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Euclidean linear conformal gravity

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Abstract. A Euclidean quantum conformal gravity in a linear approximation is considered. It is assumed that its solution possesses an exact conformal symmetry. The main difficulty, which is a characteristic not only of gravitation theory but of all gauge theories as well, consists in the fact that in all previous formulations the conformal invariance can be stated unambiguously only in the purely gauge sector. In the present new formulation this difficulty is absent. This is obtained by means of new transformation laws for the metric tensor field $h_{\mu\nu}$ and energy–momentum tensor $T_{\mu\nu}$. Previous transformation laws corresponded to the indecomposable representations of the conformal group and this fact was the origin of the difficulty mentioned. Explicit expressions are obtained for the propagator of the field $h_{\mu\nu}$ as well as for the three-point functions including $h_{\mu\nu}$ or $T_{\mu\nu}$ and the matter fields. It is shown that the equations of linear conformal gravity are the consequences of the conformal invariance. They are the manifestation of a mathematical fact of equivalence of the conformal group representations attributed in our approach to the fields $h_{\mu\nu}$, $T_{\mu\nu}$ and the Weyl tensor.

1. Introduction

The modern version of quantum gravity (see e.g. Adler 1982 and references therein) is based on introducing terms quadratic in the Riemann curvature into the Lagrangian (Utiyama and DeWitt 1962). The theory arising in this case appears to be not only renormalised (Stelle 1977, Fradkin and Tseytlin 1981, 1982a), but also asymptotically free (Fradkin and Tseytlin 1981, 1982a). One of the promising versions of this theory is conformal gravity (and particularly its super-symmetric extension, conformal supergravity (Van Nieuwenhuizen 1981 and references therein; Fradkin and Tseytlin 1982b, c)), where all terms in the Lagrangian which are quadratic in the Riemann curvature enter as the combinations

$$\Omega = C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau}$$

where $C_{\mu\nu\sigma\tau}$ is the Weyl tensor,

$$C_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau} - \tfrac{1}{2}(g_{\mu\sigma}R_{\nu\tau} + g_{\nu\tau}R_{\mu\sigma} - g_{\mu\tau}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\tau}) + \tfrac{1}{6}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})R,$$

and $g_{\mu\nu}(x)$ is the metric tensor. A full Lagrangian including matter field interactions has the form

$$L = \Omega + L^{(m)}, \tag{1.1}$$

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where $L^{(m)}$ is a matter Lagrangian. The Lagrangian (1.1) is supposed to describe the gravitational interaction at short distances, while the Einstein gravity acts at large distances. It can appear, for instance, as an induced theory after dynamical symmetry breaking. Being interested only in short distances, we can consider the conformal invariance as an unbroken symmetry of the interaction (1.1).

Surely, the question of existence of this symmetry depends essentially on which types of the matter field interactions enter (1.1). For a large variety of models the conformal invariance is broken in perturbation theory by the trace anomalies. This takes place even for those matter interactions which are conformally invariant in the limit of flat spacetime. There are, however, two possibilities to construct the conformal invariant gravity: (a) To select types of the matter fields and their interactions which do not possess anomalies. Such a variant of the theory is possible, in particular, in the framework of conformal supergravity (Fradkin and Tseytlin 1983). (b) For some interactions in flat spacetime the conformal invariance is restored after summation of the whole perturbation expansion. It is possible at the definite values of a coupling constant which coincide with zeros of the Gell–Mann–Low function. An analogous property can in principle appear in conformal gravity at the appropriate choice of the matter fields. In this case the demand of conformal invariance fixes not only the coupling constants of the matter fields but the (dimensionless) gravitation coupling constant. To formulate such a theory it is suitable to use the skeleton expansion. Each skeleton graph consists of full vertices and propagators for which the exact conformally invariant expressions must be substituted.

In this paper we start a successive analysis of conformal gravity in the group theoretical framework. As the first step we shall take linear gravity. The next step is to take into account terms nonlinear in the metric. The main problem, as we shall show, is in the choice of correct transformation laws for the metric and energy–momentum tensors. To take the terms nonlinear in the metric into account one demands more general transformation laws suitable for nonabelian gauge theories. We hope to do so in forthcoming papers. Here we limit ourselves to an analysis of the aforementioned transformation laws in linear gravitation, which appears as an abelian gauge theory. We demand its invariance (besides the Poincaré group) under the conformal transformations of coordinates

$$x_\mu \to \lambda x_\mu, \qquad x_\mu \to R x_\mu = x_\mu / x^2. \tag{1.2}$$

For this theory the following results are obtained in the paper.

(1) It is shown that the usual transformation law for the conformal tensor under the R-inversion (1.2) is wrong in the case of the metric and the energy–momentum tensors, and a new transformation law is found.

(2) Explicit expressions for the conformally invariant propagator of the metric tensor and for three-point Green functions including the metric tensor or the energy–momentum tensor are found.

(3) Conformal invariance of skeleton graphs which include such Green functions is proved.

(4) It is illustrated that the classical equations of conformal gravitation that relate the metric tensor, the Weyl tensor and the energy–momentum tensor result from conformal invariance and express the conditions of equivalence for corresponding representations of the conformal group.

Let us emphasise that the area of application of the results concerning the energy–momentum tensor is not restricted to conformal gravitation. The new transformation law for the energy–momentum tensor under the R-inversion (1.2) is true for any conformally invariant field theory. This result is of particular importance in the investigation of the conformal Green functions of the matter fields (without taking gravitation into account) beyond the scope of the perturbation theory approach started previously by Fradkin and Palchik (1978). In particular, as has been shown by Fradkin and Palchik (1982, 1983), the new transformation law allows one to deduce the field dimension spectrum in the operator expansion of products of the fundamental fields.

A good deal of the paper is devoted to questions associated with a more general problem of the formulation of the conformal invariance in gauge theories. The thing is that for a long time it was possible to formulate the conformal invariance in a purely gauge sector only (see e.g. Todorov *et al* (1978) and references therein). In recent papers of the authors (Palchik 1981, 1983, Kozhevnikov *et al* 1983, Fradkin *et al* 1982) the solution of this difficulty was found for the case of quantum electrodynamics. Since the present paper is a generalisation of these results, we shall illustrate the nature of the difficulty and the general idea of our method using Euclidean QED as a more simple example.

Let us assume that the Euclidean potential A_μ and current j_μ are transformed under the R-inversion (1.2) as the usual conformal vectors

$$A_\mu(x) \xrightarrow{R} U_R^A A_\mu(x) = (x^2)^{-1} \eta_{\mu\nu}(x) A_\nu(Rx), \tag{1.3}$$

$$j_\mu(x) \xrightarrow{R} U_R^j j_\mu(x) = (x^2)^{-3} \eta_{\mu\nu}(x) j_\nu(Rx), \tag{1.4}$$

where

$$\eta_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu / x^2. \tag{1.5}$$

Let us consider the photon propagator $D_{\mu\nu}(x_{12}) = \langle 0|TA_\mu(x_1)A_\nu(x_2)|0\rangle$. The requirement of the invariance of $D_{\mu\nu}$ with respect to (1.3) leads, as is known, to the purely longitudinal propagator

$$D^l_{\mu\nu}(x_{12}) \sim \partial_\mu \partial_\nu \ln x_{12}^2. \tag{1.6}$$

This result is a consequence of the well known fact that the transformation (1.3) leaves invariant the subspace of longitudinal functions. Indeed, if we subject the longitudinal function $A^l_\mu(x) = \partial_\mu \varphi(x)$ to the transformation (1.3) we get again the longitudinal function $U_R^A A^l_\mu(x) = (x^2)^{-1} \eta_{\mu\nu}(x) \partial_\nu^{Rx} \varphi(Rx) = \partial_\mu^x \varphi(Rx)$. Thus, (1.3) can be considered as the transformation law for longitudinal functions only. It is essential that any transverse function acquires the longitudinal part after the transformation (1.3). The subspace of transverse functions, consequently, is not invariant under (1.3) and it is necessary to construct another representation of the conformal group in this subspace.

As a result, we arrive at the following construction. An arbitrary function $A_\mu(x)$ is represented as a sum of the transverse and longitudinal parts $A_\mu(x) = A^{\text{tr}}_\mu(x) + A^l_\mu(x)$ and each term in this sum transforms under a distinct representation. Analogous considerations are valid in the case of transformation (1.4) as well; for the latter the invariaint subspace consists of the transverse functions. A new realisation of the transformation laws for A_μ and j_μ has been constructed by Palchik (1981, 1983) and investigated in detail by Kozhevnikov *et al* (1983), and by Fradkin *et al* (1982). The results looks as follows:

$$A_\mu(x) \xrightarrow{R} \tilde{U}_R^A A_\mu(x) = [(I-P)U_R^A(I-P) + U_R^A P] A_\mu(x), \tag{1.7}$$

$$j_\mu(x) \xrightarrow{R} \tilde{U}_R^j j_\mu(x) = [U_R^j(I-P) + PU_R^j P] j_\mu(x), \tag{1.8}$$

where $P = \partial_\mu \partial_\nu / \Box$ is the longitudinal projection operator $Pf_\mu(x) = (\partial_\mu \partial_\nu/\Box) f_\nu(x)$.

From the mathematical point of view the transformation laws (1.3), (1.4) correspond to the indecomposable representations belonging to the exceptional integer points (Dobrev *et al* 1977, Klimyk 1979); see also §§ 3, 4. The longitudinal character of the propagator (1.6) is a mathematical consequence of the indecomposability of the representation (1.3).

The new transformation laws (1.7), (1.8) correspond to direct sums of the irreducible unitary representations, each of which is represented by separate terms in (1.7), (1.8). As a result, for the propagator of the field A_μ obtained from (1.7), we have a non-trivial expression which includes the transverse part (Palchik 1981, 1983, Kozhevnikov *et al* 1983, Fradkin *et al* 1982):

$$D_{\mu\nu}(x) = (4\pi^2)^{-1} [(\delta_{\mu\nu} - \partial_\mu \partial_\nu/\Box)(x^2)^{-1} - \beta(\partial_\mu\partial_\nu/\Box)(x^2)^{-1}] \tag{1.9}$$

where β is the gauge parameter.

Let us turn back to conformal gravity. As is well known, the term Ω in (1.1) is invariant under the metric transformation $g_{\mu\nu}(x) \to \omega(x) g_{\mu\nu}(x)$. This fact permits one to put the constraint $\det|g_{\mu\nu}(x)| = 1$. Correspondingly, in a linear theory we have

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + h_{\mu\nu}(x), \qquad \text{where } h_{\mu\mu}(x) = 0.$$

Here $\delta_{\mu\nu}$ is the flat space metric, $h_{\mu\nu}$ are small. Thus we shall treat $h_{\mu\nu}$ as the traceless symmetric tensor of zero dimension.

For the Weyl tensor we find:

$$C_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau} - \tfrac{1}{2}(\delta_{\mu\sigma} R_{\nu\tau} + \delta_{\nu\tau} R_{\mu\sigma} - \delta_{\mu\tau} R_{\nu\sigma} - \delta_{\nu\sigma} R_{\mu\tau}) + \tfrac{1}{6}(\delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma}) R, \tag{1.10}$$

where $R_{\mu\nu\sigma\tau} = \frac{1}{2}(\partial_\mu \partial_\tau h_{\nu\sigma} + \partial_\nu \partial_\sigma h_{\mu\tau} - \partial_\mu \partial_\sigma h_{\nu\tau} - \partial_\nu \partial_\tau h_{\mu\sigma})$. In the approximation linear in $h_{\mu\nu}$ we obtain from (1.1): $L_{\text{lin}} = \Omega_{\text{lin}} - \frac{1}{2} h_{\mu\nu} T_{\mu\nu}$, where $T_{\mu\nu}$ is the matter energy–momentum tensor. We shall consider it to be a traceless symmetrical tensor with the scale dimension four. Varying L_{lin} over $h_{\mu\nu}$ we get

$$\Box^2 h_{\mu\nu} - \Box \partial_\mu \partial_\sigma h_{\nu\sigma} - \Box \partial_\nu \partial_\sigma h_{\mu\sigma} + \tfrac{2}{3} \partial_\mu \partial_\nu \partial_\sigma \partial_\lambda h_{\sigma\lambda} + \tfrac{1}{3} \delta_{\mu\nu} \Box \partial_\lambda \partial_\sigma h_{\lambda\sigma} = \tfrac{1}{2} T_{\mu\nu}. \tag{1.11}$$

This equation remains invariant under gauge transformations

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \partial_\mu h_\nu(x) + \partial_\nu h_\mu(x) - \tfrac{1}{2}\delta_{\mu\nu} \partial_\lambda h_\lambda(x). \tag{1.12}$$

We shall study the Euclidean formulation of conformal gravitation. Our approach can be formulated for Minkowski space as well, but this requires some complications associated with more complex structure of indecomposable representations of the SO(4, 2) group. In particular, the analysis of these representations for the fields A_μ and j_μ carried out by Palchik (1980) has shown that there are three invariant subspaces in the space of each of the representations, while in the Euclidean case there is only one invariant subspace (Dobrev *et al* 1977, Klimyk 1979). As a result, in the pseudo-Euclidean case a more complicated construction would be needed for substantiation of the transformation laws (1.7), (1.8). Similar complications appear also in pseudo-Euclidean conformal gravitation (see also Bineger *et al* (1982), where another approach is considered).

Thus, we shall consider a pair of Euclidean conformal fields $h_{\mu\nu}$ and $T_{\mu\nu}$. Analogously to the fields A_μ, j_μ the sum of their dimensions is equal to the dimension of the space. If we consider them as the usual conformal tensors (see § 2), we will meet the same difficulty as in electrodynamics: the propagator of the field $h_{\mu\nu}$ has the form of a pure gauge and can be represented as the expectation value of the 'longitudinal' fields

$$h^l_{\mu\nu}(x)=\partial_\mu h_\nu(x)+\partial_\nu h_\mu(x)-\tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda h_\lambda. \tag{1.13}$$

In § 3 we shall find the correct transformation laws of the fields $h_{\mu\nu}$ and $T_{\mu\nu}$ which make it possible to avoid the above difficulty and allow a non-trivial propagator of the field $h_{\mu\nu}$. These transformation laws appear as a consequence of indecomposability of representations corresponding to $h_{\mu\nu}$ and $T_{\mu\nu}$, and have just the same form as (1.7), (1.8), while the projection operator on the subspace of the functions of the form (1.13) enters as the operator P (see also Fradkin and Palchik 1983).

Modified expressions for the invariant three-point Green functions including the fields $h_{\mu\nu}$ or $T_{\mu\nu}$ and the matter fields are obtained in § 4. As is known, these Green functions are determined up to constants by conformal invariance. New expressions for them result from modification of the transformation laws for the fields $h_{\mu\nu}$ and $T_{\mu\nu}$.

In § 5 we shall show that the field equation (1.11) as well as the relation (1.10) between the Weyl tensor and the metric tensor are consequences of the transformation laws of the fields $h_{\mu\nu}$ and $T_{\mu\nu}$. This is a manifestation of the mathematical fact of an equivalence of the conformal group representations associated with the Weyl, metric and energy–momentum tensors (see also Fradkin and Palchik 1983).

2. Preliminary remarks and statement of the problem

Representations of the conformal group of Euclidean space (isomorphic to the SO(5, 1) group) are labelled by the values of scale dimension d and of spin s. Everywhere, except in § 5, we shall consider only the representations for which $s=2$. They are realised in the space of symmetrical traceless tensors of the second rank. We are interested in the elementary representations of the exceptional integer points, where the dimension takes on definite integer values. The values $d=0$ and $d=4$, corresponding to the fields $h_{\mu\nu}$ and $T_{\mu\nu}$, belong to this class. However, we begin with the description of some general properties of the conformal group representations.

Let $f^d_{\mu\nu}(x)$ be a symmetrical traceless tensor with an arbitrary value of d. The transformation law under the inversion $Rx_\mu=x_\mu/x^2$ looks as follows:

$$f^d_{\mu\nu}(x)\to(x^2)^{-d}\eta_{\mu\tau}(x)\eta_{\nu\sigma}(x)f^d_{\tau\sigma}(Rx). \tag{2.1}$$

The full group of conformal transformations includes besides (1.2) also rotations and translations. We restrict ourselves to a consideration of transformation (2.1) since modifications of the transformation laws in the case of integer d concern only this transformation.

For each non-exceptional value of d there is a corresponding irreducible representation. Let us denote it by Q^d. The representations Q^d and Q^{4-d} are equivalent (see e.g. Fradkin and Palchik 1978, Todorov *et al* 1978 and references therein)

$$Q^d \sim Q^{4-d}. \tag{2.2}$$

Let G^d be an intertwining operator that transforms Q^{4-d} into Q^d. The intertwining operator kernel defines the invariant scalar product in the representation space. The kernel has the form

$$G^d_{\mu\nu\sigma\tau}(x_1x_2) \sim (x_{12}^2)^{-d}[\eta_{\mu\sigma}(x_{12})\eta_{\nu\tau}(x_{12}) + \eta_{\mu\tau}(x_{12})\eta_{\nu\sigma}(x_{12}) - \tfrac{1}{2}\delta_{\mu\nu}\delta_{\sigma\tau}] \tag{2.3}$$

where $\eta_{\mu\nu}(x)$ is the tensor (1.5). When deducing (2.3) the form of transformation (2.1) is essential (see e.g. Fradkin and Palchik 1978, Todorov *et al* 1978). The invariant scalar product on the representation space looks as follows:

$$\begin{aligned}(f,\varphi) &= \int \mathrm{d}x_1\,\mathrm{d}x_2\, f^{4-d}_{\mu\nu}(x_1)G^d_{\mu\nu\sigma\tau}(x_1x_2)\varphi^{4-d}_{\sigma\tau}(x_2)\\ &= \int \mathrm{d}x_1\,\mathrm{d}x_2\, f^d_{\mu\nu}(x_1)G^{4-d}_{\mu\nu\sigma\tau}(x_1x_2)\varphi^d_{\sigma\tau}(x_2),\end{aligned} \tag{2.4}$$

where

$$f^d_{\mu\nu}(x_1) = \int \mathrm{d}x_2\, G^d_{\mu\nu\sigma\tau}(x_1x_2) f^{4-d}_{\sigma\tau}(x_2). \tag{2.4a}$$

Here we have the relation

$$G^{4-d}(x_1x_2) = (G^d)^{-1}(x_1x_2). \tag{2.5}$$

Let $\phi^d_{\mu\nu}(x)$ be a quantum field with the dimension d. The conformally invariant propagator for the field $\phi_{\mu\nu}(x)$ coincides with the invariant kernel (2.3)

$$\langle 0|\phi_{\mu\nu}(x_1)\phi_{\sigma\tau}(x_2)|0\rangle \sim G^d_{\mu\nu\sigma\tau}(x_1x_2). \tag{2.6}$$

From the above it is seen that from the mathematical point of view the question of the existence of the conformally invariant propagator is the question of the possibility to introduce the invariant scalar product in the representation space.

This is impossible to do in the case of the exceptional integer point representations. The reason is that these representations are indecomposable, and statements expressed by equations (2.2) and (2.4)–(2.6) are not true for them. In the space of each indecomposable representation there is an invariant subspace; and any complement to it is not invariant. The bilinear form with invariant kernel (2.3) is degenerate on the invariant subspace. As a result, the invariant propagator determined according to (2.6) is also degenerate. This is the cause of the difficulty in conformal gauge theories mentioned in the introduction. In the case of the electromagnetic potential A_μ and the metric tensor $h_{\mu\nu}$ the propagator is degenerate on invariant subspaces of the transverse functions (see § 3 for details) and for this reason it reduces to a pure gauge. In particular, for the potential we have the longitudinal expression (1.6), and for the propagator of the metric tensor we find, assuming that $d=0$ in (2.3),

$$\begin{aligned}D^l_{\mu\nu\sigma\tau}(x_{12}) &= (2\pi)^{-2}[\eta_{\mu\sigma}(x_{12})\eta_{\nu\tau}(x_{12}) + \eta_{\mu\tau}(x_{12})\eta_{\nu\sigma}(x_{12}) - \tfrac{1}{2}\delta_{\mu\nu}\delta_{\sigma\tau}]\\ &= \partial_\mu D_{\nu,\sigma\tau}(x_{12}) + \partial_\nu D_{\mu,\sigma\tau}(x_{12}) - \tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda D_{\lambda,\sigma\tau}(x_{12}),\end{aligned} \tag{2.7}$$

where

$$D_{\mu,\sigma\tau}(x)=\tfrac{1}{2}[x_\sigma\eta_{\mu\tau}(x)+x_\tau\eta_{\mu\sigma}(x)]. \tag{2.8}$$

This propagator has the form (1.13) and is a pure gauge.

The way out of this difficulty follows directly from the above analysis: each indecomposable representation should be replaced with the direct sum of a pair of irreducible unitary representations induced by the indecomposable representation. Next we should assume that the fields with integer values of the scale dimension such as A_μ, j_μ or $h_{\mu\nu}$, $T_{\mu\nu}$ are transformed according to direct sums of the unitary representations. This leads to modification of the transformation law (see (1.7), (1.8)). The invariant propagator that corresponds to the new law is non-degenerate (see (1.9)), and defines the invariant scalar product on the space of the direct sum of the representations. An explicit realisation of this program for QED is given in Palchik (1981, 1983), Fradkin *et al* (1982) and Kozhevnikov *et al* (1983). In § 3 a similar construction for conformal gravitation is discussed.

3. Transformation laws for the metric tensor and the energy–momentum tensor under conformal inversion

An explicit realisation for the pair of irreducible unitary representations induced by each indecomposable representation should be constructed. One of these two unitary representations acts in the invariant subspace of the indecomposable representation space, and the other in the quotient space (see e.g. Barut and Raczka 1977).

Let us consider the representations that correspond to the fields $h_{\mu\nu}$ and $T_{\mu\nu}$. Setting $d=0$ and $d=4$ in (3.1) we have

$$h_{\mu\nu}(x)\xrightarrow{R}U_R^h h_{\mu\nu}(x)=\eta_{\mu\sigma}(x)\eta_{\nu\tau}(x)h_{\sigma\tau}(Rx), \tag{3.1}$$

$$T_{\mu\nu}(x)\xrightarrow{R}U_R^T T_{\mu\nu}(x)=(x^2)^{-4}\eta_{\mu\sigma}(x)\eta_{\nu\tau}(x)T_{\sigma\tau}(Rx). \tag{3.2}$$

Let us consider representation (3.1). Let M_h be its representation space. The invariant subspace $M_h^l\subset M_h$ in this case consists of the tensors of the form

$$h^l_{\mu\nu}(x)=\partial_\mu h_\nu(x)+\partial_\nu h_\mu(x)-\tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda h_\lambda(x). \tag{3.3}$$

Really, it can be easily verified that every tensor of the form (3.3) preserves its form under transformation (2.2). We have

$$U_R^h h^l_{\mu\nu}(x)=\partial_\mu\tilde{h}_\nu(x)+\partial_\nu\tilde{h}_\mu(x)-\tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda\tilde{h}_\lambda(x)$$

where $\tilde{h}_\mu(x)=x^2\eta_{\mu\lambda}(x)h_\lambda(Rx)$ and U_R^h is defined by (2.2). The unitary representation that acts in M_h^l is irreducible (Dobrev *et al* 1977, Klimyk 1979). Let us denote it by Q_h^l. Another irreducible unitary representation (denoted by $Q_h^{\rm tr}$) acts in the quotient space $M_h^{\rm tr}\sim M_h/M_h^l$. As a result, instead of the initial indecomposable representation that corresponds to transformation (3.1) we may consider the reducible unitary representation

$$\tilde{Q}_h=Q_h^{\rm tr}\oplus Q_h^l \tag{3.4}$$

which acts on the direct sum of the spaces

$$\tilde{M}_h=M_h^{\rm tr}\oplus M_h^l. \tag{3.5}$$

A similar situation takes place for representation (3.2). Let us denote by M_T the space of this representation. The invariant subspace $M_T^{tr} \subset M_T$ consists now of transverse tensors. Indeed, taking into account the fact that for any symmetrical traceless tensor $T_{\rho\lambda}(x)\eta_{\mu\rho}(x)\partial_\mu[(x^2)^{-1}h_{\nu\lambda}(x)]=0$ and the relation $\partial_\mu[(x^2)^{-3}\eta_{\mu\rho}(x)]=0$ we find

$$\partial_\mu T_{\mu\nu}(x)=0 \rightarrow \partial_\mu T'_{\mu\nu}(x)=0$$

where $T'_{\mu\nu}(x)=U_R^T T_{\mu\nu}(x)$. As before, let us consider two irreducible unitary representations Q_T^{tr} and Q_T^l. Here Q_T^{tr} acts in the invariant subspace M_T^{tr}, and Q_T^l on the quotient space $M_T^l \sim M_T/M_T^{tr}$. The direct sum of these representations

$$\tilde{Q}_T = Q_T^{tr} \oplus Q_T^l \tag{3.6}$$

acts on the space

$$\tilde{M}_T = M_T^{tr} \oplus M_T^l. \tag{3.7}$$

Let us find an explicit realisation of representations (3.4) and (3.6). As mentioned above, the transformation laws for the functions transforming according to these representations differ from (3.1) and (3.2). These laws cannot be determined unambiguously. They depend on a special choice of representatives in the equivalence classes belonging to the quotient spaces M_T^l and M_h^{tr}. The most suitable choice is the following. In each equivalence class from M_T^l we choose the function represented similar to (3.3)

$$T_{\mu\nu}^l(x)=\partial_\mu T_\nu(x)+\partial_\nu T_\mu(x)-\tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda T_\lambda(x) \tag{3.8}$$

where $T_\mu(x)$ is some vector. Then the space M_T^l is realised as the space of functions of the form (3.8), and the space (3.7) as the space of pairs of tensors $\{T_{\mu\nu}^{tr}(x), T_{\mu\nu}^l(x)\}$. The next step is to consider instead of pairs of tensors their sums

$$T_{\mu\nu}(x)=T_{\mu\nu}^{tr}(x)+T_{\mu\nu}^l(x) \tag{3.9}$$

and to construct a realisation of the representation (3.6) in this space. For this it is sufficient to choose a law of transformation of the tensors $T_{\mu\nu}(x)$ such that separate terms in (3.9) do not mix under all conformal transformations. This requirement determines unambiguously the transformation law, see (3.11). The final result which will be obtained consists in finding the explicit realisation of the direct sum of representations (3.6) in the space of all tensor functions $T_{\mu\nu}$.

One should proceed in a similar manner in the case of the space (3.5). Let us choose a transverse representative $h_{\mu\nu}^{tr}$ in each equivalence class from M_h^{tr}. The space M_h^{tr} will be realised then as the space of transverse tensors $h_{\mu\nu}^{tr}$, and the direct sum of the spaces (3.5) as a set of pairs $\{h_{\mu\nu}^{tr}(x), h_{\mu\nu}^l(x)\}$. Then we can pass to the sums

$$h_{\mu\nu}(x)=h_{\mu\nu}^{tr}(x)+h_{\mu\nu}^l(x) \tag{3.10}$$

and choose the law of transformation of tensors $h_{\mu\nu}(x)$ so that the two terms in the RHS of (3.10) do not mix under conformal transformations.

The desired transformation laws can easily be determined. Let g be an arbitrary conformal transformation and U_g^h be operators of the indecomposable representation Q_h. The operators of the unitary representation (3.4) acting on the space of tensors (3.10) can be represented as

$$\tilde{U}_g^h=(I-P^l)U_g^h(I-P^l)+U_g^hP^l \tag{3.11}$$

where P^l is the projection operator on the invariant subspace M_h^l consisting of tensors of the form (3.3) and I is unity,

$$P^l\tilde{M}_h = P^l M_h^l = M_h^l, \tag{3.12}$$

$(I-P^l) = P^{\text{tr}}$ is the projection operator on the subspace M_h^{tr} consisting of the transverse tensors

$$(I-P^l)\tilde{M}_h = (I-P^l)M_h^{\text{tr}} = M_h^{\text{tr}}. \tag{3.13}$$

The explicit expression and properties of the projector P^l are given below.

Expression (3.11) has the following structure. The first term corresponds to the irreducible representation Q_h^{tr}, realised in the space of transverse tensors: at first the transverse representative of the equivalence class is selected by the action of the projection operator $(I-P^l)$; then this representative is subjected to the action of the operator U_g^h, which transforms it into an arbitrary representative of another equivalence class; then the transverse representative is again selected. As a result, the operator $(I-P^l)U_g(I-P^l)$ gives the action of the representation on the subspace of the transverse tensors. The second term in (3.11), U_gP^l, corresponds to the irreducible representation Q_h^l acting in the invariant subspace of tensors of the type (1.13) that is projected by the operator P^l. It is important that each term in (3.10) is transformed independently under the action of the operator $\tilde{U}_g^h$.

It can easily be seen that the operators $\tilde{U}_g^h$, defined according to (3.11), satisfy the group law, if the operators U_g^h of the indecomposable representation have this property. We find

$$\tilde{U}_{g_1}^h\tilde{U}_{g_2}^h = (I-P^l)U_{g_1}^h(I-P^l)^2U_{g_2}^h(I-P^l) + U_{g_1}^hP^lU_{g_2}^hP^l$$
$$+(I-P^l)U_{g_1}^h(I-P^l)U_{g_2}^hP^l + U_{g_1}^hP^l(I-P^l)U_{g_2}^h(I-P^l).$$

Taking into account the invariance of the subspace M_h^l under U_g^h we have

$$P^lU_g^hP^l = U_gP^l, \qquad (I-P^l)U_g^hP^l = 0.$$

As a result we find

$$\tilde{U}_{g_1}^h\tilde{U}_{g_2}^h = (I-P^l)U_{g_1}^hU_{g_2}^h(I-P^l) + U_{g_1}^hU_{g_2}^hP^l = \tilde{U}_{g_1g_2}^h.$$

In this equality we have allowed for the relation $U_{g_1}^hU_{g_2}^h = U_{g_1g_2}^h$.

Let us consider representations associated with the energy–momentum tensor. Let U_g^T be operators of the indecomposable representation Q_T. Then the operators $\tilde{U}_g^T$ of the unitary representation (3.6) acting in the space of tensors (3.9) have the form

$$\tilde{U}_g^T = U_g^T(I-P^l) + P^lU_g^TP^l. \tag{3.14}$$

In order to show this expression to be valid, one may repeat with obvious changes all the considerations concerning expression (3.11).

Now let us give an explicit expression for the projection operator P^l

$$P^l = P^l_{\mu\nu\sigma\tau} = \partial_\mu P_{\nu,\sigma\tau} + \partial_\nu P_{\mu,\sigma\tau} - \tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda P_{\lambda,\sigma\tau} \tag{3.15}$$

where

$$P_{\mu,\sigma\tau} = \tfrac{1}{6}[-2\partial_\mu\partial_\sigma\partial_\tau/\Box^2 + 3(\delta_{\mu\tau}\partial_\sigma/\Box + \delta_{\mu\sigma}\partial_\tau/\Box) - \delta_{\sigma\tau}\partial_\mu/\Box].$$

Being a projection operator, it has the following properties:

$$P^l_{\mu\nu\rho\lambda}P^l_{\rho\lambda\sigma\tau} = P^l_{\mu\nu\sigma\tau}, \qquad P^l_{\mu\nu\sigma\tau} = P^l_{\sigma\tau\mu\nu} = P^l_{\nu\mu\sigma\tau}, \qquad P^l_{\mu\mu\sigma\tau} = 0.$$

For any symmetrical traceless tensor $f_{\mu\nu}$ we have

$$P^l f_{\mu\nu} \equiv P^l_{\mu\nu\sigma\tau} f_{\sigma\tau} = \partial_\mu f_\nu + \partial_\nu f_\mu - \tfrac{1}{2}\delta_{\mu\nu}\partial_\lambda f_\lambda \tag{3.16}$$

where $f_\mu = \ddot{P}_{\mu,\sigma\tau} f_{\sigma\tau}$

Then we introduce the traceless operator

$$I = I_{\mu\nu\sigma\tau} = \tfrac{1}{2}(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma} - \tfrac{1}{2}\delta_{\mu\nu}\delta_{\sigma\tau}). \tag{3.17}$$

The following identities hold:

$$If_{\mu\nu} \equiv I_{\mu\nu\sigma\tau} f_{\sigma\tau} = f_{\mu\nu}, \qquad (I - P^l) f_{\mu\nu} = P^{\mathrm{tr}} f_{\mu\nu} \equiv P^{\mathrm{tr}}_{\mu\nu\sigma\tau} f_{\sigma\tau} = f^{\mathrm{tr}}_{\mu\nu} \tag{3.18}$$

where $\partial_\mu f^{\mathrm{tr}}_{\mu\nu} = 0$.

The last equation is the consequence of the relation

$$\partial_\mu P^{\mathrm{tr}}_{\mu\nu\sigma\tau} = \partial_\mu (I_{\mu\nu\sigma\tau} = P^l_{\mu\nu\sigma\tau}) = 0$$

that could easily be verified by direct calculation using (3.15), (3.17).

It can readily be seen that this complex structure of transformations (3.11) and (3.14) takes place only for transformations including the conformal inversion (1.2). Only special conformal transformations that are expressed through space translations and the conformal inversion (1.2) belong to this class of transformations. If we restrict ourselves to the Weyl subgroup that includes rotations, translations and dilatations, then for such transformations $P^l U_g = U_g P^l$ and the operators $\tilde{U}_g$ coincide with U_g. So, the new transformation laws of the fields $h_{\mu\nu}$ and $T_{\mu\nu}$ are needed only for the conformal inversion R for which $P^l U_R \neq U_R P^l$

$$h_{\mu\nu}(x) \xrightarrow{R} [(I - P^l) U^h_R (I - P^l) + U^h_R P^l] h_{\mu\nu}(x), \tag{3.19}$$

$$T_{\mu\nu}(x) \xrightarrow{R} [U^T_R (I - P^l) + P^l U^T_R P^l] T_{\mu\nu}(x), \tag{3.20}$$

where the action of the operators U^h_R and U^T_R is defined according to (3.1) and (3.2).

Finally, let us consider the invariant propagator of the field $h_{\mu\nu}$ determined by the new transformation law (3.19). Unlike the case of the indecomposable representations Q_h, Q_T, the new representations $\tilde{Q}_h$ and $\tilde{Q}_T$ are unitary and equivalent to each other (for more details see § 5). General statements (2.2), (2.4)–(2.6) are valid for these representations as well. The propagator $D_{\mu\nu\sigma\tau}$ of the metric tensor $h_{\mu\nu}(x)$ can be identified with the intertwining operator kernel. We have for it

$$D_{\mu\nu\sigma\tau}(x) = D^{\mathrm{tr}}_{\mu\nu\sigma\tau}(x) + \alpha D^l_{\mu\nu\sigma\tau}(x) \tag{3.21}$$

where the second term is given by expression (2.7), α is the gauge parameter, and the first term is equal to (see § 5 for detail)

$$D^{\mathrm{tr}}_{\mu\nu\sigma\tau}(x) = (I_{\mu\nu\sigma\tau} - P^l_{\mu\nu\sigma\tau})\Box^{-2}\delta(x) = -16\pi^{-2}(I_{\mu\nu\sigma\tau} - P^l_{\mu\nu\sigma\tau}) \ln x^2 \tag{3.22}$$

where the operator $P^l_{\mu\nu\sigma\tau}$ is given by (3.15). Detailed calculations that lead to (3.22) are presented in § 5.

The scalar product on the direct sum of spaces $M_T^{\rm tr} \oplus M_T^l$ consisting of tensors (3.9) is given by the expression

$$(T,T) = \int \mathrm{d}x_1\,\mathrm{d}x_2\, T_{\mu\nu}(x_1) D_{\mu\nu\sigma\tau}(x_{12}) T_{\sigma\tau}(x_2) \tag{3.23}$$

$$= \int \mathrm{d}x_1\,\mathrm{d}x_2\, T^{\rm tr}_{\mu\nu}(x_1) D^{\rm tr}_{\mu\nu\sigma\tau}(x_{12}) T^{\rm tr}_{\sigma\tau}(x_2) + \int \mathrm{d}x_1\,\mathrm{d}x_2\, T^{l}_{\mu\nu}(x_1) D^{l}_{\mu\nu\sigma\tau}(x_{12}) T^{l}_{\sigma\tau}(x_2). \tag{3.24}$$

Here the first term in (3.24) is a scalar product on the space $H_T^{\rm tr}$ and the second term is one on H_T^l.

4. Invariant three-point Green functions

Let us consider the Green functions, that include the fields $h_{\mu\nu}$ and $T_{\mu\nu}$. As is known, the three-point Green functions are determined by the requirement of invariance up to arbitrary factors. In this section we discuss what changes arise in explicit expressions for these Green functions resulting from the new transformation laws of the fields $h_{\mu\nu}$ and $T_{\mu\nu}$. Let

$$G^h_{\mu\nu}(x_1x_2x_3) = \langle 0|\psi_1^\delta(x_1)\psi_2^d(x_2)h_{\mu\nu}(x_3)|0\rangle \tag{4.1}$$

be a Green function containing two arbitrary tensor or spinor fields ψ_1 and ψ_2 with the scale dimension δ and d. Here and below tensor and spinor indices on these fields are omitted. The invariance conditions of the function (4.1) are expressed by some functional equations resulting from the field transformation laws and invariance of the vacuum. Let us consider the invariance condition under the new transformation law (3.19). We find for the function (4.1)

$$G^h_{\mu\nu}(x_1x_2x_3) = (\ldots)[P^{\rm tr}_{\mu\nu\alpha\beta}(\partial/\partial x_3)\eta_{\alpha\sigma}(x_3)\eta_{\beta\tau}(x_3)P^{\rm tr}_{\sigma\tau\rho\lambda}(\partial/\partial Rx_3) + \eta_{\mu\sigma}(x_3)\eta_{\nu\tau}(x_3)P^l_{\sigma\tau\rho\lambda}(\partial/\partial Rx_3)]G^h_{\rho\lambda}(Rx_1Rx_2Rx_3) \tag{4.2}$$

where $(\ldots)$ is a set of factors associated with the transformation laws of the fields ψ_1 and ψ_2 (see e.g. Fradkin and Palchik 1978, Todorov *et al* 1978). Analogously for the Green function

$$G^T_{\mu\nu}(x_1x_2x_3) = \langle 0|\psi_1^\delta(x_1)\psi_2^d(x_2)T_{\mu\nu}(x_3)|0\rangle \tag{4.3}$$

we find from (3.20)

$$G^T_{\mu\nu}(x_1x_2x_3) = (\ldots)[(x_3^2)^{-4}\eta_{\mu\sigma}(x_3)\eta_{\nu\tau}(x_3)P^{\rm tr}_{\sigma\tau\rho\lambda}(\partial/\partial Rx_3) + P^l_{\mu\nu\alpha\beta}(\partial/\partial x_3)(x_3^2)^{-4}\eta_{\alpha\sigma}(x_3)\eta_{\beta\tau}(x_3)P^l_{\sigma\tau\rho\lambda}(\partial/\partial Rx_3)]G^T_{\rho\lambda}(Rx_1Rx_2Rx_3). \tag{4.4}$$

Equations (4.2) and (4.4) are complicated integral equations because the projectors $P^{\rm tr}$ and P^l include terms of $\Box^{-1}$ and $\Box^{-2}$ type. However, these equations can be solved using the known solutions of more simple functional equations corresponding to the old transformation laws (3.1) and (3.2). Such equations look as follows (see Fradkin and Palchik 1978 or Todorov *et al* 1978 and references therein):

$$C_{\mu\nu}(x_1x_2x_3)=(\ldots)\eta_{\mu\sigma}(x_3)\eta_{\nu\tau}(x_3)C_{\sigma\tau}(Rx_1Rx_2Rx_3), \quad (4.2a)$$

$$\tilde{C}_{\mu\nu}(x_1x_2x_3)=(\ldots)(x_3^2)^{-4}\eta_{\mu\sigma}(x_3)\eta_{\nu\tau}(x_3)\tilde{C}_{\sigma\tau}(Rx_1Rx_2Rx_3), \quad (4.2b)$$

where $C_{\mu\nu}$ and $\tilde{C}_{\mu\nu}$ are Green functions of types (4.1) and (4.3) but which are invariant relative to the old transformation laws (3.1) and (3.2). At the end of this section simple expressions will be obtained for the functions $G^h_{\mu\nu}$ and $G^T_{\mu\nu}$ which appear as a result of solving of equations (4.2) and (4.4) in terms of the known functions $C_{\mu\nu}$ and $\tilde{C}_{\mu\nu}$, see (4.11) and (4.12).

We start with the discussion of some general properties of these functions. Let us consider the Green function $C_{\mu\nu}$. In the general case this Green function may contain a few independent conformally invariant structures (see e.g. Fradkin and Palchik 1978, Todorov *et al* 1978 and references therein). The complete set of such independent structures can be divided into two groups. Let us denote the structures from the first group as $C^l_{i,\mu\nu}(x_1x_2x_3)$, where i is the structure number, and from the second as $C_{k,\mu\nu}(x_1x_2x_3)$. Let us refer all the structures that can be represented as

$$C^l_{i,\mu\nu}(x_1x_2x_3)=\partial^{x_3}_{\mu}C_{i,\nu}(x_1x_2x_3)+\partial^{x_3}_{\nu}C_{i,\mu}(x_1x_2x_3)-\tfrac{1}{2}\delta_{\mu\nu}\partial^{x_3}_{\lambda}C_{i,\lambda}(x_1x_2x_3) \quad (4.5)$$

to the first group. Existence of these structures is connected to the invariance of the function subspace (3.3) relative to transformation (3.1). In particular, in the case of two scalar fields $\psi_1=\psi_2=\varphi^d$ there is only the structure

$$C_{\mu\nu}(x_1x_2x_3)=[1/(x_{12}^2)^{d+1}](x_{13}^2x_{23}^2)[\lambda^{x_3}_{\mu_1}(x_1x_2)\lambda^{x_3}_{\mu_2}(x_1x_2)-\text{trace}]$$

where

$$\lambda^{x_3}_{\mu}(x_1x_2)=(x_{13})_\mu/x_{13}^2-(x_{23})_\mu/x_{23}^2, \quad (4.6)$$

represented in the form (4.5) with the function $C_\mu(x_1x_2x_3)$ equal to

$$C^{\text{sc}}_{\mu}(x_1x_2x_3)=-\tfrac{1}{2}[x_{13}^2x_{23}^2/(x_{12}^2)^{d+1}]\ln(x_{13}^2/x_{23}^2)\lambda^{x_3}_{\mu}(x_1x_2).$$

We refer all the structures $C_{k,\mu\nu}$ that cannot be represented in the form (4.5) to the second group. These structures exist in cases where the fields ψ_1 and ψ_2 are spinors or tensors. Note that the functions $C_{k,\mu\nu}$ are not transverse as well. They can be considered as representatives of the equivalence classes from M_h/M^l_h.

Similarly, the Green function $\tilde{C}_{\mu\nu}$ that includes the energy–momentum tensor can also contain a few invariant structures forming two groups. We attribute all transverse functions $\tilde{C}^{\text{tr}}_{i,\mu\nu}(x_1x_2x_3)$

$$\partial^{x_3}_{\mu}\tilde{C}^{\text{tr}}_{i,\mu\nu}(x_1x_2x_3)=0 \quad (4.7)$$

to the first group and all the other structures $\tilde{C}_{k,\mu\nu}(x_1x_2x_3)$ that do not possess this property to the second group. Existence of the transverse functions is connected to the existence of the subspace M^{tr}_T which is invariant under the transformation (3.2). The structures $\tilde{C}_{k,\mu\nu}(x_1x_2x_3)$ are representatives of the equivalence classes from M_T/M^{tr}_T.

An example of such invariant structures is presented below. Let $\psi^\delta_1=P^\delta_{\mu_1\ldots\mu_s}(x)$ be a tensor field of rank s, symmetric and traceless in all indices, and $\psi^d_2=\varphi_d(x_2)$ be a scalar field. Then there are three independent invariant structures (see Fradkin and Palchik 1975):

$$C_{\mu\nu,\mu_1\ldots\mu_s}(x_1x_2x_3)=\frac{1}{(\frac{1}{2}x_{13}^2)^{(\delta-d-s+D-2)/2}}\frac{1}{(\frac{1}{2}x_{23}^2)^{(d-\delta+s+D-2)/2}}\frac{1}{(\frac{1}{2}x_{12}^2)^{(\delta+d-s-D+2)/2}}$$
$$\times\Bigg\{A\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2)\lambda^{x_3}_{\mu\nu}(x_1x_2)+B\frac{1}{x_{13}^2}\sum_{k=1}^{s}\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_3x_2)$$
$$\times\left[\eta_{\mu_k\mu}(x_{13})\lambda^{x_3}_{\nu}(x_1x_2)+\eta_{\mu_k\nu}(x_{13})\lambda^{x_3}_{\mu}(x_1x_2)+\frac{2}{D}\delta_{\mu\nu}\frac{x_{12}^2}{x_{23}^2}\lambda^{x_1}_{\mu_k}(x_2x_3)\right]$$
$$+C\frac{1}{(x_{13}^2)^2}\sum_{k,r=1}^{s}\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\hat{\mu}_r\ldots\mu_s}(x_3x_2)\bigg[\eta_{\mu_k\mu}(x_{13})\eta_{\mu_r\nu}(x_{13})$$
$$-\frac{1}{D}\delta_{\mu\nu}\delta_{\mu_k\mu_r}\bigg]\Bigg\},\tag{4.8}$$

where A, B and C are arbitrary constants, D is the space dimension, $\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_2x_3)=\lambda^{x_1}_{\mu_1}(x_2x_3)\ldots\lambda^{x_1}_{\mu_s}(x_2x_3)-$traces and $\lambda^{x_1}_{\mu}(x_2x_3)$ is given by expression (4.6). Taking the divergence of this expression we obtain

$$\partial_\mu C_{\mu\nu,\mu_1\ldots\mu_s}(x_1x_2x_3)$$
$$=\frac{x_{12}^2}{x_{13}^2x_{23}^2}\left(\frac{1}{(\frac{1}{2}x_{13}^2)^{(\delta-d-s+D-2)/2}}\frac{1}{(\frac{1}{2}x_{23}^2)^{(d-\delta+s+D-2)/2}}\frac{1}{(\frac{1}{2}x_{12}^2)^{(\delta+d-s-D+2)/2}}\right)$$
$$\times\Bigg(A_1\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2)\lambda^{x_3}_{\nu}(x_1x_2)+B_1\frac{1}{x_{13}^2}\sum_{k=1}^{s}\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_3x_2)\eta_{\mu_k\nu}(x_{13})$$
$$-\text{traces}\Bigg),\tag{4.9}$$

where

$$\begin{aligned}A_1&=D^{-1}[(D-1)(\delta-d)+s]A-s[(D-2)/D](D-\delta+d+s)B,\\B_1&=-D^{-1}A+(\delta-d-2s/D)B+(s-1)(\delta-d-s-D+2)C.\end{aligned}\tag{4.10}$$

Assuming $A_1=B_1=0$ and calculating A, B and C from here, we obtain from (4.8) two independent transverse structures $\tilde{C}^{\text{tr}}_{\mu\nu}$ and one structure $\tilde{C}_{\mu\nu}$.

Let us present now the general expressions of the Green functions (4.1) and (4.3) which are invariant under the new transformations laws (3.19), (3.20). According to § 3 we need to pass to the transverse representatives in the equivalence classes from M_h/M_h^l and to the representatives of the type (3.8) in the equivalence classes from M_T/M_T^{tr}. For this we use the projection operators P^l and P^{tr} introduced in § 3. As a result we have

$$G^h_{\mu\nu}(x_1x_2x_3)=P^{\text{tr}}_{\mu\nu\sigma\tau}(\partial/\partial x_3)C_{\sigma\tau}(x_1x_2x_3)+C^l_{\mu\nu}(x_1x_2x_3),\tag{4.11}$$
$$G^T_{\mu\nu}(x_1x_2x_3)=\tilde{C}^{\text{tr}}_{\mu\nu}(x_1x_2x_3)+P^l_{\mu\nu\sigma\tau}(\partial/\partial x_3)\tilde{C}_{\sigma\tau}(x_1x_2x_3),\tag{4.12}$$

where summation over all the structures of each group is meant and the projection operators P^{tr} and P^l are given by formulae (3.18) and (3.15). The invariance of these expressions can also be demonstrated by direct calculation from (4.2) and (4.4) as was done by Palchik (1983) for QED.

It is obvious that the skeleton graphs constructed from these Green functions are invariant under the transformations (3.19), (3.20). Indeed, skeleton graphs which contain the field $h_{\mu\nu}$ on an internal line have the form of the invariant scalar product (3.23) in the space of the representation

$$\int dx_3\, dx_4\, G^T_{\mu\nu}(x_1x_2x_3)D_{\mu\nu\sigma\tau}(x_3x_4)G^T_{\sigma\tau}(x_4x_5x_6)$$

and thus they are invariant.

5. Equivalence of representations and field equations

As noted in § 2, the usual equivalence of the representations Q^d and Q^{4-d} does not take place at the exceptional integer points. In this case, however, the partial equivalence (Dobrev *et al* 1977, Klimyk 1979)

$$Q^{\rm tr}_T \sim Q^{\rm tr}_h, \qquad Q^l_T \sim Q^l_h \tag{5.1}$$

is fulfilled which can be expressed by the following relations:

$$\alpha^{-1}h^l_{\mu\nu}(x) = \int dy\, D^l_{\mu\nu\sigma\tau}(x-y)T^*_{\sigma\tau}(y) \tag{5.2}$$

where α is an arbitrary factor,

$$T^{\rm tr}_{\mu\nu}(x) = \int dy\, \Pi^{\rm tr}_{\mu\nu\sigma\tau}(x-y)h^*_{\sigma\tau}(y). \tag{5.3}$$

Here $h^l_{\mu\nu}(x) \in M^l_h$, $T^{\rm tr}_{\mu\nu} \in M^{\rm tr}_T$. The representatives of the equivalence classes from the quotient spaces enter the right-hand side: $T^*_{\sigma\tau}$ is any representative of the equivalence class from $M_T/M^{\rm tr}_T$, $h^*_{\sigma\tau}$ is any representative of the class from M_h/M^l_h. $D^l_{\mu\nu\sigma\tau}$ and $\Pi^{\rm tr}_{\mu\nu\sigma\tau}$ are the kernels of the invariant intertwining operators. The kernel $D^l_{\mu\nu\sigma\tau}$ is given by equation (2.7). The transverse kernel $\Pi^{\rm tr}_{\mu\nu\sigma\tau}$ also can be found from (2.3) by taking the limit $d=4$.

This limit can conveniently be implemented in the following way. Let us put $d=4+\varepsilon$ in (2.3). Keeping the terms dominant in ε we have

$$\begin{aligned}[\eta_{\mu\sigma}(x)\eta_{\nu\tau}(x)+\eta_{\mu\tau}(x)\eta_{\nu\sigma}(x)-\tfrac{1}{2}\delta_{\mu\nu}\delta_{\sigma\tau}](x^2)^{-d}\\ =\tfrac{3}{20}H^{\rm tr}_{\mu\nu\sigma\tau}(\partial/\partial x)(x^2)^{-2-\varepsilon}+O(\varepsilon)\end{aligned} \tag{5.4}$$

where $H^{\rm tr}_{\mu\nu\sigma\tau}$ is the transverse differential operator

$$\begin{aligned}H^{\rm tr}_{\mu\nu\sigma\tau} = \tfrac{2}{3}\partial_\mu\partial_\nu\partial_\sigma\partial_\tau - \tfrac{1}{2}(\delta_{\mu\sigma}\partial_\nu\partial_\tau + \delta_{\mu\tau}\partial_\nu\partial_\sigma + \delta_{\nu\sigma}\partial_\mu\partial_\tau\\ + \delta_{\nu\tau}\partial_\mu\partial_\sigma)\Box + \tfrac{1}{3}(\delta_{\mu\nu}\partial_\sigma\partial_\tau + \delta_{\sigma\tau}\partial_\mu\partial_\nu)\Box\\ + \tfrac{1}{2}(\delta_{\mu\sigma}\delta_{\nu\tau} + \delta_{\mu\tau}\delta_{\nu\sigma} - \tfrac{2}{3}\delta_{\mu\nu}\delta_{\sigma\tau})\Box^2.\end{aligned} \tag{5.5}$$

It has the following properties:

$$\partial_\mu H^{\rm tr}_{\mu\nu\sigma\tau} = 0, \qquad H^{\rm tr}_{\mu\nu\sigma\tau} = H^{\rm tr}_{\sigma\tau\mu\nu} = H^{\rm tr}_{\nu\mu\sigma\tau}, \qquad H^{\rm tr}_{\mu\mu\sigma\tau} = 0.$$

It can be represented as

$$H^{\rm tr}_{\mu\nu\sigma\tau} = P^{\rm tr}_{\mu\nu\sigma\tau}\Box^2 = (I_{\mu\nu\sigma\tau} - P^l_{\mu\nu\sigma\tau})\Box^2 \tag{5.6}$$

where P^{tr} and P^{l} are the projection operators introduced in § 3. The right-hand side of (5.4) is singular in the limit $\varepsilon=0$. Let us multiply this expression by $\varepsilon=(d-4)$ and take the limit $\varepsilon=0$. Allowing for the relation $\varepsilon/(x^2)^{2+\varepsilon}|_{\varepsilon\to 0}=-\pi^2\delta(x)$ we obtain the expression

$$\Pi^{\text{tr}}_{\mu\nu\sigma\tau}(x)=H^{\text{tr}}_{\mu\nu\sigma\tau}(\partial/\partial x)\delta(x). \tag{5.7}$$

Let us consider the partial equivalence of (5.2) and (5.3) as applied to the realisation of representations Q^{tr}_{h} and Q^{l}_{T} described in § 3. For this we substitute the longitudinal representative $T^{*}_{\mu\nu}=T^{l}_{\mu\nu}$ of the equivalence class from H_T/H^{tr}_T into (5.2) and the transverse representative $h^{*}_{\sigma\tau}(x)=h^{\text{tr}}_{\sigma\tau}(x)$ of the equivalence class from H_h/H^{l}_{h} into (5.3):

$$\alpha^{-1}h^{l}_{\mu\nu}(x)=\int \mathrm{d}y\, D^{l}_{\mu\nu\sigma\tau}(x-y)T^{l}_{\sigma\tau}(y), \tag{5.8}$$

$$T^{\text{tr}}_{\mu\nu}(x)=\int \mathrm{d}y\, \Pi^{\text{tr}}_{\mu\nu\sigma\tau}(x-y)h^{\text{tr}}_{\sigma\tau}(y), \tag{5.9}$$

where $T^{l}_{\sigma\tau}$ is the tensor function of the form (3.8). The transverse operator Π^{tr} is non-degenerate if considered only on the transverse tensor space. Hence, relation (5.9) can be inverted

$$h^{\text{tr}}_{\mu\nu}(x)=\int \mathrm{d}y\, D^{\text{tr}}_{\mu\nu\sigma\tau}(x-y)T^{\text{tr}}_{\sigma\tau}(y) \tag{5.10}$$

where $D^{\text{tr}}(x)=(\Pi^{\text{tr}})^{-1}(x)$. From (5.7) and (5.6) we find

$$D^{\text{tr}}_{\mu\nu\sigma\tau}(x)=(I_{\mu\nu\sigma\tau}-P^{l}_{\mu\nu\sigma\tau})\Box^{-2}\delta(x)$$

from which equation (3.22) follows.

Consider the sums of tensors (3.9) and (3.10). Adding (5.8) and (5.10) and allowing for the relation $\int D^{l}_{\mu\nu\sigma\tau}T^{\text{tr}}_{\sigma\tau}=\int D^{\text{tr}}_{\mu\nu\sigma\tau}T^{l}_{\sigma\tau}=0$ we find

$$h_{\mu\nu}(x)=\int \mathrm{d}y\, D_{\mu\nu\sigma\tau}(x-y)T_{\sigma\tau}(y). \tag{5.11}$$

This relation contains the same information as the relations (5.8) and (5.10) and expresses the equivalence of the direct sums of representations (3.4) and (3.6). The intertwining operator kernel $D_{\mu\nu\sigma\tau}$ is given by expression (3.21).

Let us show that equation (5.11) is coincident with the dynamical equation for the metric tensor in the linear approximation. We introduce the classical fields $h^{\text{cl}}_{\mu\nu}(x)$ and $T^{\text{cl}}_{\mu\nu}(x)$. These can be defined as the convolutions of the Green functions including the quantum fields $h_{\mu\nu}(x)$ and $T_{\mu\nu}(x)$ with the test functions in all arguments but x. Note that the 'classical' energy–momentum tensor defined in this way is not transverse: $\partial_\mu T^{\text{cl}}_{\mu\nu}(x)\neq 0$ due to the Ward identities for the corresponding Green functions. The sets of classical fields $h^{\text{cl}}_{\mu\nu}$ and $T^{\text{cl}}_{\mu\nu}$ defined in this way form the spaces of equivalent representations (3.4) and (3.6) realised as the spaces of functions (3.9) and (3.10) and, thus, satisfy equation (5.11). Taking the inverse of this equation we obtain

$$(I_{\mu\nu\sigma\tau}-P^{l}_{\mu\nu\sigma\tau})\Box^{2}h^{\text{cl}}_{\sigma\tau}(x)+\alpha^{-1}\Pi^{l}_{\mu\nu\sigma\tau}(\partial/\partial x)h^{\text{cl}}_{\sigma\tau}(x)\sim T_{\mu\nu}(x) \tag{5.12}$$

where

$$\Pi^{l}_{\mu\nu\sigma\tau}=\partial_\mu\Pi_{\nu\sigma\tau}(\partial/\partial x)+\partial_\nu\Pi_{\mu\sigma\tau}(\partial/\partial x)-\delta_{\mu\nu}\partial_\lambda\Pi_{\lambda\sigma\tau}(\partial/\partial x) \tag{5.12a}$$

is defined by the equation

$$\Pi^{l}_{\mu\nu\sigma\tau}(\partial/\partial x)D^{l}_{\sigma\tau\rho\lambda}(x)=P^{l}_{\mu\nu\rho\lambda}(\partial/\partial x)\delta(x).$$

We find from here

$$\Pi_{\mu\sigma\tau}=\tfrac{1}{288}[22\partial_\mu\partial_\sigma\partial_\tau-g(\delta_{\mu\tau}\partial_\sigma\Box+\delta_{\mu\sigma}\partial_\tau\Box)-\delta_{\sigma\tau}\partial_\mu\Box]. \tag{5.12b}$$

Note that the first term in the left-hand side of (5.12) coincides up to a factor with the left-hand side of the linear conformal gravitation equation (1.11). The second term is a gauge term. It corresponds to adding the gauge term†

$$L_{\text{gauge}}=(1/2\alpha)h_{\mu\nu}(x)\Pi^{l}_{\mu\nu\sigma\tau}(\partial/\partial x)h_{\sigma\tau}(x) \tag{5.13}$$

to the Lagrangian where $\Pi^{l}_{\mu\nu\sigma\tau}$ is determined in (5.12*a*, *b*).

Therefore, the conformal gravitation equations for classical fields are the consequence of the condition of equivalence of the representations associated with the metric and energy–momentum tensor.

Equations for quantum fields cannot be obtained from group theoretical considerations alone. The quantum equations would mean that these equations take place for any Green functions. Using the formulae for § 4 it can be easily verified that these equations really hold (up to a factor) for the three-point Green functions

$$\{[I_{\mu\nu\sigma\tau}-P^{l}_{\mu\nu\sigma\tau}(\partial/\partial x)]\Box^2_x+\alpha^{-1}\Pi^{l}_{\mu\nu\sigma\tau}(\partial/\partial x)\}\langle 0|h_{\sigma\tau}(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle$$
$$=\tfrac{1}{2}\langle 0|T_{\mu\nu}(x)\psi(x_1)\bar{\psi}(x_2)|0\rangle.$$

However, the equations for the higher Green functions are fulfilled only under the condition of equality of the kernels of partial wave expansions of the Green functions containing $h_{\mu\nu}$ and $T_{\mu\nu}$. For conformal QED such an investigation was carried out by Fradkin *et al* (1982). All these results can be generalised to the case of conformal gravitation without any alterations.

Note, in conclusion, that the mathematical problems concerning the Weyl tensor have not been discussed. The analysis of the corresponding representations is given by Dobrev *et al* (1977) and Klimyk (1979). It can be shown that in this case the equivalence conditions of representations also coincide with the conformal gravitation equations in the linear approximation, see (1.10) and (1.11). Equation (1.11) can be written in the form

$$\partial_\nu\partial_\tau C_{\mu\nu\sigma\tau}=-\tfrac{1}{8}T^{\text{tr}}_{\mu\sigma}$$

which expresses the equivalence condition for the Weyl and energy–momentum tensors. The gauge term is omitted in this equation.

We have shown, therefore, that the linear conformal gravitation equations for classical fields and the three-point Green functions result from the equivalence conditions of the conformal group representations corresponding to the metric, Weyl and energy–momentum tensors. Note that these equations contain no non-trivial dynamical information until an explicit expression for the matter field energy–momentum tensor

† Note that the longitudinal part of the metric field propagator does not vanish at infinity since the scale dimension of this field is zero. Hence the gauge term (5.13) determines the propagator up to an arbitrary factor which also has a zero dimension. To make an unambiguous determination of the longitudinal part it is necessary to take conformal invariance into account which leads to the longitudinal part (2.7).

is given. The above group theoretical derivation of equations only shows that the metric, Weyl and energy–momentum tensors in the linear theory appear as a single object to which any of three equivalent representations can be put into correspondence.

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