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# RENORMALIZABLE ASYMPTOTICALLY FREE QUANTUM THEORY OF GRAVITY

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We prove that higher derivative quantum gravity is asymptotically free in *all* essential coupling constants by the calculation of one-loop counterterms (correcting the previous result of Julve and Tonin) and the solution of the corresponding renormalization group (RG) equations. Strong arguments are presented in favour of the possibility that renormalizable asymptotically free gravity establishes asymptotic freedom for the effective mass parameters and non-gauge couplings in grand unified gauge theories. We also analyse the RG equations in the Einstein theory with $\Lambda$ term and in the higher derivative conformal invariant theories. Among other topics discussed are the algorithm for the divergences of the determinant of the fourth-order differential operator, the consistent renormalization of the boundary terms in the action, the one-loop $\beta$-function in the fourth derivative vector gauge theory and the RG equations in the "$g\phi^4+\eta R\phi^2$" theory.

## 1. Introduction

In recent years there has been a revival of interest in gravitational theory with the lagrangian containing quadratic as well as linear terms in the curvature (see [1–11] and references therein). This theory is renormalizable [4] but possesses a ghost pole in the tree propagator. However, it is important to realize that the question of unitarity (or about the asymptotic states) is a dynamical one and should be discussed accounting for radiative corrections and probably non-perturbatively (cf. [6–12]). We consider this theory as a possible alternative to the attempts to construct a unified ultraviolet (UV) *finite* model on the basis of extended supergravity, leading after the inclusion of the matter described by a grand unified theory (GUT), to a complete *renormalizable* theory of all interactions in nature.

The GUTs based on a vector gauge theory have solved the problem of gauge couplings by indicating their dynamical origin (the meaning of asymptotic freedom (AF) is that bare couplings are zero in the local limit) but have given no self-consistent answer to the question of the bare values of masses and non-gauge ($\phi^4$, Yukawa) coupling constants. One may conjecture that gravity provides the AF behaviour for all interactions, but in the Einstein theory this is principally possible only in a non-perturbative approach. We shall see that renormalizable higher derivative gravity treated in the context of perturbation theory is not only AF in all its essential coupling constants but is also likely to solve the problems of bare masses and couplings by establishing AF behaviour for them.

To prove asymptotic freedom in renormalizable quantum gravity* we consider in this paper the calculation of the one-loop counter-terms (correcting the previous result of ref. [7]) and solve the corresponding renormalization group (RG) equations.

In sect. 2 we quantize the theory using the background field method (see e.g. [17, 18]) and stressing the role of the operator $H$ involved in "averaging over gauges" (see also [19] about the analogous operator in the supergravity case). The corresponding factor $(\det H)^{1/2}$ gives additional counter-terms missed in [7]. Finally, we obtain the one-loop effective action, involving the determinants of the 4th order differential operators $\Delta_4$. In sect. 3 we work out the corresponding counter-terms using a convenient "$\Delta_4$ algorithm" (established in appendix B). We also stress the necessity of the *independent* calculation of counter-terms in the higher derivative conformal invariant theories and thus correct the expression for the Weyl theory $\beta$-function proposed in [10] by naively taking the limiting case of the (independently erroneous) result of ref. [7].

Sect. 4 is devoted to the RG analysis of the theory. We first discuss the RG equations for essential couplings in the Einstein theory with $\Lambda$ term stressing the different roles of the "volume" $k_V$ and boundary $k_S$ gravitational constants and taking into account the contribution of the boundary $C_4$ coefficient in the heat kernel expansion. Then we prove that the RG equations in higher derivative gravity predict the asymptotically free behaviour for all essential couplings in this theory. Among the consequences of the asymptotic freedom we mention the possible application of the Lee-Wick-type mechanism for restoration of unitarity (though, of course, the question of unitarity cannot be settled only by the ultraviolet analysis which is the topic of the present paper).

In sect. 5 we discuss the one-loop RG equations for a system of higher derivative gravity plus self-interacting matter fields. We study the general structure of the set of RG equations (illustrating the analysis by the example of the "$g\phi^4 + \eta R\phi^2$" scalar field) and show that the property of the asymptotic freedom of gravity itself leads to the asymptotically free solutions for matter coupling constants and effective masses.

Our notations are summarized in appendix A. In appendix B we establish the algorithm for divergences of the determinant of the covariant 4th-order operator $\Delta_4$ using only the well-known $\Delta_2$ algorithm [17, 20–22]. The principal advantage of our method over the diagram one [7] is the possibility of obtaining the "topological" and boundary terms in the divergences (for example, we present the expression for the boundary part of the logarithmic counterterm or the $C_4$ coefficient for the $\Delta_4$ case as well as for $\Delta_2$). In appendix C we discuss the problem of establishing limits on counter-terms through the example of the higher (fourth) derivative vector gauge theory. We compute the corresponding one-loop $\beta$-function and show that limits in

* For various earlier proposals about AF in quantum gravity see [13–16] and especially [9, 10].

the counter-terms are regular by taking into account suitable finite terms in the effective action.

## 2. Background field method of quantization of higher derivative gravity

The theory under consideration has the following bare euclidean action (for notations see appendix A):

$$I=\int_{\mathrm{M}}\mathfrak{L}_{\mathrm{V}}\sqrt{g}\,\mathrm{d}^4x+\int_{\partial\mathrm{M}}\mathfrak{L}_{\mathrm{S}}\sqrt{\gamma}\,\mathrm{d}^3x, \tag{2.1}$$

$$\mathfrak{L}_{\mathrm{V}}=-\frac{1}{k_{\mathrm{V}}^2}(R-2\Lambda)+aW+\tfrac{1}{3}bR^2+\alpha_{\mathrm{V}}R^*R^*+\kappa D^2R, \tag{2.2}$$

$$\mathfrak{L}_{\mathrm{S}}=-\frac{1}{k_{\mathrm{S}}^2}2K+\alpha_{\mathrm{S}}\Omega_{\mathrm{S}}. \tag{2.3}$$

Here the $\Lambda$ term and boundary terms in the Einstein action $K$ [23, 24] and in the Euler number $\Omega_{\mathrm{S}}$ [25] are included for renormalizability of the theory under the natural asymptotically flat boundary conditions. Under these conditions we may disregard the $R^*R^*$ and $D^2R$ terms in (2.2) and also some other boundary terms (see also [26, 27] for comments about higher derivative gravity action). The $K$ term in (2.3) is also important for correspondence with Einstein theory on the level of tree $S$-matrix [12, 28, 29]. We assume that for renormalized constants

$$k_{\mathrm{V}}(\mu)=k_{\mathrm{S}}(\mu)=k, \qquad k^2=16\pi G. \tag{2.4}$$

Thus we may use a *single* constant $k$ $(=k_{\mathrm{V}}+\mathrm{O}(\hbar))$ in all perturbative expressions and also, e.g., in the trace of the classical field equations

$$R-4\Lambda=-2k^2D^2R. \tag{2.5}$$

The classical action is positive if $a>0$, $b>0$, $\Lambda>3/8bk^2$, but the second condition seems to be unphysical. Really, if $\Lambda\simeq 0$ (cf. [4]),

$$\mathfrak{L}_{\mathrm{V}}\simeq h\{\tfrac{1}{4}a(-\Box)[m^2+(-\Box)]P^{2^+}+b(-\Box)[m'^2+(-\Box)]P^{0^+}\}h,$$

$$m^2=(ak^2)^{-1}, \qquad m'^2=(-2bk^2)^{-1}, \tag{2.6}$$

where $P^{2^+}$ and $P^{0^+}$ are the transverse spin projectors and thus $b>0$ leads to the presence of the $0^+$ tachyon (and so to oscillations of the static potential and instabilities of solutions [5]). Hence the condition for correspondence with general relativity apparently implies $b<0$ and, as a consequence, the same problem of indefiniteness of the euclidean action in the $0^+$ sector [30] (cf., however, ref. [31]).

To quantize the theory using the background field method one first chooses some background covariant gauge $\chi_\mu[g, \hat{g}] = \zeta_\mu(x)$, $\hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ (see e.g. [17, 18]) and then "averages over gauges" with the help of a suitable covariant positive operator $H_{\mu\nu}[g]$. The resulting effective action is

$$Z[g] = \mathrm{e}^{-I_{\rm eff}} = (\det H)^{-1/2} \int \mathrm{d}h\, \mathrm{d}c\, \mathrm{d}\bar{c}\, \mathrm{e}^{-\tilde{I}[h, g, c, \bar{c}]}, \tag{2.7}$$

$$\tilde{I} = I[g+h] - \frac{\delta I}{\delta g_{\mu\nu}} h_{\mu\nu} + \tfrac{1}{2}\chi_\mu H^{\mu\nu}\chi_\nu + \bar{c}^\mu \tilde{\Delta}_{G\mu\nu} c^\nu, \tag{2.8}$$

$$\tilde{\Delta}_{G\mu\nu} = H_{\mu\lambda}\Delta_G{}^\lambda{}_\nu, \qquad \Delta_{G\mu\nu} = \frac{\delta\chi_\mu}{\delta h_{\lambda\rho}} g_{\nu(\lambda} D_{\rho)}(\hat{g}), \tag{2.9}$$

where $\Delta_G[g, \hat{g}]$ and $\tilde{\Delta}_G[g, \hat{g}]$ are the ghost and "modified" ghost operators. The part of (2.2) quadratic in $h_{\mu\nu}$ has the form [7]

$$\begin{aligned}\mathfrak{L}_2 &= \tfrac{1}{4}a\left(D^2\bar{h}_{\mu\nu}\right)^2 + \tfrac{3}{16}b\left(D^2\varphi\right)^2 + \cdots \\ &= \tfrac{1}{2}a\left\{\tfrac{1}{2}\left(D^2\bar{h}_{\mu\nu}\right)^2 - \tfrac{1}{2}\beta\left(D^2\varphi\right)^2 - \tfrac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \gamma B_0^2\right\} \\ &\quad + \text{terms with less derivatives},\end{aligned} \tag{2.10}$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}\varphi, \qquad \varphi = h^\mu_\mu, \tag{2.11}$$

$$\beta = \frac{3}{4}\frac{\omega}{1+\omega}, \qquad \gamma = \tfrac{2}{3}(1+\omega), \qquad \omega = -\frac{b}{a}, \tag{2.12}$$

$$\mathcal{F}_{\mu\nu} = D_\mu P_\nu - D_\nu P_\mu, \qquad P_\mu = D_\lambda \bar{h}^\lambda_\mu, \tag{2.13}$$

$$B(\xi_1, \xi_2, \xi_3) = D_\lambda P^\lambda - \beta D^2\varphi - \xi_1 R_{\mu\nu}\bar{h}^{\mu\nu} - \xi_2 m^2\varphi - \xi_3 R\varphi, \tag{2.14}$$

$$B_0 = B(1, 0, \tfrac{1}{4}), \qquad m^2 = (ak^2)^{-1}. \tag{2.15}$$

The gauge-breaking term should provide the "diagonality" of the highest derivative term in the operator of the second functional derivative of the action (cf. [7]):

$$\mathfrak{L}_{\rm g.b.} = a\left(\tfrac{1}{4}\mathcal{F}_{\mu\nu}^2 + \tfrac{1}{2}\gamma B^2\right). \tag{2.16}$$

The corresponding ghost term was obtained in [7] from the condition of BRST invariance of the analog of (2.8). It should be stressed that this condition is insensitive to the presence of the $(\det H)^{-1/2}$ factor in (2.7) and that is why this essential factor was missed in [7]. To justify the use of (2.16) one must reveal its "$\chi H \chi$" structure and check the locality of (2.8). This is possible in the following way:

$$\mathcal{L}_{\text{g.b.}} = \tfrac{1}{2} a \chi_\mu H^{\mu\nu} \chi_\nu , \qquad \chi_\mu = P_\mu^\perp + D_\mu D^{-2} B , \tag{2.17}$$

$$P_\mu^\perp = \Pi_{\mu\nu}^\perp P^\nu , \qquad \Pi_{\mu\nu}^\perp = g_{\mu\nu} - D_\mu D^{-2} D_\nu , \tag{2.18}$$

$$H_{\mu\nu} = -g_{\mu\nu} D^2 + D_\nu D_\mu - \gamma D_\mu D_\nu . \tag{2.19}$$

Note that $\chi_\mu$ (and $\Delta_G$) is local only when all gauge parameters $\xi_i$ in (2.14) are zeros:

$$\chi_\mu = D_\nu \bar{h}^\nu_\mu - \beta D_\mu \varphi \tag{2.20}$$

($\beta = \frac{1}{4}$ corresponds to the usual background harmonic gauge). Using (2.9) and (2.17)–(2.19) we get the *local* operator

$$\tilde{\Delta}_{G\mu\nu} = 2 \left[ \left( -g_{\mu\lambda} D^2 + D_\lambda D_\mu \right) \frac{\delta P^\lambda}{\delta h_{\alpha\beta}} - \gamma D_\mu \frac{\delta B}{\delta h_{\alpha\beta}} \right] g_{\nu(\alpha} D_{\beta)} . \tag{2.21}$$

The "one-loop" term in (2.8) has the form

$$\tilde{I}_2 = \tfrac{1}{2} a \left[ \tfrac{1}{2} \bar{h} \Delta_4^{(\bar{h})} \bar{h} - \tfrac{1}{2} \beta \varphi \Delta_4^{(\varphi)} \varphi + \bar{h} \Delta_2^{(\bar{h}\varphi)} \varphi \right] + \bar{c} \tilde{\Delta}_G c , \tag{2.22}$$

where $\Delta_2^{(\bar{h}\varphi)} = \Omega_1^{\mu\nu} D_\mu D_\nu + \Omega_2^\mu D_\mu + \Omega_3$ and $\Delta_4^{(\bar{h})}$, $\Delta_4^{(\varphi)}$ and $\tilde{\Delta}_G$ have the following general structure (cf. [7])

$$\Delta_4 = D^4 + V^{\mu\nu} D_\mu D_\nu + 2 N^\mu D_\mu + U , \tag{2.23}$$

where $D_\mu$, $V_{\mu\nu} = V_{\nu\mu}$, $N_\mu$ and $U$ act on a certain vector bundle over space-time. Defining the combined operator $\hat{\Delta}$ on $(\bar{h}\varphi)$ and changing the variables

$$\bar{h}_{\mu\nu} \to \sqrt{\frac{2}{a}}\, \bar{h}_{\mu\nu} , \qquad \varphi \to \sqrt{-\frac{2}{a\beta}}\, \varphi , \tag{2.24}$$

we get the one-loop approximation for (2.7):

$$Z^{(1)}[g] = \left( \det \mu^{-4} \hat{\Delta} \right)^{-1/2} \left( \det \mu^{-4} \tilde{\Delta}_G \right) \left( \det \mu^{-2} H \right)^{-1/2} , \tag{2.25}$$

where we adopted [in (2.24)] the prescription of ref. [30] for the definition of the functional integral over $\varphi$ ($\beta > 0$ for $b < 0$).

## 3. One-loop counter-terms

### 3.1. GENERAL LAGRANGIAN CASE

Here we present the expressions for the one-loop counter-terms in the theory (2.1)–(2.3) (with $a \neq 0$, $b \neq 0$) generalizing (by the $R^*R^*$ and boundary terms) and correcting the previous result of ref. [7]. We find agreement up to the contribution of $H_{\mu\nu}$ in (2.25) missed in [7]. Calculations are based on the convenient "$\Delta_4$ algorithm" (B.14).

We shall use the following notations for coefficients in the infinite part of the effective action for various background metric dependent operators [cf. (B.4)]:

$$\bar{b}_0 = N_{\rm tot}, \qquad \bar{b}_2 = \rho_1 R + \rho_2, \qquad \bar{c}_2 = 2K\rho_{1\rm S}, \tag{3.1}$$

$$\bar{b}_4 = \beta_1 R^*R^* + \beta_2 W + \tfrac{1}{3}\beta_3 R^2 + \beta_4 R + \beta_5 + \beta_6 D^2 R, \tag{3.2}$$

$$\bar{c}_4 = \beta_{1\rm S}\Omega_{\rm S} + \nu_1 R^{\mu\nu}\bar{K}_{\mu\nu} + 2K(\nu_2 R + \nu_3) + \cdots . \tag{3.3}$$

Always in the following

$$\beta_{1\rm S} = \tfrac{1}{12}\nu_1 = \tfrac{1}{180}N_{\rm tot}, \qquad \rho_{1\rm S} = \tfrac{1}{6}N_{\rm tot},$$

$$\nu_2 = \tfrac{1}{6}\rho_1 - \tfrac{1}{360}N_{\rm tot}, \qquad \nu_3 = \tfrac{1}{6}\rho_2. \tag{3.4}$$

For example, one can find [using (B.5)–(B.10)] that, in the one-loop approximation for Einstein gravity [$a = 0$, $b = 0$ in (2.2)] and for matter fields in an external metric,

$$\beta_1 = \tfrac{53}{45}N_2 - \tfrac{13}{180}N_1^{(0)} - \tfrac{1}{15}N_1 + \tfrac{7}{360}N_{1/2} + \tfrac{1}{180}N_0, \tag{3.5}$$

$$\beta_2 = \tfrac{7}{10}N_2 + \tfrac{1}{5}N_1^{(0)} + \tfrac{13}{60}N_1 + \tfrac{1}{10}N_{1/2} + \tfrac{1}{60}N_0, \tag{3.6}$$

$$\beta_3 = \tfrac{3}{4}N_2 + \tfrac{1}{24}N_1 + \tfrac{1}{24}N_0(1 - 12\eta)^2, \tag{3.7}$$

$$\beta_4 = -\tfrac{26}{3}\Lambda N_2 + \tfrac{1}{2}m_1^2 N_1 - \tfrac{1}{3}m_{1/2}^2 N_{1/2} - \tfrac{1}{6}m_0^2 N_0(1 - 12\eta), \tag{3.8}$$

$$\beta_5 = 20\Lambda^2 N_2 + \tfrac{3}{2}m_1^4 N_1 - 2m_{1/2}^4 N_{1/2} + \tfrac{1}{2}m_0^4 N_0, \tag{3.9}$$

$$\beta_6 = -\tfrac{19}{15}N_2 - \tfrac{1}{10}N_1^{(0)} - \tfrac{1}{15}N_1 + \tfrac{1}{30}N_{1/2} + \left(\tfrac{1}{30} - \tfrac{1}{6}\eta\right)N_0, \tag{3.10}$$

$$\rho_1 = -\tfrac{23}{3}N_2 - \tfrac{2}{3}N_1^{(0)} - \tfrac{1}{2}N_1 + \tfrac{1}{3}N_{1/2} + \tfrac{1}{6}(1-12\eta)N_0, \tag{3.11}$$

$$\rho_2 = 20\Lambda N_2 - 3m_1^2 N_1 + 4m_{1/2}^2 N_{1/2} - m_0^2 N_0, \tag{3.12}$$

$$\rho_{1S} = \tfrac{1}{6}N_{\rm tot}, \qquad N_{\rm tot} = 2N_2 + 2N_1^{(0)} + 3N_1 - 4N_{1/2} + N_0, \tag{3.13}$$

where $N_s$ is the number of fields with spin $s$ and mass $m_s$, $N_1^{(0)}$ is the number of massless gauge fields and the $\eta R\phi^2$ term is assumed to be present for all scalars. The harmonic gauge gravity contribution in $b_4$ is the "off-shell" extension of the result of [32] and agrees with the results of refs. [20, 33] when $\Lambda = 0$. The $N_2$ part of (3.11) was first obtained in [12, 13] (and differs from that of [16]). For the matter contribution in $b_4$ see e.g. [34, 35, 22].

The infinite part of the effective action in (2.25) is expressed by (B.4) with

$$A_p^{\rm tot} = A_p(\hat{\Delta}) + A_p(H) - 2A_p(\tilde{\Delta}_{\rm G}). \tag{3.14}$$

The expressions for $b_4(\tilde{\Delta})$ and $b_4(\tilde{\Delta}_{\rm G})$ are calculated straightforwardly using (2.23) and (B.14) [the dependence on $\xi_i$ in (2.14) is cancelled in the combination (3.14)]. To obtain the contribution of $H_{\mu\nu} = -g_{\mu\nu}D^2 + R_{\mu\nu} - (\gamma-1)D_\mu D_\nu$, (2.19), it is sufficient to note that it coincides with the covariant spin-one operator in a general gauge and thus $b_4(H)$ is independent of $\gamma$* and can be calculated, e.g., for $\gamma = 1$ using (B.5), (B.7):

$$\bar{b}_4(H):\quad \beta_1 = -\tfrac{11}{180}, \qquad \beta_2 = \tfrac{7}{30}, \qquad \beta_3 = \tfrac{1}{12}, \qquad \beta_4 = \beta_5 = 0, \qquad \beta_6 = -\tfrac{1}{30}. \tag{3.15}$$

Let us note in passing that it is possible to obtain the $b_4$ coefficient for the following generalization of $H_{\mu\nu}$ (see [36]):

$$\Delta_{\mu\nu}^{(\zeta)} = -g_{\mu\nu}D^2 + X_{\mu\nu} + \zeta D_\mu D_\nu, \tag{3.16}$$

$$\begin{aligned}\bar{b}_4(\Delta^{(\zeta)}) = &-\tfrac{11}{180}R^*R^* + \left(\tfrac{1}{24}s^2 - \tfrac{37}{120}\right)W + \left(\tfrac{5}{48}s^2 - \tfrac{1}{8}s - \tfrac{5}{48}\right)\tfrac{1}{3}R^2 \\ &+ \left(\tfrac{1}{24}s^2 - \tfrac{1}{6}s + \tfrac{7}{24}\right)X_{\mu\nu}^2 + \left(-\tfrac{1}{12}s^2 + \tfrac{1}{6}s + \tfrac{1}{4}\right) \\ &\times X_{\mu\nu}R^{\mu\nu} + \tfrac{1}{48}(s+1)^2X^2 + \left(-\tfrac{1}{24}s^2 - \tfrac{1}{8}\right)RX, \\ s \equiv (\zeta-1)^{-1}, &\qquad X = X_\mu^\mu. \end{aligned} \tag{3.17}$$

* In other words, one may use decomposition $u_\mu = u_\mu^\perp + u_\mu^\parallel$, $D_\mu u_\mu^\perp = 0$ and prove that $\det_u H = \det_{u^\perp} H(\gamma=0)\cdot \det D^2 \cdot \gamma^N$, $N = b_4(D^2)$.

If $X_{\mu\nu} = -R_{\mu\nu}$, $\zeta = 2(\beta - \frac{1}{4})$, $\Delta^{(\zeta)}$ coincides with the ghost operator in (2.9) in the general covariant gauge (2.20) (note that the case of $\zeta = 1$ can be treated as that of the massive vector field operator, cf. [35]). Thus we can find the result for the general gauge ghost contribution in the divergences without computing any diagram (cf. [37]) but simply using the fact that multiplication of two special second-order operators $H$ and $\Delta_G$ with the non-diagonal highest derivative terms gives the fourth-order operator $\tilde{\Delta}_G$, diagonal in highest derivatives and hence with a simply computable $b_4$ coefficient.

Finally we get according to (3.14) and (3.2)

$$\bar{b}_4^{\text{tot}}: \qquad \beta_1 = \tfrac{413}{180}, \qquad \beta_2 = \tfrac{133}{10}, \qquad \beta_3 = \tfrac{10}{3}\omega^2 + 5\omega - \tfrac{1}{12}, \tag{3.18}$$

$$\tilde{\beta}_4 = \tfrac{10}{3}\omega - \frac{1}{4\omega} - \tfrac{13}{6}, \qquad F_1 = \tfrac{56}{3} + \frac{2}{3\omega}, \qquad F_2 = \tfrac{5}{2} + \frac{1}{8\omega^2},$$

$$\tilde{\beta}_4 = m^{-2}\beta_4, \qquad \tilde{\beta}_5 = m^{-4}\beta_5 \equiv F_1\bar{\lambda} + F_2, \qquad \bar{\lambda} = a\Lambda k_{\text{V}}^2. \tag{3.19}$$

Using the results of appendix B we also obtain the expressions for $b_2^{\text{tot}}$, $c_2^{\text{tot}}$ and $c_4^{\text{tot}}$ in (3.1), (3.3), (3.4),

$$\rho_1 = -\tfrac{10}{3}\omega - 5, \qquad \rho_2 = -m^2\left(5 + \frac{1}{2\omega}\right), \qquad N_{\text{tot}} = 8. \tag{3.20}$$

It is important to realize that one cannot put $a = 0$ or $b = 0$ in (3.18) because these limits (in contrast with the $k_{\text{V}}^{-2} \to 0$ and $\Lambda \to 0$ limits) change the order (from fourth to second) of the operators involved (e.g. $\beta_1$ and $\beta_2$ have a "step-function" of $b$). That is why one cannot establish the result for $\beta_2$, for example, in the Weyl theory [$b = 0$, $\Lambda = 0$, $k_{\text{V}}^{-2} = 0$ in (2.2)] simply by taking the value of $\beta_2$ in (3.18) as was incorrectly proposed in [10] (on the basis of the result of [7]).

### 3.2. CONFORMAL INVARIANT THEORIES

Let us consider the following two theories (see e.g. [38–40] and references therein):

$$\mathfrak{L}_1 = aW + \alpha_{\text{V}} R^* R^*, \tag{3.21}$$

$$\mathfrak{L}_2 = -6\phi\left(-D^2 + \tfrac{1}{6}R\right)\phi + 2\lambda\phi^4 + aW + \alpha_{\text{V}} R^* R^*. \tag{3.22}$$

The Weyl theory (3.21) is useful in the context of the induced gravity approach [41, 11, 10] and for conformal supergravity [42], while (3.22) is a generalization of conformal off-mass-shell extension of the Einstein theory [39]. Both actions are invariant (up to boundary terms) under $g'_{\mu\nu} = \sigma^2 g_{\mu\nu}$, $\phi' = \sigma^{-1}\phi$. Let us begin with the background field method quantization of (3.22) with the action

$$I[g, \phi] = I_0[\tilde{g}], \qquad \tilde{g}_{\mu\nu} = g_{\mu\nu}\phi^2 k_{\text{V}}^2, \qquad \lambda = \Lambda k_{\text{V}}^2, \tag{3.23}$$

where $I_0$ corresponds to (2.1), (2.2) with $b=0$. Introducing the classical and quantum fields $\hat{g}_{\mu\nu}=g_{\mu\nu}+h_{\mu\nu}$, $\phi=\phi+\psi$ and choosing the background conformal and coordinate gauges

$$\psi=0, \qquad \chi_\mu[\tilde{g},\hat{\tilde{g}}]=\zeta_\mu(x), \tag{3.24}$$

we preserve the background conformal invariance and get for the effective action (integrating over $\psi$, changing the variable $\tilde{h}\to h$ and omitting all $L^4$ divergent factors)

$$I_{\rm eff}[g,\phi]=I_{{\rm eff}_0}[\tilde{g}]. \tag{3.25}$$

As a consequence, one must first calculate the effective action (or, e.g., its infinite part) for $I_0$ and then make the substitution $g_{\mu\nu}\to\tilde{g}_{\mu\nu}$ as in (3.23). When $b=0$ in (2.2), the operator on $\varphi$ in (2.10) is of *second* order and therefore one must analyse the infinities in the $\varphi$ and $\bar{h}$ sectors independently. This is possible in the gauge (2.14), (2.16) with $\xi_3=0$ and $\xi_2=-\frac{3}{8}$ (where the mixed $\varphi D_\mu D_\nu \bar{h}^{\mu\nu}$ terms are absent). Then the $\Delta_2^{(\varphi)}$ contribution is computed with the help of (B.6)–(B.10) and we finally get instead of (3.18) and (3.20)

$$\bar{b}_4^{\rm tot}: \quad \beta_1=\tfrac{103}{45}, \qquad \beta_2=\tfrac{797}{60}, \qquad \beta_3=-\tfrac{11}{24},$$

$$\tilde{\beta}_4=-\tfrac{13}{6}+\tfrac{2}{9}\bar{\lambda}, \qquad \tilde{\beta}_5=\tfrac{8}{9}\bar{\lambda}^2+\tfrac{56}{3}\bar{\lambda}+\tfrac{5}{2},$$

$$\rho_1=-\tfrac{11}{2}, \qquad \rho_2=m^2\left(\tfrac{4}{3}\bar{\lambda}-5\right), \qquad N_{\rm tot}=7. \tag{3.26}$$

The counter-terms for (3.22) are obtained by the substitution $g_{\mu\nu}\to g_{\mu\nu}\phi^2k^2$ in (3.2)–(3.4), (3.26). In view of $\beta_3\neq 0$ the theory is *renormalizable only on-mass-shell*, $R(\tilde{g})=4\Lambda$ ($R^2(\tilde{g})\sqrt{\tilde{g}}$ is a possible conformal invariant counter-term)*.

The main question in quantization of (3.21) is how to establish the background conformal invariance of the $\chi_\mu$ gauge and hence of the effective action (working in the one-loop approximation we disregard the problem of conformal anomalies, i.e. the breaking of the above invariance by a regularization, cf., however, [39]). This problem is solved (cf. [39]) by using the following gauges [compare with (3.24)]:

$$\varphi=0, \qquad \chi_\mu[\tilde{g},\hat{\tilde{g}}]=\zeta_\mu(x), \qquad \tilde{g}_{\mu\nu}=g_{\mu\nu}\Phi^2(g), \tag{3.27}$$

where $R(\tilde{g})=0$, i.e. $\Phi(g)=1-(-D^2+\frac{1}{6}R)^{-1}\frac{1}{6}R$ if $g_{\mu\nu}$ is asymptotically flat. Integrating over $\varphi$ we get for (3.21): $I_{\rm eff}[g]=I_{{\rm eff}_0}[\tilde{g}]$, where $I_{{\rm eff}_0}$ corresponds to the

* This is due to a "miraculous" cancellation of contributions of two different diagrams (see [36]) and thus is in no way a trivial result as it is in the Einstein theory where the absence of the $F_{\mu\nu}^2$ terms follows simply from dimensional considerations [45].

theory (3.21) quantized in the gauge $\chi_\mu[g, g+\bar{h}]=\zeta_\mu$ with an *arbitrary* background metric. As a result, the Weyl theory is one-loop renormalizable ($R^2$ terms disappear after $g\to\tilde{g}$). Putting $b$, $\Lambda$, $k_V^{-2}=0$, we see that $\varphi$ drops from (2.10) and so in the $\varphi$-independent gauge [(2.14), (2.16) with $\xi_2=\xi_3=0$] we get

$$\bar{b}_4^{\text{tot}}:\qquad \beta_1=\tfrac{137}{60},\qquad \beta_2=\tfrac{199}{15},\qquad \beta_3=-\tfrac{1}{2},\qquad \beta_4=\beta_5=0,$$

$$\rho_1=-\tfrac{17}{3},\qquad \rho_2=0,\qquad N_{\text{tot}}=6, \tag{3.28}$$

or after the change $g\to\tilde{g}$ in (3.27)

$$\bar{b}_4^{\text{tot}}=\tfrac{137}{60}R^*R^*+\tfrac{199}{15}W,\qquad \bar{b}_2^{\text{tot}}=0. \tag{3.29}$$

This value of $\beta_2$ in the Weyl theory was checked through the independent calculation of the one-loop $\beta$-function in $N=1$ conformal supergravity [42]. Note also a simple relation of the $\beta_2$ values for the theories (2.2), (3.22) and (3.21) in (3.18), (3.26) and (3.28), respectively: $\beta_2(2.2)-2\cdot\frac{1}{60}=\beta_2(3.22)-\frac{1}{60}=\beta_2(3.21)$, where $\frac{1}{60}$ is simply the contribution of one scalar ($\varphi$) degree of freedom [cf. (3.6)].

## 4. Renormalization group equations and asymptotic freedom

### 4.1. EINSTEIN THEORY WITH $\Lambda\neq 0$

It seems useful first to make several remarks about the one-loop RG equations for the theory (2.2) with $a=b=0$.

Consider an arbitrary renormalizable euclidean theory with the action $I=\Sigma_i a_i I_i$ and the infinite part of the one-loop effective action $I_\infty=-\frac{1}{2}\log(L^2/\mu^2)\Sigma_i\beta^{(i)}I_i$ [cf. (B.4)] (we assume that $L^4$ divergences are cancelled by the measure, while $L^2$ ones are simply subtracted and here we neglect boundary terms). Here $a_i$ are coupling constants and $I_i$ are invariants constructed from fields. Then the RG equations are (see e.g. [43])

$$\dot{a}_i\equiv\frac{\mathrm{d}a_i}{\mathrm{d}t}=\beta^{(i)}(a_k),\qquad t=\frac{1}{2(4\pi)^2}\log\frac{\mu'^2}{\mu^2}. \tag{4.1}$$

It is useful to introduce the concept of essential coupling constants (cf. [43]), i.e. some combinations of $a_i$ which are invariant under renormalization, possess the gauge-independent RG equations and which actually contribute in the $S$-matrix (or in the effective action, calculated from the solution of the field equations). The RG equations for essential couplings can be obtained either by combining (4.1) or by the use of field equations in $I$ and $I_\infty$ [36]. The second procedure is valid also for only on-shell (one-loop) renormalizable theories like the Einstein theory. The *essential* *couplings* here are dimensionless, $\lambda = \Lambda k_V^2$, $\alpha_V$, $\alpha_S$ and $k_S$ [we assume the condition (2.4)] satisfying the following RG equations [see (3.1)–(3.13)]*:

$$\dot{\alpha}_V = \beta_1, \qquad \dot{\alpha}_S = \beta_{1S}, \qquad \left(\dot{k_S^{-2}}\right) = \sigma_S \lambda k_S^{-2}, \tag{4.2}$$

$$\dot{\lambda} = -\tfrac{1}{2}\sigma\lambda^2, \qquad \text{or} \qquad \lambda(t) = \frac{\lambda(0)}{1 + \frac{1}{2}\sigma\lambda(0)t}, \tag{4.3}$$

where

$$\sigma_S = -4\nu_2 - \Lambda^{-1}\nu_3 = \tfrac{9}{5}, \qquad \sigma = \tfrac{4}{3}\beta_2 - \tfrac{16}{3}\beta_3 - 4\Lambda^{-1}\beta_4 - \Lambda^{-2}\beta_5 = \tfrac{58}{5}$$

(on the field equations $\bar{b}_4 = \beta_1 R^*R^* - \sigma\Lambda^2$ in agreement with the result of [32]). As a consequence,

$$k_S^2(t) = k_S^2(0)\left[\frac{\lambda(t)}{\lambda(0)}\right]^p, \qquad p = \frac{2\sigma_S}{\sigma} \simeq 0.3, \tag{4.4}$$

and we conclude that we have *asymptotic freedom* (AF) for $\lambda > 0$ and $k_S^2$ (and also for $\alpha_V^{-1} > 0$, $\alpha_S^{-1} > 0$). The AF for $\lambda$ can be treated as a manifestation of a negligible value of the effective $\Lambda$ term at very small distances, implying that space-time *is not becoming "foamier"* with the decreasing of the scale (contrary to the proposals in [24, 44, 32]). For correspondence with observations one must have $\lambda(t) < \lambda_0 \sim 10^{-122}$ on all the scales less than the size of the Universe. Then the apparent "infrared" pole in (4.3) will occur at inconceivably large distances and therefore is unphysical. As a result, one can, in principle, match the asymptotic freedom at microscopic distances with the needed small value of $\Lambda$ at large distances and thus justify the flat-space expansion in quantum gravity.

#### 4.2. HIGHER DERIVATIVE GRAVITY

The essential couplings for the theory (2.1)–(2.3) are again $\lambda = \Lambda k_V^2$, $k_S$, $\alpha_V$, $\alpha_S$ and also $a$, $b$ and $\kappa$. The RG equations can be derived using (4.1), (3.1)–(3.4):

$$\dot{a} = \beta_2, \qquad \dot{b} = \beta_3, \qquad \dot{\lambda} = 2\beta_4 k^2\lambda + \tfrac{1}{2}\beta_5 k^4, \tag{4.5}$$

or in terms of $\omega = -b/a$ and $\bar{\lambda} = a\lambda$ [see (3.19) for notations]

$$\omega' \equiv a(t)\frac{d\omega}{dt} = -\beta_2\omega - \beta_3,$$

$$\bar{\lambda}' = \left(2\tilde{\beta}_4 + \beta_2 + \tfrac{1}{2}F_1\right)\bar{\lambda} + \tfrac{1}{2}F_2. \tag{4.6}$$

* Let us stress that it is the knowledge of the $c_4$ coefficient (B.10) that gives the possibility to obtain the RG equations for $k_S$ and $\alpha_S$ and also that the logarithmic renormalization of $k_S$ is necessary even if only matter is quantized, $(\dot{k_S^{-2}}) = -\nu_3$ [cf. (3.3), (3.11)–(3.13)].

To establish the renormalizability of the boundary terms (2.3) one must use the field equations and their trace (2.5) in (3.3) and neglect (due to the boundary conditions) the $KRR$ and $KD^2R$ terms. This yields the following result [cf. (4.2)]:

$$\left(\dot{k}_S^{-2}\right) = a^{-1} d k_S^{-2}, \qquad d = -4\nu_2\bar{\lambda} - m^{-2}\nu_3. \tag{4.7}$$

As follows from (4.5) and (3.18) one has the *asymptotically free* solution for $a^{-1} \equiv f^2 > 0$ in the theory (2.2):

$$a(t) = a(0) + \beta_2 t, \qquad f^2(t) = f^2(0)\left[1 + \beta_2 f^2(0) t\right]^{-1}. \tag{4.8}$$

From (4.6) and (3.18) we get

$$\omega' = A\omega^2 + B\omega + C, \qquad A = -\tfrac{10}{3}, \qquad B = -\tfrac{183}{10}, \qquad C = \tfrac{1}{12}, \tag{4.9}$$

with the following regular solution

$$\omega(t) = \frac{\omega_1 Q(t) - \omega_2}{Q(t) - 1}, \qquad Q(t) = \left|\frac{a(t)}{a_0}\right|^p, \tag{4.10}$$

where

$$\omega_{1,2} = -\frac{1}{2A}\left[B \pm \sqrt{D}\right], \qquad \omega_1 > \omega_2, \qquad D = B^2 - 4AC > 0,$$

$$p = \sqrt{D}/\beta_2 > 0, \qquad a_0 = \text{const},$$

i.e.

$$\omega_1 \simeq 0.0046, \qquad \omega_2 \simeq -5.4946, \qquad p \simeq 1.36. \tag{4.11}$$

Assuming $\omega > 0$ [i.e. $b < 0$, corresponding to the absence of the $0^+$ tachyon in (2.6)] we have $\omega(t)$ decreasing to the stable fixed point $\omega_1 = \omega(\infty)$ with $t \to \infty$, thus implying, due to (4.8), AF behaviour for $b^{-1} < 0$. The equation for $\bar{\lambda}$ in (4.6) $\bar{\lambda}' \equiv S_1(t)\bar{\lambda} + S_2(t)$ has the following general solution:

$$\bar{\lambda} = \Phi\left[c + \int^t a^{-1} S_2 \Phi \, dt'\right], \qquad \Phi(t) = \exp\left(\int^t S_1 a^{-1} \, dt'\right), \tag{4.12}$$

where $c = \text{const}$ and $S_1$, $S_2$ are defined from (4.6) and (3.18). When $t \to \infty$ we get

$$\bar{\lambda}(t) \simeq \bar{\lambda}_\infty + c[a(t)]^q, \qquad \bar{\lambda} \equiv \lambda \cdot a,$$

$$\bar{\lambda}_\infty = -\frac{S_2(\infty)}{S_1(\infty)} \simeq 166, \qquad q = \frac{S_1(\infty)}{\beta_2} \simeq -1.1, \tag{4.13}$$

and so $\bar{\lambda}(\infty)=\bar{\lambda}_\infty>0$, leading in view of (4.8) to AF for $\lambda>0$. Note that there is also the *stable* "fixed point" solution $\omega(t)=\omega_1$, $\bar{\lambda}(t)=\bar{\lambda}_\infty$. Observe also that "off-mass-shell" $\Lambda|_{t\to\infty}\sim t^{\nu}$, $k_{\rm V}^2|_{t\to\infty}\sim t^{-\nu-1}$, where in the gauge we used $\nu\simeq 2.9$. In the UV limit one can solve (4.7) as follows ($d(\infty)\simeq 670$):

$$k_{\rm S}^2(t)\simeq k_{\rm S}^2(0)\left[\frac{a(0)}{a(t)}\right]^r, \qquad r=\frac{d}{\beta_2}\simeq 50, \tag{4.14}$$

and conclude AF behaviour for the *dimensional* gravitational constant [cf. (4.4)]. As a result *the theory* (2.1)–(2.3) *is asymptotically free in all essential coupling constants*.

According to (3.26) and (3.28) the conformal invariant theories (3.21) and (3.22) are also asymptotically free in $a^{-1}$. Using (3.26) it is possible to obtain the "on-shell" RG equation for $\lambda$ in (3.22):

$$\bar{\lambda}'=\tfrac{8}{3}\beta_3\bar{\lambda}^2+\left(2\tilde{\beta}_4+\beta_2\right)\bar{\lambda}+\tfrac{1}{2}\tilde{\beta}_5=-\tfrac{1}{3}\bar{\lambda}^2+\tfrac{1097}{60}\bar{\lambda}+\tfrac{5}{4}\,. \tag{4.15}$$

This equation is analogous to (4.9) and has a regular solution like (4.10) with $\bar{\lambda}_1\simeq 54.87$, $\bar{\lambda}_2\simeq -0.023$, $p\simeq 1.98$ and so $\bar{\lambda}(\infty)=\bar{\lambda}_1$ implies the *AF regime for* $\lambda>0$. This result should be compared with the well-known "zero-charge" behaviour ($\dot{\lambda}=\lambda^2$) in flat space-time with the conclusion that interaction with gravity drastically changes the behaviour of the $\phi^4$ coupling constant in the model (3.22) (essentially due to the AF in $a^{-1}$).

### 4.3. CONSEQUENCES OF ASYMPTOTIC FREEDOM

Let us consider first the ultraviolet (UV) (or "weak coupling") limit. The flat-space expansion of (2.2),

$$g_{\mu\nu}=\delta_{\mu\nu}+fh_{\mu\nu}, \qquad a^{-1}=f^2,$$

$$\mathcal{L}_{\rm V}\simeq -\frac{1}{k^2}\left(f^2h\Box h+\cdots\right)+\left(h\Box^2h+{\rm O}(f)\right)-\omega\left(h\Box^2h+{\rm O}(f)\right), \tag{4.16}$$

implies that the main terms in the action are the kinetic parts of $W$ and $R^2$ and that the AF in $f$ (4.8) leads to the validity of the perturbation theory in $f$. A possible mechanism restoring unitarity [cf. (2.6)] is a summation of radiative corrections which may shift the ghost poles off the real axis [12, 15, 8–10]. The inverse euclidean propagator in the UV limit can be written as

$$2^+: \qquad \frac{1}{4k^2}p^2\left(1+\frac{k^2\beta_2}{32\pi^2}p^2\log\frac{p^2}{\mu^2}+\cdots\right),$$

$$0^+: \qquad -\frac{1}{2k^2}p^2\left(1-\frac{k^2\beta_3}{16\pi^2}p^2\log\frac{p^2}{\mu^2}+\cdots\right), \tag{4.17}$$

and thus our results ($\beta_2>0$, $\beta_3(\omega(\infty))<0$) *do indicate* the absence of real poles.

For a complete analysis of the unitarity problem one needs some additional information about the infrared (IR) and intermediate regions. [The IR-limit RG equations *are different from* that of the UV ones because of the presence of the Einstein ("mass") term in (2.2)*.] Such an analysis lies beyond the scope of the present paper. However, one can believe that the above UV solutions for $a$, $b$ and $\lambda$ have some sense in the intermediate region and qualitatively reproduce the "tendencies" in the behaviour of these couplings. For example, the AF for $f$ suggests the IR growth of $f$ (and thus of the $2^+$ ghost and the $0^+$ particle "masses", cf. [6, 7]) and that the Einstein term in (2.2) prevails at large distances. The apparent IR growth of $\lambda$ can probably be cured by the choice of $c$ in (4.13) [and also of the position of the IR pole in (4.8)] and (or) by the study of the IR properties of the theory.

## 5. Renormalization group equations in the presence of matter

In this section we discuss the one-loop counterterms and RG equations for a system of matter fields with a symbolic lagrangian

$$\mathcal{L}_{\mathrm{m}}=\frac{1}{4e^2}\left(F^a_{\mu\nu}\right)^2+\tfrac{1}{2}D_\mu\phi D_\mu\phi+\tfrac{1}{2}m_0^2\phi^2+\tfrac{1}{4}g\phi^4$$

$$+\eta R\phi^2+i\bar{\psi}\hat{D}\psi+\chi\bar{\psi}\psi\phi, \tag{5.1}$$

interacting with higher derivative gravity (2.1)–(2.3). The total theory is renormalizable** (contrary to the Einstein theory with matter [45]) and may serve as a model of a unified renormalizable theory. The analysis of the corresponding system of RG equations is interesting because gravity essentially changes the behaviour of all running coupling constants and because in the presence of gravity masses become true coupling constants with separate RG equations. A natural question is whether it is possible to obtain the asymptotically free solutions for all matter couplings $(e, g, \eta, \chi)$ and masses as well as for gravitational couplings in (2.2).

* This statement can be illustrated using the example discussed in appendix C: taking $L=\infty$ and $L=0$ in (C.5) we get different results. Thus the IR $\beta$-function for $a^{-1}$ can be obtained by assuming $L\to 0$ in (C.7).

** It is important to stress the necessity of the $R\phi^2$ term for renormalizability (cf. [46]).

Using the background field method it is possible to work out the general structure of the one-loop system of RG equations. The "gravitational" part of this system is again (4.5) [or (4.6) and (4.7)] where $\beta_i$ must be substituted by $\hat{\beta}_i = \beta_i + \beta_i^{(m)}$ with the matter contributions $\beta_i^{(m)}$ given by (3.5)–(3.13). Thus the total $\hat{\beta}_2$ coefficient is always *positive* and we conclude that matter fields *support* the asymptotic freedom in $a^{-1}$ (see also [9]). Moreover, it is easy to prove [36] that interaction of the gauge fields with higher-derivative gravity does not lead to additional $F_{\mu\nu}^2$ counter-terms*. That is why the gauge field $\beta$-function has its flat space value (e.g. implying AF in $e$). Equations for $g$, $\chi$ and $\eta$ can be written in the form

$$\bar{g}'_3 = A_1\bar{g}_3^2 + A_2\bar{g}_2^2 + A_3\bar{g}_1^2 + \bar{g}_3\left[A_4\bar{g}_1 + A_5\bar{g}_2 + B_1 + \omega^{-1}\left(B_2 + B_3\eta + B_4\eta^2\right)\right]$$
$$+\eta^2\left[B_5 + \omega^{-2}\left(B_6 + B_7\eta + B_8\eta^2\right)\right] + \bar{g}_1\eta^2\left(B_9 + \omega^{-1}B_{10}\right), \tag{5.2}$$

$$\bar{g}'_2 = D_1\bar{g}_2^2 + D_2\bar{g}_1\bar{g}_2 + \bar{g}_1\left[E_1 + E_2\eta + \omega^{-1}(E_3 + E_4\eta)\right], \tag{5.3}$$

$$\eta' = \eta\left[C_1 + C_2\omega + \omega^{-1}\left(C_3 + C_4\eta + C_5\eta^2\right) + C_6\bar{g}_1 + C_7\bar{g}_2 + C_8\bar{g}_3\right]$$
$$+ C_{10}\bar{g}_1 + C_{11}\bar{g}_2 + C_{12}\bar{g}_3, \tag{5.4}$$

where $\bar{g}_1 = ae^2$, $\bar{g}_2 = a\chi^2$, $\bar{g}_3 = ag$ and $B_i, E_i, C_i = \text{const}$, while $A_i$, $D_i$ have their flat-space values. There are also equations for masses of the type $\overline{M}' = d_1\overline{M} + d_2\overline{M}^2 + \cdots$, $\overline{M} = Ma$, where the *essential dimensionless coupling constant* $M \equiv m_0^2 k_V^2$ plays the role of the "effective mass". From the structure of (5.2)–(5.4) it follows that $\eta$, $\bar{g}_3$ and $\bar{g}_1$, $\bar{g}_2$ have (in principle) *UV fixed points* and therefore $g$ and $\chi$ have asymptotically free ($\sim a^{-1}$) behaviour. There are also fixed points for $\bar{\lambda}$ and $\overline{M}$ and thus, again, the universal "$a^{-1}$" regime for $\lambda$ and $M$. The only open question is whether all asymptotic ($t \to \infty$) values are consistent with the existence of fixed points and have proper signs. One may conjecture that in a realistic model these values will be the correct ones.

Now let us explicitly study the case when only scalar field terms are present in (5.1). Then it is possible to obtain the following system of RG equations ($N_0$ is the number of scalars)

$$\dot{a} = 13.3 + \tfrac{1}{60}N_0,$$
$$\omega' = -\tfrac{10}{3}\omega^2 - \left(18.3 + \tfrac{1}{60}N_0\right)\omega + \tfrac{1}{12} - \tfrac{1}{24}N_0(1 - 12\eta)^2, \tag{5.5}$$

$$\bar{g}'_3 = 2(N_0 + 8)\bar{g}_3^2 + \bar{g}_3\left[\left(18.3 + \tfrac{1}{60}N_0\right) + \omega^{-1}\left(\tfrac{1}{2} - 16\eta + 72\eta^2\right)\right]$$
$$+\eta^2\left[10 + \tfrac{1}{2}\omega^{-2}(1 - 12\eta)^2\right], \tag{5.6}$$

* This is due to a "miraculous" cancellation of contributions of two different diagrams (see [36]) and thus is in no way a trivial result as it is in the Einstein theory where the absence of the $F_{\mu\nu}^2$ terms follows simply from dimensional considerations [45].

$$\eta' = \eta\left[-\tfrac{10}{3}\omega + \omega^{-1}\left(\tfrac{1}{3} - 7\eta + 12\eta^2\right)\right] - \tfrac{1}{6}(N_0+2)(1-12\eta)\bar{g}_3, \tag{5.7}$$

$$\overline{M}' = \overline{M}\left[\left(18.3 + \tfrac{1}{60}N_0 + \tfrac{10}{3}\omega\right) + \omega^{-1}\left(\tfrac{1}{6} - 8\eta + 12\eta^2\right)\right]$$
$$+2(N_0+2)\overline{M}\bar{g}_3 - \tfrac{1}{6}N_0(1-12\eta)\overline{M}^2 - \eta\left[10 + \tfrac{1}{2}\omega^{-1}(1-12\eta)\right], \tag{5.8}$$

$$\bar{\lambda}' = \left(18.3 + \tfrac{1}{60}N_0 + \tfrac{20}{3}\omega - \tfrac{1}{6}\omega^{-1}\right)\bar{\lambda} + \tfrac{5}{4} + \tfrac{1}{16}\omega^{-2}$$
$$-\tfrac{1}{3}N_0(1-12\eta)\overline{M}\bar{\lambda} + \tfrac{1}{4}N_0\overline{M}^2. \tag{5.9}$$

Though this system does not possess "fixed point" solutions with physical signs* ($\omega > 0,\ g > 0$) there are $t$-dependent solutions with $\omega(t)|_{t\to\infty} \to 0$ and $\bar{g}_3(t)|_{t\to\infty} \to \bar{g}_\infty > 0$, $\overline{M}(t)|_{t\to\infty} \to \overline{M}_\infty < 0$, implying asymptotically free behaviour for $g$ and $M$ (note that $m_0^2 < 0$ in the case of a spontaneous symmetry breaking).

One may observe that the above results can be applied to the following *renormalizable globally* scale invariant theory (cf. [38])

$$\mathcal{L} = \eta R\phi^2 + \tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi + \tfrac{1}{4}g\phi^4 + aW + \tfrac{1}{3}bR^2. \tag{5.11}$$

The corresponding RG equations are *identical* with (5.5)–(5.7). The idea is to generate the dynamical symmetry breaking a la Coleman-Weinberg (such that $\eta\langle\phi^2\rangle = -1/k^2$) due to higher derivative gravitational terms in (5.11) (just like proposed in [47] in the case of Einstein theory). The one-loop effective potential has the general form $V = \frac{1}{4}g\phi^4 + \gamma\phi^4[\log(\phi^2/\mu^2) - \frac{25}{6}]$ (remember that the theory is renormalizable) making this idea rather natural (cf. [53])**.

In conclusion we again want to point out that renormalizable asymptotically free quantum gravity may solve the problem of bare masses and non-gauge couplings by establishing their asymptotically free behaviour.

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## Note added in proof

As was pointed out in the text, the AF for $a^{-1}$ supports the idea of restoration of unitarity at the quantum level. The resulting acausality in the propagation of wave packets is reminiscent of that of the Lee-Wick model. However, in their case acausalities were expected to show up at $10^{-15}$ cm scales (contradicting experiment), while in our case they occur at Planck lengths where the question of causality is a subtle one.

* This fact should not be considered as raising doubt about the above conjecture because it is the existence of the fixed point for the *total* system (5.2)–(5.4) that is important for applications.

** The value of $\gamma$ is [cf. (5.6)] $\gamma = (1/128\pi^2)\{18g^2 + [\frac{28}{3}f^2 + (\frac{1}{3} - 16\eta + 72\eta^2)\sigma^2]g + [10f^4 + \frac{1}{2}\sigma^4(1 - 12\eta)^2]\eta^2\}$, where $f^2 = a^{-1}$, $\sigma^2 = -b^{-1}$, $N_0 = 1$.

Next let us remark that the possibility that IR free matter couplings (i.e. Yukawa and quartic) become UV free in the presence of renormalizable gravity does not affect the low energy behaviour in theories with scalars. These couplings presumably remain IR free at distances larger than Planck length where gravitational effects become irrelevant.

## Appendix A

### NOTATIONS

We use the following notations:

$$R^{\lambda}{}_{\mu\nu\rho}=\partial_\nu\Gamma^{\lambda}{}_{\mu\rho}-\cdots,\qquad R_{\mu\nu}=R^{\lambda}{}_{\mu\lambda\nu},\qquad \operatorname{sign} g_{\mu\nu}=+4, \tag{A.1}$$

$$W=R^2_{\mu\nu}-\tfrac{1}{3}R^2,\qquad R^*R^*=\tfrac{1}{4}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\lambda\rho\gamma\delta}R^{\lambda\rho}_{\mu\nu}R^{\gamma\delta}_{\alpha\beta}. \tag{A.2}$$

In the four-dimensional case one has for the square of the Weyl tensor

$$C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho}=R^*R^*+2W. \tag{A.3}$$

If $n_\mu$ is an outward directed unit normal to the boundary $\partial M$ of $M^4$ and $\gamma_{\mu\nu}=g_{\mu\nu}-n_\mu n_\nu$ is the induced metric on $\partial M$, then the second fundamental form of $\partial M$ and its trace are given by

$$K_{\mu\nu}=(D_\lambda n_\rho)\gamma^\lambda_\mu\gamma^\rho_\nu,\qquad K=K_{\mu\nu}g^{\mu\nu}. \tag{A.4}$$

The Euler number of $M^4$ has the following representation [25]:

$$\chi=\chi_V+\chi_S=\frac{1}{32\pi^2}\left(\int_M R^*R^*\sqrt{g}\,d^4x+\int_{\partial M}\Omega_S\sqrt{\gamma}\,d^3x\right),$$

$$\Omega_S=\tfrac{1}{4}R_{\lambda\mu\nu\rho}K^{\lambda\nu}n^\mu n^\rho+16\det{}^{(3)}K_{\mu\nu},$$

$$\det{}^{(3)}K_{\mu\nu}=\frac{1}{3!}\varepsilon^{\lambda\mu\nu\rho}\varepsilon^{\alpha\beta\gamma\delta}n_\lambda n_\alpha K_{\mu\beta}K_{\nu\gamma}K_{\rho\delta},\qquad \gamma=\det{}^{(3)}\gamma_{\mu\nu}. \tag{A.5}$$

## Appendix B

### DIVERGENCES OF THE DETERMINANT OF THE FOURTH-ORDER DIFFERENTIAL OPERATOR

Let us first give the known results for $(\log\det\Delta_2)_\infty$, where $\Delta_2$ is an elliptic differential operator defined on some vector bundle over a riemannian manifold $M^n$.

Then the standard formulae are valid (see e.g. [17, 21, 48]):

$$\left(\operatorname{tr} e^{-s\bar{\Delta}_2}\right)_{s\to 0} \simeq \sum_{p=0}^{\infty} \left(\mu^{-2}s\right)^{(p-n)/2} A_p, \qquad \bar{\Delta}_2 \equiv \mu^{-2}\Delta_2, \tag{B.1}$$

$$A_p = B_p + C_p = \int_{\mathrm{M}} b_p \sqrt{g}\, \mathrm{d}^n x + \int_{\partial\mathrm{M}} C_p \sqrt{\gamma}\, \mathrm{d}^{n-1}x, \tag{B.2}$$

$$\log\det \bar{\Delta}_2 = -\int_{\varepsilon}^{\infty} \frac{\mathrm{d}s}{s} \operatorname{tr}\exp\left(-s\bar{\Delta}_2\right), \qquad \varepsilon \to +0, \tag{B.3}$$

and thus for the infinite part of the Bose field effective action we have (for $n=4$)

$$I_\infty = \tfrac{1}{2}\left(\log\det\bar{\Delta}_2\right)_\infty = -\tfrac{1}{2}\left(\tfrac{1}{2}A_0 L^4 + \tfrac{2}{3}A_1 L^3 + A_2 L^2 + 2A_3 L + A_4 \log\left(L^2/\mu^2\right)\right),$$

$$L/\mu = \varepsilon^{-1}, \qquad A_{2p+1} = C_{2p+1}. \tag{B.4}$$

In the important special case

$$\Delta_2 = -D^2 + X \tag{B.5}$$

($D_\mu$ is the $g_{\mu\nu}$ covariant derivative on the base indices and the $A_\mu$ covariant derivative on the internal ones, $[D_\mu, D_\nu] = F_{\mu\nu}$) the following results are known [17, 20–22]:

$$b_p \equiv (4\pi)^{-n/2}\bar{b}_p, \qquad \bar{b}_0 = \operatorname{tr}\mathbb{1}, \qquad \bar{b}_2 = \operatorname{tr}\left(\mathbb{1}\cdot\tfrac{1}{6}R - X\right), \tag{B.6}$$

$$\bar{b}_4 = \operatorname{tr}\left(\tfrac{1}{12}F_{\mu\nu}F^{\mu\nu} + \mathbb{1}\cdot E + \tfrac{1}{2}X^2 - \tfrac{1}{6}RX - \tfrac{1}{6}D^2X\right), \tag{B.7}$$

$$E = \tfrac{1}{180}R^*R^* + \tfrac{1}{60}W + \tfrac{1}{72}R^2 + \tfrac{1}{30}D^2R. \tag{B.8}$$

For $b_6$ see [21]. For the boundary coefficients $c_p = (4\pi)^{-n/2}\bar{c}_p$ (for the Dirichlet problem case) one has [49, 29, 50]

$$\bar{c}_0 = 0, \qquad \bar{c}_1 = -\tfrac{1}{2}\sqrt{\pi}\operatorname{tr}\mathbb{1}, \qquad \bar{c}_2 = \tfrac{1}{6}2K\operatorname{tr}\mathbb{1}. \tag{B.9}$$

For the result for $c_3$ in the case of $X = \eta R$ see [50]. However, *it is the $c_4$ coefficient* that is essential for "*logarithmic*" renormalization of surface terms in the action. Using the method of "doubling" of $\mathrm{M}^n$ (cf. [49]) we obtained (cf., however, [51])

$$\bar{c}_4 = \operatorname{tr}\left\{\mathbb{1}\left(\tfrac{1}{180}\Omega_{\mathrm{S}} + \tfrac{1}{15}R^{\mu\nu}\bar{K}_{\mu\nu} + \tfrac{1}{20}RK + \cdots\right) - \tfrac{1}{3}XK + \cdots\right\}$$

$$\equiv \bar{c}_4^{(0)} - \tfrac{1}{3}\operatorname{tr}X\cdot K + \cdots, \tag{B.10}$$

where $\bar{K}_{\mu\nu}=K_{\mu\nu}-\frac{1}{4}g_{\mu\nu}K$ (see appendix A) and dots stand for analogs of the $D^2R$ and $D^2X$ terms in (B.7). The main idea is to use (B.7) on the doubled manifold extracting the boundary corrections on $M^n$ with the help of the equalities like $R_{\mu\nu}(g_1\oplus g_2)=R_{\mu\nu}(g_1)\oplus R_{\mu\nu}(g_2)\oplus 2(K_{1\mu\nu}-K_{2\mu\nu})\delta(\partial M)$ and the fact that the $R^*R^*$ term in (B.8) is supplemented up to $\chi$ in (A.5).

Now consider the elliptic covariant fourth-order differential operator $\Delta_4$ defined on the same vector bundle as $\Delta_2$. Then for $I_\infty=\frac{1}{2}(\log\det\bar{\Delta}_4)_\infty$ one has the same expression as (B.4), $A_p$ being some invariants of $\Delta_4$*. Suppose that for some $\Delta_4$

$$\Delta_4=\Delta_2\Delta_2',\qquad \det\Delta_4=\det\Delta_2\det\Delta_2'. \tag{B.11}$$

Comparing expressions like (B.4) we get

$$A_p(\Delta_4)=A_p(\Delta_2)+A_p(\Delta_2'),\qquad p=0,\dots,4, \tag{B.12}$$

and thus one can in principle gain some information about $A_p(\Delta_4)$ from $A_p(\Delta_2)$. This information fortunately turns out to be complete in the case of $\Delta_4$ in (2.23) we are interested in. Here on dimensional and invariance grounds we have

$$\bar{b}_4=\mathrm{tr}\Big(a_1F_{\mu\nu}^2+\mathbb{1}\cdot\tilde{E}+a_2V_{\mu\nu}^2+a_3V^2+a_4V^{\mu\nu}R_{\mu\nu}+a_5V\cdot R+a_6U$$
$$+a_7D_\mu D_\nu V^{\mu\nu}+a_8D^2V+a_9D_\mu N^\mu\Big),$$
$$V=V_\mu^\mu,\qquad \tilde{E}=\tilde{E}(g_{\mu\nu}). \tag{B.13}$$

If $\Delta_2$ and $\Delta_2'$ in (B.11) are of the form (B.5), $V_{\mu\nu}=-g_{\mu\nu}(X+X')$, and we get using (B.12) and (B.7), (B.8):

$$\tilde{E}=2E,\qquad a_1=\tfrac{1}{6},$$
$$a_4+4a_5=a_7+4a_8=\tfrac{1}{6},\qquad a_6=-a_9=-1,\qquad 4(a_2+4a_3)=\tfrac{1}{2}.$$

Another special case of (B.11) we need is $\Delta_4=D^2(A_+)D^2(A_-)$, $A_{\pm\mu}=A_\mu\pm Q_\mu$, where $Q_\mu\to SQ_\mu S^{-1}$ under the change of a basis in the fiber. Again using (B.12), (B.7), (B.8) we finally get

$$\bar{b}_4=\mathrm{tr}\Big(\tfrac{1}{6}F_{\mu\nu}^2+2\cdot\mathbb{1}\cdot E+\tfrac{1}{24}V_{\mu\nu}^2+\tfrac{1}{48}V^2$$
$$-\tfrac{1}{6}V^{\mu\nu}R_{\mu\nu}+\tfrac{1}{12}VR-U+\tfrac{1}{3}D_\mu D_\nu V^{\mu\nu}+\tfrac{1}{12}D^2V\Big). \tag{B.14}$$

Here we assumed that $\Delta_4$ is a self-adjoint variant of (2.23) ($=D^4+D_\mu V^{\mu\nu}D_\nu+D_\mu N^\mu$

* Note that the Seeley's coefficients [48] for $\Delta_4$ are defined by $(\mathrm{tr}\,e^{-s\bar{\Delta}_4})_{s\to0}\simeq\sum_{p=0}^{\infty}(\mu^{-4}s)^{(p-n)/4}\bar{A}_p$ and hence $A_p=\bar{A}_p$ only for $p=n$.

$+N^\mu D_\mu + U$). The only difference is in total derivative terms proportional to $D^2R$ in our case. We disregarded $D^2R$ counter-terms in this paper (though they may be important in the problem of conformal anomalies for gravity itself [33], they are inessential under natural boundary conditions and their consideration is consistent only taking account of the corresponding boundary terms like $n^\mu D_\mu R$ in $c_4$). Following the method discussed above we also found that for $\Delta_4$ [cf. (B.6), (B.9), (B.10)]

$$\bar{b}_0 = 2\,\mathrm{tr}\,\mathbb{1}, \qquad \bar{b}_2 = \mathrm{tr}\left(\mathbb{1}\cdot\tfrac{1}{3}R + \tfrac{1}{4}V\right), \qquad \bar{c}_2 = \tfrac{1}{3}2K\,\mathrm{tr}\,\mathbb{1}, \tag{B.15}$$

$$\bar{c}_4 = 2\bar{c}_4^{(0)} + \mathrm{tr}\left(-\tfrac{1}{3}V^{\mu\nu}\bar{K}_{\mu\nu} + \tfrac{1}{12}VK + \cdots\right). \tag{B.16}$$

## Appendix C

### ONE-LOOP $\beta$-FUNCTION IN HIGHER DERIVATIVE VECTOR GAUGE THEORY AND THE QUESTION OF LIMITS IN COUNTER-TERMS

Consider the following SO($N$) gauge theory in flat euclidean space-time ($\gamma =$ const):

$$\mathcal{L} = \frac{1}{4e^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{4M^2}\left[\left(D_\mu F^a_{\alpha\beta}\right)^2 + \gamma f^{abc}F^a_{\mu\nu}F^b_{\mu\lambda}F^c_{\nu\lambda}\right]. \tag{C.1}$$

The one-loop approximation for (C.1) has some similarity with that for the gravity lagrangian (2.2). The superrenormalizable theory (C.1) (cf. [52]) is in fact analogous not to (2.2) but to the superrenormalizable $R^2 + (DR)^2$ theory [the lagrangian (2.2) cannot be considered as a regularization of the Einstein theory – we cannot switch off the $R^2$ terms without breaking renormalizability]. If $A^a_\mu$ and $\phi^a_\mu$ are the background and the quantum fields, the relevant one-loop part of (C.1) is

$$\mathcal{L}_2 = \frac{1}{2M^2}\phi\tilde{\Delta}\phi - \frac{1}{2M^2}\chi H\chi,$$

$$\chi^a = D_\mu\phi^a_\mu, \qquad H = -D^2 + e^{-2}M^2, \qquad \tilde{\Delta} = \Delta_4 + M^2e^{-2}\Delta_2, \tag{C.2}$$

where $\Delta_2$ and $\Delta_4$ have the structure (B.5) and (2.23). Averaging over gauges $\chi^a = \zeta^a(x)$ we get

$$Z = \left(\det M^{-2}H\right)^{1/2}\left(\det\mu^{-4}\tilde{\Delta}\right)^{-1/2}\left(\det\mu^{-2}\Delta_G\right), \qquad \Delta_G = -D^2, \tag{C.3}$$

and finally, with the help of (B.7) and (B.14),

$$\begin{aligned} \bar{b}_4^{\mathrm{tot}} &= -\tfrac{1}{4}\beta\left(F^a_{\mu\nu}\right)^2 - \tfrac{1}{2}\left(M^2e^{-2}\right)^2N, \\ \beta &= \left(3\gamma^2 + 30\gamma - \tfrac{23}{3}\right)C_2, \qquad C_2 \equiv C_2(\mathrm{SO}(N)), \\ e^{-2}(L) &= e^{-2}(\mu) + \frac{\beta}{32\pi^2}\log\frac{L^2}{\mu^2}. \end{aligned} \tag{C.4}$$

The theory (C.1) is asymptotically free in $e^2$ when $\beta>0$ (in the Yang-Mills case $M=\infty$ and $\beta=\beta_0=\frac{22}{3}C_2$), i.e. when $\gamma<-10.2$ or $\gamma>0.2$.

Taking the limit $M\to\infty$ in the full effective action in (C.3) we regularly get the Yang-Mills case result (this is obvious from the representation:

$$Z=\left(\det\mu^{-2}\Delta_G\right)^{1/2}\left(\det\mu^{-4}\Delta\right)^{-1/2},\qquad \Delta=\tilde{\Delta}_\perp,\qquad \tilde{\Delta}_\parallel=H\cdot\Delta_G,$$

where in (C.2) $\mathfrak{L}_2\equiv(1/2M^2)\phi\Delta\phi$). Using the analogous representation for $Z$ in the case of (2.2) (e.g. in the gauge $D_\mu\bar{h}^{\mu\nu}=0$) we obtain regularity of the limits $a\to0$, $b\to0$ in the full gravitational effective action.

A *non-trivial* question is how to establish correspondence between the limiting (e.g. for Einstein theory) and initial (for renormalizable theory (2.2)) expressions for the *counter-terms*. The general answer is that limits in counter-terms can be made regular by taking into account terms of the same structure in the initially *finite* part of the effective action. Let us consider $\tilde{\Delta}_4=\Delta_2+M^{-2}\Delta_4$ [with notations as in (B.5) and (2.23)] and evaluate those diagrams which are finite or infinite only for $M\neq\infty$ [cf. (B.4)]:

$$\mathfrak{L}_\infty=-\tfrac{1}{2}\left\{\tfrac{1}{12}F_{\mu\nu}^2\log\left[\frac{L^2}{\mu^2}\left(1+\frac{L^2}{M^2}\right)\right]+\tfrac{1}{2}\left(X+\frac{1}{M^2}U\right)^2\log\left(\frac{L^2/\mu^2}{1+L^2/M^2}\right)\right.$$
$$\left.+\left[\tfrac{1}{24}V_{\mu\nu}^2+\tfrac{1}{48}V^2-M^2\left(X+\frac{1}{M^2}U\right)\right]\log\left(1+\frac{L^2}{M^2}\right)+\cdots\right\}.\qquad\text{(C.5)}$$

This expression provides interpolation between divergences of $\log\det\tilde{\Delta}_4$ in the case of $M\neq\infty$ [cf. (B.7)], (B.14) and $M=\infty$ (B.7) ($M\to\infty$ is taken *before* $L\to\infty$). Using (C.5) in the case of (C.2) we get instead of (C.4) the expression, valid for any $M$,

$$e^{-2}(L)=e^{-2}(\mu)+\frac{\beta_0}{32\pi^2}\log\frac{L^2}{\mu^2}+\frac{\beta-\beta_0}{32\pi^2}\log\left(1+e^2\frac{L^2}{M^2}\right).\qquad\text{(C.6)}$$

In the gravitational case $M$ is substituted by $m$ and $m'$ [cf. (2.6)] and, e.g., in the limit $a\to0$ we get from (3.18) the counter-terms in Einstein theory in the gauge (2.20) with the analog of (C.6) being

$$a(L)=a(\mu)+\frac{\beta_{02}}{32\pi^2}\log\frac{L^2}{\mu^2}+\frac{\beta_2-\beta_{02}}{32\pi^2}\log\left(1+a(L)k^2L^2\right).\qquad\text{(C.7)}$$

Note that here taking the limit implies $a(L)\to0$ on *both* sides of (C.7) (there is no bare $W$ term in the Einstein theory).

## References


[1] R. Utiyama and B.S. De Witt, J. Math. Phys. 3 (1962) 608
[2] S. Deser, Proc. Conf. on Gauge theory and modern field theory, ed. R. Arnowitt and P. Nath (MIT Press, Cambridge, 1975)
[3] S. Weinberg, Proc. 17th Int. Conf. on High-energy physics, ed. J.R. Smith (Chilton, Didcot, 1974)
[4] K.S. Stelle, Phys. Rev. D16 (1977) 953
[5] K.S. Stelle, Gen. Rel. Grav. 9 (1978) 353
[6] A. Salam and J. Strathdee, Phys. Rev. D18 (1978) 4480
[7] J. Julve and M. Tonin, Nuovo Cim. 46B (1978) 137
[8] E. Tomboulis, Phys. Lett. 70B (1977) 361
[9] E. Tomboulis, Phys. Lett. 97B (1980) 77
[10] B. Hasslacher and E. Mottola, Phys. Lett. 99B (1981) 221
[11] S.L. Adler, *in* The high energy limit, 1980 Erice Lectures, ed. A. Zichichi (Plenum, 1981); Revs. Mod. Phys. (1982), to be published
[12] E.S. Fradkin and G.A. Vilkovisky, Bern preprint (1976)
[13] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. 77B (1978) 262
[14] E.S. Fradkin and G.A. Vilkovisky, Proc. 18th Int. Conf. on High-energy physics (Tbilisi, 1976) vol. 2, sec. C, p. 28
[15] E.S. Fradkin, Trieste preprint-IC/79/67 (1979)
[16] H. Terazawa, Gen. Rel. Grav. 12 (1980) 93
[17] B.S. De Witt, Dynamical theory of groups and fields (Gordon and Breach, 1965); *in* Recent developments in gravitation, ed. M. Levy and S. Deser (Plenum, 1979); *in* Quantum gravity, 2nd Oxford Symp., ed. C. Isham et al. (Oxford U.P., 1981)
[18] R. Kallosh, Nucl. Phys. B78 (1974) 293;
M. Grisaru, P. van Nieuwenhuizen and C.C. Wu, Phys. Rev. D12 (1975) 3203
[19] N.K. Nielsen, Nucl. Phys. B140 (1978) 499;
R. Kallosh, Nucl. Phys. B141 (1978) 141
[20] G. 't Hooft and M. Veltman, Ann. Inst. H. Poincaré A20 (1974) 69
[21] P.B. Gilkey, J. Diff. Geom. 10 (1975) 601
[22] S.M. Christensen and M.J. Duff, Nucl. Phys. B154 (1979) 301;
S.M. Christensen, Ph.D. dissertation (1975)
[23] J.W. York, Phys. Rev. Lett. 28 (1972) 1082;
G.W. Gibbons and S.W. Hawking, Phys. Rev. D15 (1977) 2752
[24] S.W. Hawking, *in* General relativity, ed. S.W. Hawking and W. Israel (Cambridge U.P., 1979)
[25] S.S. Chern, Ann. Math. 46 (1945) 674
[26] D.M. Capper and D.P. Kimber, J. Phys. A13 (1980) 3671
[27] T.S. Bunch, J. Phys. A14 (1981) L139
[28] R. Gastmans, R. Kallosh and C. Truffin, Nucl. Phys. B133 (1978) 417
[29] S.M. Christensen and M.J. Duff, Phys. Lett. 79B (1978) 213
[30] G.W. Gibbons, S.W. Hawking and M.J. Perry, Nucl. Phys. B138 (1978) 141
[31] K.I. Macrae, Phys. Lett. 103B (1981) 103
[32] S.M. Christensen and M.J. Duff, Nucl. Phys. B170 [FS1] (1980) 480
[33] R. Critchley, Phys. Rev. D18 (1978) 1849
[34] L. Parker, *in* Recent developments in gravitation, ed. M. Levy and S. Deser (Plenum, 1979)
[35] G.W. Gibbons, Lect. Notes Math. 676 (1978) 513
[36] E.S. Fradkin and A.A. Tseytlin, Higher derivative quantum gravity: one-loop counterterms and asymptotic freedom, Lebedev Inst. preprint N70 (1981)
[37] R.E. Kallosh, O.V. Tarasov and I.V. Tyutin, Nucl. Phys. B137 (1978) 145
[38] P.G.O. Freund, Ann. of Phys. 84 (1974) 440;
A. Zee, Phys. Rev. Lett. 42 (1979) 417; 44 (1980) 703;
L. Smolin, Nucl. Phys. B160 (1979) 253
[39] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. 73B (1978) 209
[40] H. Terazawa and K. Akama, Phys. Lett. 97B (1980) 81

[41] A.D. Sakharov, Sov. Phys. Dokl. 12 (1968) 1040;
O. Klein, Phys. Scr. 9 (1974) 69;
S.L. Adler, Phys. Rev. Lett. 44 (1980) 1567;
K. Akama, Y. Chikashige, T. Matsuki and H. Terazawa, Prog. Theor. Phys. 60 (1980) 1900
[42] E.S. Fradkin and A.A. Tseytlin, One-loop $\beta$-function in conformal supergravities, Lebedev Inst. preprint N213 (1981); Phys. Lett. 110B (1982) 117
[43] S. Weinberg, *in* General relativity, ed. S.W. Hawking and W. Israel (Cambridge U.P., 1979)
[44] P.K. Townsend, Phys. Rev. D15 (1977) 2795
[45] S. Deser, *in* Quantum gravity, ed. C.J. Isham et al. (Oxford U.P., 1975)
[46] D. Freedman, I. Muzinich and E. Weinberg, Ann. of Phys. 87 (1974) 95;
L.S. Brown and J.C. Collins, Ann. of Phys. 130 (1980) 215
[47] L. Smolin, Phys. Lett. 93B (1980) 95
[48] R. Seeley, Proc. Symp. Pure math. 10 (1971) 288
[49] H.P. McKean and I.M. Singer, J. Diff. Geom 1 (1967) 43
[50] G. Kennedy, R. Critchley and J.S. Dowker, Ann. of Phys. 125 (1980) 346
[51] G. Kennedy, J. Phys. A11 (1978) L173
[52] A.A. Slavnov, Nucl. Phys. B31 (1971) 301;
L.D. Faddeev and A.A. Slavnov, Introduction to the theory of gauge fields (Benjamin, Reading, Mass., 1980)
[53] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888