T^{ij}_{\mu\nu} + \sqrt{2}\,e^{-1}\hat{F}^{ij}_{\mu\nu}(B). \tag{6.10}$$

Thus we shall consider $T_{\mu\nu}$ and B_μ as independent variables. Dots in $\mathcal{L}_{O(N)}$ stand for other auxiliary fields (like $S, P, \ldots$). Though the complete non-linear expression for (6.9) can be written only for $N = 1, 2$ (while the linear version of $\mathcal{L}_{U(N)}$ is also known for $N = 3, 4$ [7]), one may believe that there exists an appropriate generalization of (6.9) for $4 < N \leq 8$ with higher spin fields [also denoted by dots in (6.9)] being auxiliary (propagating) in the Poincaré (conformal) supergravity lagrangians (cf. ref. [5b]). One probably should not worry about troubles with propagating higher spin fields because being present in $\mathcal{L}_{U(N)}$ they are part of the ghost multiplet and thus unphysical. Hence the reasoning and conclusion presented below for $N \leq 4$ may possibly be extended also to the $N > 4$ case. In order to find the coefficient of the $F^2_{\mu\nu}(B)$ term in the one-loop infinities for (6.9) it is sufficient to consider the situation when only B_μ has a non-trivial background (cf. the method of sect. 3 and ref. [23]). Choosing the same gauge as for pure conformal supergravity (and gauging out the background part of the compensating scalar multiplet in $\mathcal{L}_{O(N)}$) we can write down the relevant one-loop part of (6.9) (e.g. for $N = 4$) in the following symbolic form:

$$\mathscr{L} = k^{-2}\{\bar{\psi}\mathscr{D}(B)\psi + \bar{\psi}F(B)\psi + e^{-2}F^2_{\mu\nu}(\hat{B})\hat{g}\hat{g}\sqrt{\hat{g}} + (T_{\mu\nu} + \sqrt{2}e^{-1}F_{\mu\nu}(\hat{B}))^2\hat{g}\hat{g}\sqrt{\hat{g}} + \bar{\Lambda}\mathscr{D}(B)\Lambda + C^*\mathscr{D}^2(B)C + \cdots\} + \alpha^{-2}\{h_{\mu\nu}\Box^2 h_{\mu\nu} + \bar{\psi}_\mu\hat{\partial}^3\psi_\mu + \bar{\Lambda}\hat{\partial}^3\Lambda + C^*\Box^2 C + \partial_\mu T^-_{\mu\nu}\partial_\rho T^+_{\rho\nu} + \cdots\}, \quad (6.11)$$

where $\mathscr{D}(B) = \partial + B$, $\hat{g} = g + h$, $\hat{B} = B + B_{\text{quant}}$. Two remarkable facts are worth noting here: (i) B_μ is *absent* in the conformal supergravity lagrangian, where all derivatives are ordinary ones and (ii) for *every* kinetic term in $\mathscr{L}_{\text{O}(N)}$ (i.e. for ψ_μ, $h_{\mu\nu}$ and *also* for the matter fields Λ and $C^\star$) there is a *corresponding higher derivative term* in $\mathscr{L}_{\text{U}(N)}$ [7], while for B_μ we have a non-trivial mixing with $T_{\mu\nu}$. As a consequence, all possible contributions of ψ_μ, Λ and C to the $F^2_{\mu\nu}(B)$ infinities *are suppressed* (e.g. log det $(\hat{\partial}^3 + \hat{\partial} + B + F(B))_\psi$ is finite). In order to see that there are no B_μ dependent infinities coming from B_{quant}, $h_{\mu\nu}$, $T_{\mu\nu}$ mixings, let us use (5.5) ($T \sim \partial\zeta$) and write (6.11) in the symbolic form

$$\mathscr{L} = k^{-2}(\zeta\Box\zeta + e^{-1}F(B)h\partial\zeta + e^{-2}F^2_{\mu\nu}(\hat{B})\hat{g}\hat{g}\sqrt{\hat{g}} + \cdots) + \alpha^{-2}(h\Box^2 h + \zeta\Box^2\zeta + \cdots). \quad (6.12)$$

It is obvious that the $h\partial\zeta$ term leads to convergent diagrams. As a final step one must use the remarkable fact that higher derivative gravity does not produce new $F^2_{\mu\nu}(B)$ infinities (see (3.1)–(3.4) and ref. [9]). The conclusion is that the $\beta(e)$ function is completely determined by B_μ self-interactions and thus is simply *equal to the flat space one-loop* SO(N) *gauge field β-function.*

This result should be compared with the corresponding one for the pure O(N) supergravity case [22, 23]. Instead of *negative* values of β for $N = 2, 3, 4$ [see (6.5)] we get the zero ($N = 2$) and two positive (leading to asymptotic freedom) ones. The explanation of this change is quite obvious: all negative gravitino and matter contributions in $\beta(e)$ (dominating for $N = 2, 3, 4$ in the O(N) case [23]) are now *suppressed* by the conformal supergravity term in (6.9), which at the same time does not alter the flat space gauge field β-function. It is worth noting that in our case the β-function is defined off-shell, while the analysis of ref. [23] (see also [22]) was based on the assumption that infinities due to non-minimal interactions (like $\bar{\psi}F\psi$) mutually cancel on-shell.

In order to have a complete picture of the one-loop renormalization of the theory (6.6) or (6.9) one must also calculate the renormalization of the gravitational constant (or the "norm" of the scalar field ϕ) and also of the cosmological Λ-term [or $\lambda\phi^4$ in (6.6)]. The general recipe of this calculation is clear: for example, one

* The numbers of Λ and C in the cases $N = 2, 3, 4$ are (0, 0), (1, 0) and (4, 1).

takes the same gauge as in the case of pure conformal supergravity [cf. (2.18)], assumes that only $g_{\mu\nu}$ and ϕ have non-trivial backgrounds and calculates the infinities of $R\phi^2$ and ϕ^4 type*. It should be taken into consideration that (as was pointed out in [9]) the $N=0$ theory (6.6) is renormalizable only *on-shell* because of the presence of the conformal invariant counterterm $[R^2\sqrt{g}]\ (g\to g\phi^2)=[R-6\phi\mathscr{D}^2\phi/\phi^{+2}]^2\sqrt{g}$, which is proportional to ϕ^4 on-shell. The same situation takes place in the $N=1$ case because the $N=1$ supersymmetric extension of R^2 is known to exist [5]. However, there is a possibility that the $N\geqslant 2$ extensions of R^2 do not exist if the O(N) symmetry *is gauged* (and thus there are no corresponding infinities in a superfield calculation with a manifest background supersymmetry). Some support for this proposal comes from the remark that if an $(R^2+\cdots)$ infinity is present, it gives (on-shell) only an additional Λ-term (but not $F^2_{\mu\nu}(B)$) renormalization which contradicts the well-known relation [27] $\Lambda k^2=-6e^2$, following from supersymmetry when O(N) is gauged.

If corresponding R^2 extensions exist for $N>1$ one is to consider the off-shell renormalizable theory obtained by a supersymmetric extension of $(R+\Lambda+R^2+\alpha^{-2}W)$ lagrangian. The presence of the R^2 term will not qualitatively change the renormalization of α (cf. ref. [9]) and very probably will not alter the above result concerning the renormalization of e^2.

Appendix A

NOTATIONS AND SOME USEFUL FORMULAE

Our spinor conventions are in general the same as in [1, 4]:

$$\gamma_{(a}\gamma_{b)}=\delta_{ab}\,,\qquad \gamma_5^2=1\,,\qquad \sigma_{ab}=\tfrac{1}{2}\gamma_{[a}\gamma_{b]}\,,$$
$$\bar{\psi}=\psi^{\mathrm{T}}C\,,\qquad \bar{\gamma}_a=C^{-1}\gamma_a^{\mathrm{T}}C=-\gamma_a\,,\qquad \bar{\gamma}_5=\gamma_5\,. \tag{A.1}$$

We use throughout the euclidean metric and

$$\epsilon_{\lambda\mu\nu\rho}\epsilon^{\lambda\mu\nu\rho}=+4!\,. \tag{A.2}$$

For the curvature invariants one has

$$(C_{\lambda\mu\nu\rho})^2=2W+R^*R^*\,, \tag{A.3}$$

$$W\equiv R^2_{\mu\nu}-\tfrac{1}{3}R^2\,, \tag{A.4}$$

where $C_{\lambda\mu\nu\rho}$ is the Weyl tensor and

$$R^{\lambda}_{\ \mu\nu\rho}=\partial_\nu\Gamma^{\lambda}_{\ \mu\rho}-\cdots\,,\qquad R_{\mu\nu}=R^{\lambda}_{\ \mu\lambda\nu} \tag{A.5}$$

(note that our Ricci tensor has opposite sign to that of refs. [1, 3]).

* Note that for the background conformal invariance of the gauge (2.18) we must substitute there $g_{\mu\nu}\to g_{\mu\nu}\phi^2$.

The functional integral over the Majorana variables is defined as follows:

$$\int \mathrm{d}\psi\, \mathrm{d}\bar{\psi}\, \mathrm{e}^{-\bar{\psi}\Delta\psi} = \sqrt{\det \Delta}\,, \tag{A.6}$$

and thus (in four dimensions) the Majorana spinor describes two Fermi degrees of freedom,

$$\sqrt{\det \overset{\leftrightarrow}{\partial}} = Z_0^{-2}\,, \qquad Z_0 \equiv \frac{1}{\sqrt{\det(-\Box)}}\,. \tag{A.7}$$

We often use in the text the "averaging over gauges" procedure [12–14] based on the trivial formula

$$\int \mathrm{d}\xi \delta[C-\xi(x)](\det H)^{\varepsilon/2}\, \mathrm{e}^{-\xi H\xi/2} = \mathrm{e}^{-CHC/2}(\det H)^{\varepsilon/2}\,, \tag{A.8}$$

where $\varepsilon = \pm 1$ if ξ is a Bose or Fermi variable. Note that the same result is obtained, of course, if one absorbs H in the gauge:

$$C \to \check{C} = C\sqrt{H}\,, \qquad \delta[\check{C}] = \delta[C](\det H)^{\varepsilon/2}\,.$$

Let us also give here the definitions of the traceless variables:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu}\varphi\,, \qquad \varphi = h^{\mu}_{\mu}\,,$$
$$\varphi_{\mu} = \psi_{\mu} - \tfrac{1}{4}\gamma_{\mu}\psi\,, \qquad \psi = \gamma_{\mu}\psi_{\mu}\,. \tag{A.9}$$

Appendix B

ALGORITHMS FOR ONE-LOOP INFINITIES

Here we want to give a summary of a number of known facts about the infinite part I_∞ of one-loop (euclidean) effective action. For one Bose field one has

$$I_\infty = \tfrac{1}{2}[\log\det(\mu^{-p}\Delta_p)]_\infty = -\tfrac{1}{2}B_4 \log\frac{L^2}{\mu^2}\,, \tag{B.1}$$

$$B_4(\Delta_p) = \int \mathrm{d}^4x\, b_4\,, \qquad b_4 = \frac{1}{(4\pi)^2}\bar{b}_4\,, \tag{B.2}$$

where $L \to \infty$ and Δ_p is the pth-order differential operator (the counter-action in dimensional regularization is $\Delta I = -I_\infty = -B_4/(n-4)$). Here and throughout this paper we discard polynomial, boundary and total derivative-like divergences. The cases of $p = 2$ and 4 are of special interest for us. If

$$\Delta_2 = -\mathscr{D}^2 + X\,, \qquad \mathscr{D} = \mathscr{D}(\mathscr{A})\,, \tag{B.3}$$

and

$$\Delta_4 = \mathscr{D}^4 + V^{\alpha\beta}\mathscr{D}_{(\alpha}\mathscr{D}_{\beta)} + U\,, \tag{B.4}$$

where $\mathscr{A}$ is a connection (with a curvature $\mathscr{F}$) on some vector bundle over space-time, one has correspondingly the "Δ_2 algorithm" [12, 16, 17] and the "Δ_4 algorithm" [9]:

$$\bar{b}_4(\Delta_2) = \mathrm{tr}\,(\tfrac{1}{12}\mathscr{F}^2_{\mu\nu} + \mathbb{1}\cdot E + \tfrac{1}{2}X^2 - \tfrac{1}{6}RX)\,,$$
$$E \equiv \tfrac{1}{180}R^*R^* + \tfrac{1}{60}W + \tfrac{1}{72}R^2\,, \tag{B.5}$$

$$\bar{b}_4(\Delta_4) = \mathrm{tr}\,(\tfrac{1}{6}\mathscr{F}^2_{\mu\nu} + 2\cdot\mathbb{1}\cdot E + \tfrac{1}{24}V^2_{\alpha\beta} + \tfrac{1}{48}V^2 - \tfrac{1}{6}V^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{12}VR - U)\,, \qquad V = V^{\alpha}_{\alpha}\,. \tag{B.6}$$

If the bare euclidean action and the infinite part of the effective action are proportional to some invariant $I_0 = \int \mathscr{L}_0\, \mathrm{d}^4x$,

$$I = \alpha^{-2}I_0\,, \qquad I_\infty = \tilde{\beta}I_0\,,$$
$$\bar{b}_4 = \beta\mathscr{L}_0\,, \qquad \tilde{\beta} = -\frac{1}{2(4\pi)^2}\beta\log\frac{L^2}{\mu^2}\,, \tag{B.7}$$

then we get the following relation for the bare coupling constant:

$$\alpha^{-2}(L) = \alpha^{-2}(\mu) + \frac{\beta}{2(4\pi)^2}\log\frac{L^2}{\mu^2}\,, \tag{B.8}$$

and thus $\beta > 0$ implies asymptotic freedom for α.

Using (B.5) one can easily obtain that the conformally invariant part of the infinities for the system of matter fields is given by (see e.g. [21])

$$\bar{b}_4 = \beta W\,, \qquad \beta = \beta_1 N_1 + \beta_{1m}N_{1m} + \beta_{1/2}N_{1/2} + \beta_0 N_0\,,$$
$$\beta_1 = \tfrac{1}{5}\,, \qquad \beta_{1m} = \tfrac{13}{60}\,, \qquad \beta_{1/2} = \tfrac{1}{20}\,, \qquad \beta_0 = \tfrac{1}{60}\,, \tag{B.9}$$

where $N_1(N_{1m})$, $N_{1/2}$, N_0 are the numbers of massless (massive) vector, Majorana spinor and real scalar fields.

Appendix C

QUANTIZATION OF THE ORDINARY GRAVITINO ON A CURVED BACKGROUND

Let us consider here the quantization of the spin $\frac{3}{2}$ field with the lagrangian

$$\mathscr{L} = \epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho\gamma_5\gamma_\sigma\mathscr{D}_\nu\psi_\mu\,, \tag{C.1}$$

trying to follow as closely as possible the well-known rule: the choice of the gauge should provide the diagonality of the highest derivative term in the action. Eq.

(C.1) can be written in the two equivalent forms

$$\mathscr{L}_1 = -\bar{\psi}_\mu(\Gamma^\rho)_{\mu\nu}\mathscr{D}_\rho\psi_\nu + \tfrac{1}{2}\bar{\psi}\hat{\mathscr{D}}\psi\,, \tag{C.2}$$

$$\mathscr{L}_2 = -\bar{\varphi}_\mu\hat{\mathscr{D}}_{\mu\nu}\varphi_\nu - \tfrac{2}{3}\bar{\chi}\hat{\mathscr{D}}^{-1}\chi + \tfrac{3}{8}\bar{\psi}'\hat{\mathscr{D}}\psi'\,, \tag{C.3}$$

where $(\Gamma^\rho)_{\mu\nu} = -\frac{1}{2}\gamma_\nu\gamma^\rho\gamma_\mu$, $\chi = \mathscr{D}_\mu\varphi_\mu$, $\psi' = \psi - \frac{4}{3}\hat{\mathscr{D}}^{-1}\mathscr{D}_\mu\varphi_\mu$ [see (A.9)]. Observing that (C.1) is Q superinvariant ($\delta\psi_\mu = \mathscr{D}_\mu\varepsilon$) if $R_{\mu\nu} = 0$ we have the possibility of choosing a gauge, e.g.,

$$(1)\quad \psi = \rho(x)\,, \qquad \text{or} \qquad (2)\quad \mathscr{D}_\mu\varphi_\mu = \zeta(x)\,. \tag{C.4}$$

Averaging over these gauges in the cases (C.2) and (C.3) respectively and integrating over ψ_μ and φ_μ and ψ we get the following expressions for the partition function [cf. (3.10), (4.5)]

$$Z_1 = \frac{\sqrt{\det(\Gamma^\mu\mathscr{D}_\mu)_\psi}}{\det\Delta_g^{(1)}\sqrt{\det\hat{\mathscr{D}}}}\,, \qquad \Delta_g^{(1)} = \hat{\mathscr{D}}\,, \tag{C.5}$$

$$Z_2 = \frac{\sqrt{\det(\hat{\mathscr{D}})_\varphi}\sqrt{\det\hat{\mathscr{D}}}}{\det\Delta_g^{(2)}\sqrt{\det\hat{\mathscr{D}}^{-1}}}\,, \qquad \Delta_g^{(2)} = -\mathscr{D}^2 - \tfrac{1}{12}R\,. \tag{C.6}$$

The approach based on (C.2), (C.5) is a standard one and gives the correct values for the number of degrees of freedom ($Z_0^{-8}/Z_0^{-4-2} = Z_0^{-2}$, see also [14]) and the anomalies [25, 26]. However, it seems somewhat unnatural: (i) the operator $(\Gamma^\rho\mathscr{D}_\rho)_{\mu\nu}$ in (C.2) is "non-diagonal" in μ, ν (the diagonal operator is obtained only after "squaring"); (ii) comparing (C.4) with (2.18b) we conclude that the first gauge in (C.4) is natural as a superconformal (S) gauge (ψ would drop from the lagrangian if it was S invariant), while in the Q-invariant case (C.1) one should use the second (Q) gauge in (C.4).

It is simple to check that (C.6) gives results the same as (C.5) for the number of degrees of freedon ($Z_0^{-6-2}/Z_0^{-8+2} = Z_0^{-2}$) and for the anomalies. In conclusion let us note that an approach to the quantization of (C.1) similar to ours (C.3), (C.4, (2)), (C.6) but less straightforward was presented in [26].

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