International Journal of Modern Physics A, Vol. 13, No. 28 (1998) 4837–4888

METHOD OF SOLVING CONFORMAL MODELS IN D-DIMENSIONAL SPACE III SECONDARY FIELDS IN $D > 2$ AND THE SOLUTION OF TWO-DIMENSIONAL MODELS

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Received 24 December 1996

We proceed with the study (started in Refs. 1 and 2) of the Hilbert space of conformal field theory in D dimensions. We discuss an infinite family of secondary fields P_s^T generated by the action of the components of energy–momentum tensor $T_{\mu\nu}$ on the fundamental (primary) field. It is shown that the states of these fields form a specific sector of the Hilbert space H which is determined by the Ward identities and $[\frac{1}{2}(D+1)(D+2)]$-dimensional conformal symmetry. We demonstrate that for $D = 2$ the subspace H coincides with the space of representation of the Virasoro algebra. Each exactly solvable model in the case of $D \geq 2$ is defined by the requirement of vanishing of a certain state $Q_s(x)|0\rangle \subset H$ analogous to the null vector of two-dimensional theory. The Green functions of the fields P_s^T are calculated in terms of the Green functions of the fundamental field. It is shown that all the Green functions of the type $\langle TP_s^T \cdots\rangle$ satisfy the anomalous Ward identities. The anomalous contributions are given by the fields $P_{s'}^T$, where $s' \leq s-1$. The fields Q_s are constructed as superpositions of secondary fields with the anomalous contribution equal to zero, i.e. having the transformation properties of primary fields.

An approach developed is based on a finite-dimensional conformal symmetry for any $D \geq 2$. Nevertheless the resulting models have the structure analogous to that of two-dimensional conformal theories. This analogy is discussed in detail. It is shown that for $D = 2$ the family of models coincides with the well-known family of conformal models based on infinite-dimensional conformal symmetry. The analysis of this phenomenon indicates the existence of the D-dimensional analog of the Virasoro algebra.

1. Introduction

In Refs. 1 and 2 the Hilbert space of conformal quantum field theory in D dimensions was considered and shown to have a specific subspace

$$H \subset M \tag{1.1}$$

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completely determined by general principles of QTP supplied with the requirement of $[\frac{1}{2}(D+1)(D+2)]$-dimensional conformal symmetry. The states of this are associated with a pair of infinite sets of tensor fields

$$P_s^T = P_{\mu_1\cdots\mu_s}^T(x)\,, \qquad P_s^j = P_{\mu_1\cdots\mu_s}^j\,, \quad s=0,1,\dots\,, \tag{1.2}$$

of scale dimensions $d_s^T = d_s^j = d+s$, where d is the dimension of fundamental (primary) field φ. The fields P_s^T are begotten by energy–momentum tensor, and the fields P_s^j by conserved currents.

The transformation properties of the fields P_s^T and P_s^j are analogous to those of secondary fields appearing in two-dimensional models.[3,4] Further on, it was shown that there exists a family of models (including Lagrangian ones) defined by the requirement that certain superpositions Q_s of the fields P_s^T or P_s^j should vanish. The coefficients in these superpositions are chosen from the conditions that $Q_s(x)$ must represent primary fields (the conditions of self-consistency of the models). Each model defined by the equation

$$Q_s(x) = 0\,, \qquad s=1,2,\dots, \tag{1.3}$$

can be solved exactly in the case of D-dimensional space $D \geq 2$. Some of these models are discussed in Refs. 1 and 2.

These analogies with two-dimensional models are discussed in much detail in the present paper.

We are concerned with the family of secondary fields in $D \geq 2$ generated by the action of energy–momentum tensor components on a (neutral) fundamental field. This family results from the operator product expansions

$$T_{\mu\nu}(x_1)\varphi(x_2) = \sum_s [P_s^T]\,, \qquad T_{\mu\nu}(x_1)P_{s_1}^T(x_2) = \sum_s [P^{T,s_1}],\dots.$$

The basis of the space H is formed by the states

$$\varphi(x)|0\rangle\,, \qquad P_s^T(x)|0\rangle\,, \qquad P_s^{T,s_1}(x)|0\rangle,\dots.$$

In Secs. 2–4 we shall find closed expressions for the Green functions of the fields $P_s^T, P_s^{T,s_1},\dots$ and study anomalous Ward identities for the Green functions of these fields. These results are obtained for the space of arbitrary dimension $D \geq 2$ (the analogous results for the fields P_s^j were obtained in Ref. 2).

In Secs. 5–7, this formalism is applied to the solution of two-dimensional models. We demonstrate that in the framework of this approach the well-known[3] family of conformal models is reproduced in the case of $D=2$. The fields P_s^T turn out to be the superpositions of secondary fields resulting from the action of Virasoro algebra generators on the fundamental field. The sector H (found in Refs. 1 and 2 for $D \geq 2$) mentioned above coincides with the representation space of the Virasoro algebra, while the states $Q_s(x)|0\rangle$ coincide with null vectors of two-dimensional conformal models.

Let us stress that our approach assumes $[\frac{1}{2}(D+1)(D+2)]$-dimensional conformal symmetry in the space of any dimension D. In the case of $D = 2$ this symmetry is six-parametric. Its generators

$$L_0, L_{\pm 1}, \bar{L}_0, \bar{L}_{\pm 1} \tag{1.4}$$

form the algebra of the group

$$\mathrm{SL}(2, R) \times \mathrm{SL}(2, R), \tag{1.5}$$

which constitutes the maximal finite-dimensional subalgebra of the Virasoro algebra. Note that the form of the Ward identities for the Green functions of the energy–momentum tensor is determined by the symmetry (1.4). Thus the commutators $[T_\pm(x_1), T_\pm(x_2)]$ of the energy–momentum tensor components are given implicitly. Therefore, in the approach concerned, the infinite-dimensional conformal symmetry arises as an auxiliary result, and is not *a priori* presumed. Hence it follows that the implementations based on the Virasoro algebra, being probably useful, are *not* necessary in the framework of our formalism. One can act as if the existence of the infinite-dimensional conformal symmetry was not known. An analogous situation is likely to be realized in the models (1.3) for $D > 2$ as well. In Secs. 5–7 and in App. B we demonstrate the equivalence of the two approaches for $D = 2$ in more detail. The analysis presented here may prove useful for the construction of a D-dimensional analog of the Virasoro algebra; see, for example, Ref. 5. In particular, the subspace (1.1) might coincide with the representation space of such an algebra (in the case where this algebra will be actually found); and Eqs. (1.3), with the conditions selecting its irreducible representations. As shown in Ref. 2, the self-consistency conditions of the models (1.3) allow one to derive, in principle, a D-dimensional analog of the Kac formula revealing the dependence of scale dimension on the central charge.[a]

2. A General Conformally Invariant Solution of the Ward Identities in D-Dimensional Space and an Irreducibility Condition for the Tensor $T_{\mu\nu}$

Consider the Green functions

$$G^T_{\mu\nu}(xx_1 \cdots x_m) = \langle T_{\mu\nu}(x)\varphi(x_1)\cdots\varphi(x_m)\rangle, \tag{2.1}$$

where $\varphi_1 \cdots \varphi_m$ are the conformal scalar fields of scale dimensions $d_1 \cdots d_m$. Conformally invariant Ward identities in D dimensions ($D \geq 2$) have the form

$$\partial^x_\nu G^T_{\mu\nu}(xx_1 \cdots x_m) = -\left\{\sum_{k=1}^{m} \delta(x - x_k)\partial^{x_k}_\mu - \partial^x_\mu \left[\sum_{k=1}^{m} \frac{d_k}{D}\delta(x - x_k)\right]\right\} G(x_1 \cdots x_m), \tag{2.2}$$

[a]For the sake of the reader's convenience we have included a brief review of the traditional approach based on infinite-dimensional conformal symmetry (see App. B), as well as the concurrent juxtaposition of the latter approach to the one being presented.

where

$$G(x_1 \cdots x_m) = \langle \varphi_1(x_1) \cdots \varphi_m(x_m) \rangle . \tag{2.3}$$

The general conformally invariant solution of the Ward identities for[b]

$$D \geq 3 \tag{2.4}$$

is the sum of two terms of a different nature:[1,2]

$$G^T_{\mu\nu}(xx_1 \cdots x_m) = \tilde{G}^T_{\mu\nu}(xx_1 \cdots x_m) + G^{T\ \mathrm{tr}}_{\mu\nu}(xx_1 \cdots x_m) , \tag{2.5}$$

where $G^{T\ \mathrm{tr}}_{\mu\nu}(xx_1 \cdots x_m)$ is the transverse function

$$\partial^x_\mu G^{T\ \mathrm{tr}}_{\mu\nu}(xx_1 \cdots x_m) = 0 , \tag{2.6}$$

and $\tilde{G}^T_{\mu\nu}$ is uniquely determined by the Ward identity. Note that for $D \geq 3$, $G^T_{\mu\nu}$ cannot be decomposed into a sum of longitudinal and transversal components in a conformally invariant fashion.

The two terms in (2.5) correspond to the two types of contributions[1] to the operator product expansions of $T_{\mu\nu}\varphi$:

$$T_{\mu\nu}(x_1)\varphi(x_2) = \sum_s [P^T_s] + \sum_s [R^T_s] . \tag{2.7}$$

The state of the fields P^T_s and R^T_s belong to mutually orthogonal subspaces[c] $M^{(m)}$ and $M^{(g)}$

$$\begin{gathered} P^T_s(x)|0\rangle \subset M^{(m)} , \qquad R^T_s(x)|0\rangle \subset M^{(g)} , \\ \langle 0|P^T_s(x_1) R^T_s(x_2)|0\rangle = 0 . \end{gathered} \tag{2.8}$$

Correspondingly, the Green functions of the metric field $h_{\mu\nu}(x)$ (which is a conformal partner[1] of the energy–momentum tensor)

$$G^h_{\mu\nu}(xx_1 \cdots x_m) = \langle h_{\mu\nu}(x)\varphi(x_1) \cdots \varphi(x_m) \rangle \tag{2.9}$$

may also be represented as a sum of two terms,

$$G^h_{\mu\nu}(xx_1 \cdots x_m) = G^{h\ \mathrm{long}}_{\mu\nu}(xx_1 \cdots x_m) + \tilde{G}^h_{\mu\nu}(xx_1 \cdots x_m) , \tag{2.10}$$

where $G^{h\ \mathrm{long}}_{\mu\nu}$ is a longitudinal conformally invariant function,

$$\begin{aligned} G^{h\ \mathrm{long}}_{\mu\nu}(xx_1 \cdots x_m) &= \partial^x_\mu G_\nu(xx_1 \cdots x_m) + \partial^x_\nu G_\mu(xx_1 \cdots x_m) \\ &\quad - \frac{2}{D}\delta_{\mu\nu}\partial^x_\lambda G_\lambda(xx_1 \cdots x_m) , \end{aligned} \tag{2.11}$$

[b] The solution of the Ward identity for $D = 2$ is considered in Sec. 4.

[c] As shown in Ref. 1, the total Hilbert space of conformal theory may be represented as a direct sum

$$M = M^{(m)} \oplus M^{(g)} ,$$

where $M^{(m)}$ includes the states of the matter fields, while $M^{(g)}$ includes the states of the gauge fields.

and the function $G^h_{\mu\nu}$ cannot be split into longitudinal and transversal components in a conformally invariant way.

In Ref. 1 the two terms in (2.5) were juxtaposed with nonequivalent representations of the conformal group

$$\tilde{Q}_T \qquad \text{and} \qquad Q_T^{\text{tr}} \tag{2.12}$$

pertaining to the energy–momentum tensor. If both the terms in (2.5) are nonzero, then the energy–momentum tensor transforms by the direct sum of irreducible representations:

$$\tilde{Q}_T \oplus Q_T^{\text{tr}} . \tag{2.13}$$

Analogously, two terms in (2.10) correspond to a direct sum of two nonequivalent representations,[1]

$$Q_h^{\text{long}} \oplus \tilde{Q}_h , \tag{2.14}$$

compliant to metric field $h_{\mu\nu}(x)$. The irreducible representations are pairwise equivalent:[6,7]

$$\tilde{Q}_T \sim Q_h^{\text{long}} , \qquad Q_T^{\text{tr}} \sim \tilde{Q}_h .$$

The equivalence conditions are expressed by the following relations between independent irreducible components of (2.5) and (2.10):

$$G^{h\ \text{long}}_{\mu\nu}(xx_1 \cdots x_m) = \int dy D^h_{\mu\nu,\rho\sigma}(x-y) \tilde{G}^T_{\rho\sigma}(yx_1 \cdots x_m) , \tag{2.15}$$

where $D^h_{\mu\nu,\rho\sigma}$ is the conformally invariant propagator of the longitudinal field $h^{\text{long}}_{\mu\nu}$,

$$\begin{aligned} D^h_{\mu\nu,\rho\sigma}(x_{12}) &= \langle h^{\text{long}}_{\mu\nu}(x_1) h^{\text{long}}_{\rho\sigma}(x_2) \rangle \\ &= C_h \left[g_{\mu\rho}(x_{12}) g_{\nu\sigma}(x_{12}) + g_{\mu\sigma}(x_{12}) g_{\nu\rho}(x_{12}) - \frac{2}{D} \delta_{\mu\nu} \delta_{\rho\sigma} \right] , \end{aligned} \tag{2.16}$$

where C_h is a constant and

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} . \tag{2.17}$$

The expression (2.16) may be represented in the form[8]

$$D^h_{\mu\nu,\rho\sigma}(x_{12}) = \partial^{x_1}_\mu D_{\nu,\rho\sigma}(x_{12}) + \partial^{x_1}_\nu D_{\mu,\rho\sigma}(x_{12}) - \frac{2}{D}\delta_{\mu\nu} \partial^{x_1}_\lambda D_{\lambda,\rho\sigma}(x_{12}) , \tag{2.18}$$

where

$$D^h_{\mu\nu,\rho\sigma}(x) = \frac{1}{2} C_h \left[x_\sigma g_{\mu\rho}(x) + x_\rho g_{\mu\sigma}(x) + \frac{2}{D} \delta_{\rho\sigma} x_\mu \right] . \tag{2.19}$$

From (2.15) and (2.18) the representation of $\tilde{G}^{\text{long}}_{\mu\nu}$ in the form (2.11) follows.

The second condition of equivalence is given by the relation

$$G^{T\ \text{tr}}_{\mu\nu}(xx_1 \cdots x_m) = \int dy D^{T,\text{tr}}_{\mu\nu,\rho\sigma}(x-y) \tilde{G}^h_{\rho\sigma}(yx_1 \cdots x_m) , \tag{2.20}$$

where $D^{T,\mathrm{tr}}_{\mu\nu,\rho\sigma}$ is the transversal propagator. The general conformally invariant expression for the propagator of the energy–momentum tensor has the form

$$D^{T}_{\mu\nu,\rho\sigma}(x_{12}) = \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle = \tilde{C}_T\left\{g_{\mu\rho}(x_{12})g_{\nu\sigma}(x_{12}) + g_{\mu\sigma}(x_{12})g_{\nu\rho}(x_{12}) - \frac{2}{D}\delta_{\mu\nu}\delta_{\rho\sigma}\right\}\frac{1}{(x_{12}^2)^D}\,. \tag{2.21}$$

In the space of odd dimension this expression is given by a well-defined function and may be represented as[8]

$$D^{T,\mathrm{tr}}_{\mu\nu,\rho\sigma}(x_{12}) \sim H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^{x_1})\frac{1}{(x_{12}^2)^{D-2}}\,, \tag{2.22}$$

where

$$H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x) = \left\{\frac{(D-2)}{(D-1)}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma - \frac{1}{2}(\delta_{\mu\rho}\partial_\nu\partial_\sigma + \delta_{\mu\sigma}\partial_\nu\partial_\rho + \delta_{\nu\rho}\partial_\mu\partial_\sigma + \delta_{\nu\sigma}\partial_\mu\partial_\rho)\Box + \frac{1}{(D-1)}(\delta_{\mu\nu}\partial_\rho\partial_\sigma\Box + \delta_{\rho\sigma}\partial_\mu\partial_\nu\Box) + \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\Box^2 - \frac{1}{(D-1)}\delta_{\mu\nu}\delta_{\rho\sigma}\Box^2\right\}, \tag{2.23}$$

$$H^{\mathrm{tr}}_{\mu\nu,\rho\sigma} = H^{\mathrm{tr}}_{\nu\mu,\rho\sigma} = H^{\mathrm{tr}}_{\rho\sigma,\mu\nu}\,, \qquad H^{\mathrm{tr}}_{\mu\mu,\rho\sigma} = 0\,.$$

$$H^{\mathrm{tr}}_{\mu\nu,\lambda\tau}(\partial^x)H^{\mathrm{tr}}_{\lambda\tau,\rho\sigma}(\partial^x) = \Box^2 H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x)\,, \tag{2.24}$$

$$\partial^x_\mu H^{\mathrm{tr}}_{\mu\nu,\rho\tau}(\partial^x) = 0\,. \tag{2.25}$$

In the even-dimensional space the expression (2.21) diverges due to the singularity of the factor $(x_{12}^2)^{-D}$. Let us redefine this propagator as follows. Introduce the conformally invariant regularization by the addition of a small anomalous correction to the dimension l_T of the field $T_{\mu\nu}$:

$$l_T = D \to l_T^\varepsilon = D + \varepsilon\,. \tag{2.26}$$

The regularized propagator $D^{T^\varepsilon}_{\mu\nu,\rho\sigma}$ results from (2.21) after the substitution of the factor $(x_{12}^2)^{-D-\varepsilon}$ for the factor $(x_{12}^2)^{-D}$. Define a new propagator for the space of even dimension $D \geq 4$ by

$$D^{T,\mathrm{tr}}_{\mu\nu,\rho\sigma}(x_{12}) = \lim_{\varepsilon\to 0}\varepsilon D^{T^\varepsilon}_{\mu\nu,\rho\sigma}(x_{12})\,. \tag{2.27}$$

Resolving the ambiguity with the help of the relation[9]

$$(x^2)^{-D-\varepsilon}\big|_{\varepsilon\to 0} = -\frac{1}{\varepsilon}\frac{\pi^{D/2}4^{-D/2}}{\Gamma(D)\Gamma\left(\frac{D+2}{2}\right)}\Box^{D/2}\delta(x)\,, \tag{2.28}$$

one gets

$$D^{T,\mathrm{tr}}_{\mu\nu,\rho\sigma}(x_{12}) \sim H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^{x_1})\Box^{\frac{D-4}{2}}\delta(x_{12}), \qquad \text{for even } D \geq 4. \tag{2.29}$$

Depending on a parity of space dimension, either (2.22) or (2.29) is used in the relation (2.20).

It is convenient to introduce the projection operators P^{tr} and P^{long} into the transversal and longitudinal sectors accordingly. Owing to (2.24) the latter is done in a manner which is natural to conformal theory, by setting

$$P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x) = \frac{1}{\Box^2}H^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x), \tag{2.30}$$

$$P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x) + P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x) = \frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \frac{2}{D}\delta_{\mu\nu}\delta_{\rho\sigma}\right). \tag{2.31}$$

One can easily check that the thus-defined operator $P^{\mathrm{long}}_{\mu\nu,\rho\sigma}$ has the properties

$$P^{\mathrm{long}}_{\mu\nu,\lambda\tau}(\partial^x)P^{\mathrm{long}}_{\lambda\tau,\rho\sigma}(\partial^x) = P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x), \tag{2.32}$$

$$P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x) = \partial^x_\mu P_{\nu,\rho\sigma}(\partial^x) + \partial^x_\nu P_{\mu,\rho\sigma}(\partial^x) - \frac{2}{D}\delta_{\mu\nu}\partial^x_\lambda P_{\lambda,\rho\sigma}(\partial^x), \tag{2.33}$$

where

$$P_{\mu,\rho\sigma}(\partial^x) = -\frac{1}{2}\left[\frac{D-2}{D-1}\partial_\mu\partial_\rho\partial_\sigma\frac{1}{\Box^2} - (\delta_{\mu\rho}\partial_\sigma + \delta_{\mu\sigma}\partial_\rho)\frac{1}{\Box} + \frac{1}{D-1}\delta_{\rho\sigma}\partial_\mu\frac{1}{\Box}\right]. \tag{2.34}$$

Furthermore, one can explicitly check that the longitudinal propagator (2.16) satisfies the relation

$$P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^{x_1})D^h_{\rho\sigma,\lambda\tau}(x_{12}) = D^h_{\mu\nu,\lambda\tau}(x_{12}). \tag{2.35}$$

As follows from (2.22), (2.29) and (2.30), the transversal propagator of the energy–momentum tensor satisfies the relation

$$P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^{x_1})D^{T,\mathrm{tr}}_{\rho\sigma,\lambda\tau}(x_{12}) = D^{T,\mathrm{tr}}_{\mu\nu,\lambda\tau}. \tag{2.36}$$

Using these relations one finds from Eqs. (2.15) and (2.20) that

$$P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x)G^{T\,\mathrm{tr}}_{\rho\sigma}(xx_1\cdots x_m) = G^{T\,\mathrm{tr}}_{\mu\nu}(xx_1\cdots x_m), \tag{2.37}$$

$$P^{\mathrm{long}}_{\mu\nu,\rho\sigma}(\partial^x)G^{h\,\mathrm{long}}_{\rho\sigma}(xx_1\cdots x_m) = G^{h\,\mathrm{long}}_{\mu\nu}(xx_1\cdots x_m). \tag{2.38}$$

Note that the remaining pair of irreducible functions, $\tilde{G}^T$ and $\tilde{G}^h$, do not satisfy similar relations. Each of these functions has both the transversal and the longitudinal components, and only the whole sums possess the property of conformal invariance. As shown in Ref. 1, the requirement of conformal invariance allows one to reconstruct the transversal part of the function $\tilde{G}^T_{\mu\nu}$ uniquely from the longitudinal part which is known from the Ward identities provided that one chooses a certain realization of the representation $\tilde{Q}_T$. The choice of the realization in this case is imposed by the orthogonality condition (2.8). As shown in Ref. 1, the latter allows one to separate out the contribution of the gravitational interaction into the Green function (2.1) in an explicit manner; see below.

A summary consists in the following. The general solution of the Ward identities (2.2) represents a sum of the two conformally invariant terms (2.5). The first one, $\tilde{G}^T_{\mu\nu}$, is uniquely determined by the Ward identity and the requirement of the conformal symmetry. The second term is transverse, and may be expressed through the Green function of the metric field by Eq. (2.20). In the space of even dimension this equation takes the form

$$G^{T\,\mathrm{tr}}_{\mu\nu}(xx_1\cdots x_m) = \Box_x^{D/2} P^{\mathrm{tr}}_{\mu\nu,\rho\sigma}(\partial^x)\langle h_{\rho\sigma}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,. \tag{2.39}$$

In four-dimensional space it coincides[8] with the equation of linear conformal gravity. The longitudinal part $G^{h\,\mathrm{long}}_{\mu\nu}$ of the Green function $G^h_{\mu\nu} = \langle h_{\mu\nu}\varphi_1\cdots\varphi_m\rangle$ does not contribute to (2.39). It is determined from Eq. (2.15) and may be calculated directly from the Ward identities:

$$G^{h\,\mathrm{long}}_{\mu\nu}(xx_1\cdots x_m) = -2\int dy D_{\mu\nu,\rho}(x-y)\partial^y_\sigma\langle T_{\rho\sigma}(y)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,, \tag{2.40}$$

where $D_{\mu\nu,\rho}(x)$ is the function (2.19):

$$D_{\mu\nu,\rho}(x-y) = P_{\rho,\lambda\tau}(\partial^x)D^h_{\mu\nu,\lambda\tau}(x-y)\,. \tag{2.41}$$

Thus, the functions

$$\tilde{G}^T_{\mu\nu} \qquad \text{and} \qquad G^{h\,\mathrm{long}}_{\mu\nu} \tag{2.42}$$

are determined by the Ward identities, the function $\tilde{G}^h_{\mu\nu}$ remains arbitrary, and the function $G^{T\,\mathrm{tr}}_{\mu\nu}$ is expressed through it by Eq. (2.39) (or a similar one for odd D). To this pair of functions (2.42) a pair of equivalent irreducible representations $\tilde{Q}_T \sim Q_h^{\mathrm{long}}$ corresponds.

According to Ref. 1, the Green functions (2.42) describe the contribution of matter fields into energy–momentum tensor, while the function $\tilde{G}^h_{\mu\nu}$, as well as the transversal function $G^{T\,\mathrm{tr}}_{\mu\nu}$ which expresses through it, are related to gravitational interaction. To this pair of functions, $G^{T\,\mathrm{tr}}_{\mu\nu}$ and $\tilde{G}^h_{\mu\nu}$, another pair of equivalent irreducible representations $\tilde{Q}^{\mathrm{tr}} \sim \tilde{Q}^h$ corresponds.

Due to this, the theories which are free of gravitation interaction are selected by the following condition: *the energy–momentum tensor transforms by the irreducible representation* $\tilde{Q}^T$. *Its Green functions* coincide with $\tilde{G}^T_{\mu\nu}$:

$$\langle T_{\mu\nu}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle = \tilde{G}^T_{\mu\nu}(xx_1\cdots x_m)\,, \tag{2.43}$$

and, in virtue of the above arguments, *are uniquely determined by the Ward identities.*

The condition of irreducibility was formulated in Ref. 1 in terms of the equation (for $s \geq 2$)

$$\int dy_1\, dy_2 \tilde{C}^l_{\mu_1\cdots\mu_s,\mu\nu}(x_1y_1y_2)\langle T_{\mu\nu}(y_2)\varphi_1(y_1)\varphi_2(x_2)\cdots\varphi_m(x_m)\rangle = 0\,, \tag{2.44}$$

where the function $\tilde{C}^{\sigma}_{\mu\nu} = \tilde{C}^{l}_{\mu_1\cdots\mu_s,\mu\nu}$ is the conformally invariant function of the type

$$\tilde{C}^{\sigma}_{\mu\nu}(x_1x_2x_3) = \tilde{C}^{l}_{\mu_1\cdots\mu_s,\mu\nu}(x_1x_2x_3) = \langle P_\sigma(x_1)\varphi_{D-d}(x_2)\tilde{h}_{\mu\nu}(x_3)\rangle \,. \tag{2.45}$$

Below the symbol σ will stand for the pair of quantum numbers of the (symmetric and traceless) conformal tensor

$$P_\sigma(x) = P^{l}_{\mu_1\cdots\mu_s}(x)\,, \qquad \sigma = (l,s)\,. \tag{2.46}$$

The amputation is equivalent to the substitution[1]

$$\sigma \to \tilde{\sigma} = (D-l,s)\,.$$

In particular, the argument x_2 is amputed in (2.45), which is equivalent to the change

$$\varphi_d(x) \to \varphi_{D-d}(x) = \varphi_{\tilde{d}}(x)\,. \tag{2.47}$$

The explicit representation of the function (2.45) is found in Refs. 1 and 10. It is not used in this discussion.

As shown in Ref. 1, the condition (2.44) is equivalent to the equations

$$G^{T\,\mathrm{tr}}_{\mu\nu}(xx_1\cdots x_m) = \tilde{G}^{h}_{\mu\nu}(xx_1\cdots x_m) = 0\,. \tag{2.48}$$

In what follows, only the models where these equations are satisfied will be concerned. It is essential that all the Green functions of metric tensor are longitudinal. In such models the propagator (2.21) is well defined in any space dimension, both the even and the odd. The reason lies in the fact that the only contractions with the propagator (2.21) in these models are

$$\int dx\,dy\,h^{\mathrm{long}}_{\mu\nu}(x)D^{T}_{\mu\nu,\rho\sigma}(x-y)h^{\mathrm{long}}_{\rho\sigma}(y)\,, \tag{2.49}$$

where $h^{\mathrm{long}}_{\mu\nu}$ is the longitudinal conformal field

$$h^{\mathrm{long}}_{\mu\nu}(x) = \partial_\mu h_\nu(x) + \partial_\nu h_\mu(x) - \frac{2}{D}\delta_{\mu\nu}\partial_\lambda h_\lambda(x)\,. \tag{2.50}$$

The transversal part of the propagator $D^{T}_{\mu\nu,\rho\sigma}$, being divergent for even D, disappears from (2.49) due to the fact that the functions $h_{\mu\nu}$ are longitudinal. However, dealing with practical calculations, one should introduce the conformally invariant regularization (2.26). The regularized propagator (2.21) reads

$$\begin{aligned} D^{T^\varepsilon}_{\mu\nu,\rho\sigma}(x_{12}) &= \langle T^{\varepsilon}_{\mu\nu}(x_1)T^{\varepsilon}_{\rho\sigma}(x_2)\rangle \\ &= \tilde{C}_T\left[g_{\mu\rho}(x_{12})g_{\nu\sigma}(x_{12}) + g_{\mu\sigma}(x_{12})g_{\nu\rho}(x_{12}) - \frac{2}{D}\delta_{\mu\nu}\delta_{\rho\sigma}\right]\frac{1}{(x_{12}^2)^{D+\varepsilon}}\,. \end{aligned} \tag{2.51}$$

Correspondingly, the Green functions of the field $h_{\mu\nu}^{\rm long}$ in the contractions (2.49) should also be generalized by an addition of the anomalous correction to the dimension $l_h = D - l_T$:

$$l_h \to l_h^\varepsilon = D - l_T^\varepsilon = -\varepsilon \,. \tag{2.52}$$

Consider the expression (2.51) in the space of even dimension $D \geq 2$. Calculating its divergence, we find that

$$\partial_\nu^{x_1} D^{T^\varepsilon}_{\mu\nu,\rho\sigma}(x_{12}) = \varepsilon \tilde{C}_T[(D-1+\varepsilon)(D+1-\varepsilon)]^{-1}\bigg\{\partial_\mu\partial_\rho\partial_\sigma - \frac{D-1+\varepsilon}{2(D+2\varepsilon)} \times (\delta_{\mu\rho}\partial_\sigma + \delta_{\mu\sigma}\partial_\rho)\Box - \frac{1}{(D+\varepsilon)^2}\delta_{\rho\sigma}\partial_\mu\Box\bigg\}(x_{12})^{-D+1-\varepsilon} \,. \tag{2.53}$$

Defining the derivative of the propagator (2.21) as the limit

$$\partial_\nu\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle = \lim_{\varepsilon\to 0}\partial_\nu^{x_1}\langle T_{\mu\nu}^\varepsilon(x_1)T_{\rho\sigma}^\varepsilon(x_2)\rangle \,, \tag{2.54}$$

we get the Ward identity

$$\partial_\nu^{x_1}\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\rangle = C_T\bigg\{\partial_\mu\partial_\rho\partial_\sigma - \frac{D-1}{2D}(\delta_{\mu\rho}\partial_\sigma + \delta_{\mu\sigma}\partial_\rho)\Box - \frac{1}{D^2}\delta_{\rho\sigma}\partial_\mu\Box\bigg\}\Box^{\frac{D-2}{2}}\delta(x_{12}) \,,$$

where C_T is an independent parameter of the theory analogous to the central charge.

3. The Green Functions of Secondary Fields P_s^T

Consider a theory satisfying Eqs. (2.44). As was already mentioned above, the Green functions $G_{\mu\nu}^T$ and $\tilde{G}_{\mu\nu}^T$ coincide,

$$G_{\mu\nu}^T(xx_1\cdots x_m) = \langle T_{\mu\nu}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle = \tilde{G}^T(xx_1\cdots x_m) \,,$$

and are completely determined by the Ward identity. To calculate them it proves helpful to apply the method of conformal partial wave expansions. The latter are shown[1] to include two terms,

$$G_{\mu\nu}^T(xx_1\cdots x_m) = \sum_\sigma \int dy_1\, dy_2\, C_{1\mu\nu}^\sigma(xx_1y_1)D_\sigma(y_{12})G_{1\sigma}^T(y_2x_2\cdots x_m) + \sum_\sigma \int dy_1\, dy_2\, C_{2\mu\nu}^\sigma(xx_1y_1)D_\sigma(y_{12})G_{2\sigma}^T(y_2x_2\cdots x_m) \,. \tag{3.1}$$

All the notations are explained in detail in Refs. 1 and 2. Two independent sets of functions, $C_{1\mu\nu}^\sigma$ and $C_{2\mu\nu}^\sigma$, are found in the next section.

A general solution to the Ward identities contains, together with (3.1), the additional term

$$\sum_\sigma \int dy_1\, dy_2\, C^{\mathrm{tr},\sigma}_{\mu\nu}(xx_1y_1)\Delta^{-1}_\sigma(y_{12})G^{T^{\mathrm{tr}}}_{3\sigma}(y_2x_2\cdots x_m)\,, \tag{3.2}$$

where

$$\begin{aligned} C^{\mathrm{tr},\sigma}_{\mu\nu}(x_1x_2x_3) &= \langle P_\sigma(x_1)\varphi(x_2)(x_2)T^{\mathrm{tr}}_{\mu\nu}(x_3)\rangle\,, \\ \partial^{x_3}_\mu C^{\mathrm{tr},\sigma}_{\mu\nu}(x_1x_2x_3) &= 0\,. \end{aligned} \tag{3.3}$$

The poles of the kernel $G^{T^{\mathrm{tr}}}_{3\sigma}$ in the σ variable determine the Green functions of the fields R^T_s; see (2.7). The condition (2.44) is equivalent to the requirement

$$G^{T^{\mathrm{tr}}}_{3\sigma}(x_1\cdots x_m) = 0\,, \qquad \text{or} \qquad R^T_s(x) = 0\,. \tag{3.4}$$

Consider the kernels $G^T_{1\sigma}$ and $G^T_{2\sigma}$ of the expansion (3.1). According to Ref. 1 they are represented by

$$G_{1\sigma}(x_1\cdots x_m) = \int dx\, dy\, \tilde{C}^\sigma_{1\mu\nu}(x_1xy)G^T_{\mu\nu}(yxx_2\cdots x_m)\,, \tag{3.5}$$

$$G_{2\sigma}(x_1\cdots x_m) = \int dx\, dy\, \tilde{C}^\sigma_{2\mu\nu}(x_1xy)G^T_{\mu\nu}(yxx_2\cdots x_m)\,, \tag{3.6}$$

where $\tilde{C}^\sigma_{1\mu\nu}(x_1x_2x_3)$ and $\tilde{C}^\sigma_{2\mu\nu}(x_1x_2x_3)$ are independent invariant functions of the type $\langle P_\sigma(x_1)\varphi_{D-d_1}(x_2)h_{\mu\nu}(x_3)\rangle$, orthogonal to the function (3.3) (see Ref. 1):

$$\int dy_1\, dy_2\, \tilde{C}^{\sigma'}_{1\mu\nu}(x_1y_1y_2)C^{\mathrm{tr},\sigma}_{\mu\nu}(x_2y_1y_2) = \int dy_1\, dy_2\, \tilde{C}^{\sigma'}_{2\mu\nu}(x_1y_1y_2)C^{\mathrm{tr},\sigma}_{\mu\nu}(x_2y_1y_2) = 0$$

for all σ', σ. Due to transversality of the function $C^{\mathrm{tr},\sigma}_{\mu\nu}$ both of the functions $\tilde{C}^\sigma_{i\mu\nu}$, $i = 1, 2$, are longitudinal,

$$\tilde{C}^\sigma_{i\mu\nu}(x_1x_2x_3) = \langle P_\sigma(x_1)\varphi_{D-d_1}(x_2)h^{\mathrm{long}}_{i,\mu\nu}(x_3)\rangle\,, \qquad i = 1, 2\,,$$

and have the form

$$\begin{aligned} \tilde{C}^\sigma_{i\mu\nu}(x_1x_2x_3) &= \partial^{x_3}_\mu B^\sigma_{i\nu}(x_1x_2x_3) + \partial^{x_3}_\nu B^\sigma_{i\mu}(x_1x_2x_3) \\ &\quad - \frac{2}{D}\delta_{\mu\nu}\partial^{x_3}_\lambda B^\sigma_{i\lambda}(x_1x_2x_3)\,, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} B^\sigma_{i\mu}(x_1x_2x_3) &= B^l_{i\mu,\mu_1\cdots\mu_s}(x_1x_2x_3) \\ &= \Bigg\{\lambda^{x_3}_\mu(x_1x_2)\lambda^{x_1}_{\mu_1\cdots\mu_s}(x_3x_2) \\ &\quad + \tilde{\alpha}_i\frac{1}{x^2_{13}}\left[\sum_{k=1}^s g_{\mu\mu_k}(x_{13})\lambda^{x_1}_{\mu_1\cdots\hat{\mu}_k\cdots\mu_s}(x_3x_2) - \text{traces}\right]\Bigg\}\tilde{\Delta}^l_h(x_1x_2x_3)\,. \end{aligned} \tag{3.8}$$

Here we have used the standard notation

$$\tilde{\Delta}^l_h(x_1x_2x_3) = (x^2_{13})^{-\frac{l+d_1-s-2-D}{2}}(x^2_{23})^{-\frac{l+d_1-s+2-D}{2}}(x^2_{12})^{-\frac{l-d_1-s+2+D}{2}}, \tag{3.9}$$

$$\begin{aligned}\lambda^{x_3}_{\mu_1\cdots\mu_s}(x_1x_2) &= \lambda^{x_3}_{\mu_1}(x_1x_2)\cdots\lambda^{x_3}_{\mu_s}(x_1x_2) - \text{traces}\,,\\ \lambda^{x_3}_{\mu}(x_1x_2) &= \frac{(x_{21})_\mu}{x^2_{12}} - \frac{(x_{31})_\mu}{x^2_{13}}\,,\end{aligned} \tag{3.10}$$

where $\hat{\mu}$ means that the index μ is dropped and the function $g_{\mu\nu}(x_{13})$ is given by the formula (2.17). The coefficients $\tilde{\alpha}_i$ in (3.8) can be found from the orthogonality condition.[1] Their precise values are redundant in what follows.

The fields P^T_s correspond[1,2] (see also Ref. 11) to the poles of one of the kernels (3.5) and (3.6) [depending on the choice of the functions $C^\sigma_{i\mu\nu}$ in (3.1)]. Let this kernel be $G^l_{2\sigma}$. In Sec. 4 we demonstrate that its physical poles are exhausted by the poles at the points $l = d + s$. The other kernel, $G^T_{1\sigma}$, has no physical poles; see Sec. 4. The Green functions of the fields P^T_s are determined[1] (see also Ref. 2) by the equations

$$\langle P^T_s(x_1)\varphi_2(x_2)\cdots\varphi_m(x_m)\rangle = \Lambda^T_s \operatorname*{res}_{\sigma=\sigma_s} G^T_{2\sigma}(x_1\cdots x_m)\,, \tag{3.11}$$

where Λ^T_s are definite constants,[6,11]

$$\sigma_s = (l_s, s)\,, \qquad l_s = d_1 + s\,.$$

Using (3.6) and (3.7) we find that

$$\begin{aligned}\langle P^T_{\mu_1\cdots\mu_s}(x_1)\varphi_2(x_2)\cdots\varphi_m(x_m)\rangle &= -2\Lambda^T_s \operatorname*{res}_{l=d_1+s}\int dy_1\,dy_2\,B^l_{2\mu,\mu_1\cdots\mu_s}(x_1y_1y_2)\partial^{y_2}_\nu\\ &\quad\times\langle T_{\mu\nu}(y_2)\varphi_1(y_1)\varphi_2(x_2)\cdots\varphi_m(x_m)\rangle\,.\end{aligned} \tag{3.12}$$

Thus the Green functions of the fields P^T_s may be calculated directly from the Ward identities.

The technical subtleties in evaluation of the right hand sides of the equations of the type (3.12) are considered in detail in Ref. 2 on an example of the fields P^j_s and in Sec. 6 on an example of two-dimensional field theory. One can show that the final representation for the Green function $\langle P^T_s\cdots\rangle$ takes the form

$$\langle P^T_{\mu_1\cdots\mu_s}(x_1)\varphi_2(x_2)\cdots\varphi_m(x_m)\rangle = \hat{P}^T_{\mu_1\cdots\mu_s}(x,\partial)\langle\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,, \tag{3.13}$$

where $\hat{P}^T_s = \hat{P}^T_{\mu_1\ldots\mu_s}$ is the tensor differential operator of the rank $s+1$:

$$\hat{P}^T_s(x,\partial) = \hat{P}_{\mu_1\cdots\mu_s}(x_1\cdots x_m;\partial^{x_1},\ldots,\partial^{x_m})\,. \tag{3.14}$$

In the general case[d] $D > 2$ the expressions for these operators are quite cumbersome. As an illustration, we write down the formula for the simplest one: $\hat{P}^T_\mu = \hat{P}^T_s|_{s=1}$. In the condensed notation ($d_1 = d_2 = d$, $d_3 = d_4 = \Delta$) one has

$$\begin{aligned}
&\hat{P}_\mu(x, \partial^x)\langle\varphi(x_1)\varphi(x_2)\chi(x_3)\chi(x_4)\rangle \\
&= \int dx_5 \bigg\{ C^d_{\mu,\sigma}(x_1x_2|x_5)\partial^{x_2}_\sigma + \frac{d}{D}\partial^{x_2}_\sigma C^d_{\mu,\sigma}(x_1x_2|x_5) \\
&\quad + C^\Delta_{\mu,\sigma}(x_1x_3|x_5)\partial^{x_3}_\sigma + \frac{\Delta}{D}\partial^{x_3}_\sigma C^\Delta_{\mu,\sigma}(x_1x_3|x_5) \\
&\quad + C^\Delta_{\mu,\sigma}(x_1x_4|x_5)\partial^{x_4}_\sigma + \frac{\Delta}{D}\partial^{x_4}_\sigma C^\Delta_{\mu,\sigma}(x_1x_2|x_5) \bigg\} \langle\varphi(x_5)\varphi(x_2)\chi(x_3)\chi(x_4)\rangle\,,
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
C^d_{\mu,\sigma}(x_1x_2|x_5) \sim \bigg\{ &\left[\frac{1}{D+2} + \tilde{\alpha}_2\right] x^2_{12} g_{\mu\sigma}(x_{12}) \Box_{x_5} \delta(x_{15}) \\
&+ \frac{2}{D+2} x^2_{12} g_{\sigma\tau}(x_{12}) \partial^{x_5}_\mu \partial^{x_5}_\tau \delta(x_{15}) \\
&+ \left[\frac{2(D-2d)}{D+2} + 2\tilde{\alpha}_2(D-2-2d)\right] g_{\mu\sigma}(x_{12})(x_{12})_\tau \partial^{x_5}_\tau \delta(x_{15}) \\
&- \frac{2(d+1)}{D+2}(x_{12})_\mu g_{\sigma\tau}(x_{12}) \partial^{x_5}_\tau \delta(x_{15}) + \frac{4(d+1)}{D+2}(x_{12})_\sigma \partial^{x_5}_\mu \delta(x_{15}) \\
&+ \left[\frac{2d(4d+2-D)}{D+2} - 2\tilde{\alpha}_2 d(D-2-2d)\right] g_{\mu\sigma}(x_{12}) \delta(x_{15}) \\
&- \frac{4d(d+1)}{D+2} \delta_{\mu\sigma} \delta(x_{15}) \bigg\}\,.
\end{aligned}$$

To obtain (3.15) we use the same technical tricks as in the derivation of the differential operators $\hat{P}^j_s$; see Ref. 2. The expression (3.15) underlies the derivation of differential equations for the Green functions $\langle\varphi\varphi\varphi\varphi\rangle$ and $\langle\varphi\varphi\chi\chi\rangle$ obtained in Ref. 1 for the models defined by the equations

$$Q^T_\mu(x) = 0\,, \qquad Q^T_\mu(x) + \beta Q^j_\mu(x) = 0\,.$$

Equations (3.12) determine the Green functions of the fields P^T_s of the first generation:

$$T_{\mu\nu}(x_1)\varphi(x_2) = \sum_\sigma [P^T_s]\,.$$

[d]The case $D = 2$ is remarkable for its simplicity: the derivatives of the highest order in the operators (3.14) cancel, and the order of the operators becomes s. Their explicit expressions are quite simple and are presented in Sec. 6.

The fields of succeeding generations are treated similarly. They arise in the operator product expansions

$$T_{\mu\nu}(x_1)P^T_{s_1}(x_2) = \sum_s [P^{s_1}_s], \qquad T_{\mu\nu}(x_1)P^{s_2}_{s_1}(x_2) = \sum_s [P^{s_1 s_2}_s], \text{ etc.} \tag{3.16}$$

Denote the field of the kth generation by

$$P^{T(k)}_s(x) = P^{s_1 s_2 \cdots s_{k-1}}_s(x)\,. \tag{3.17}$$

The set of such fields spans the basis of the operator product expansions

$$T_{\mu_1\nu_1}(x_1)T_{\mu_2\nu_2}(x_2)\cdots T_{\mu_k\nu_k}(x_k)\varphi(x)\,. \tag{3.18}$$

The Green functions for any field (3.17) may be evaluated directly from the Ward identities and are determined by equations of the type (3.12):

$$\begin{aligned}\langle P^{T(k)}(x_1)\varphi_2(x_2)\cdots\varphi_m(x_m)\rangle = \Lambda^{(k)}_s \int dy_1\, dy_2\, B^{(k)l,\sigma_1\cdots\sigma_{s-1}}_{\mu,\sigma_s}(x_1 y_1 y_2)\partial^{y_2}_\nu \\ \times \langle T_{\mu\nu}(y_2)P^{T(k-1)}_s(y_1)\varphi(x_2)\cdots\varphi_m(x_m)\rangle\,.\end{aligned} \tag{3.19}$$

The formulation of the conditions leading to the appearance of the entire set of secondary fields P^T_s in the operator product expansion $T_{\mu\nu}\varphi$ did not belong to the list of problems we intended to solve in this article. One can show that for $D > 2$ the solution depends on the behavior of the product $T_{\mu\nu}(x_1)T_{\rho\lambda}(x_2)$ at neighboring points. As shown in Refs. 1 and 12, the expansion may have three kinds of operator contributions,

$$T_{\mu\nu}(x_1)T_{\rho\lambda}(x_2) = [C_T] + [P_T] + [T_{\mu\nu}(x)] + \cdots\,, \tag{3.20}$$

where $[C_T]$ is the c-number contribution and P_T is the field of scale dimension $l_P = D - 2$:

$$P_T = P^{D-2}_T(x)\,. \tag{3.21}$$

Note that the field $P_T(x)$ can be easily shown to be a primary field, while the field $T_{\mu\nu}(x)$ belongs to a family of secondary fields generated P_T. The c-number contribution to (3.20) is absent for odd D.[1,12]

In the work to follow we shall show that the theories which do not imply the introduction of the field $P_T(x)$ have just several fields P^T_s depending on the choice of anomalous contributions to the third term of (3.20). The theories with the field $P_T(x)$ being present comprise the complete spectrum of fields P^T_s for $s = 1, 2, \ldots$. (The field P^T_μ of dimension $l_1 = d + 1$ may exist only under a definite choice of the anomalous contribution to $[T_{\mu\nu}]$; see Ref. 1). In such theories [with $P_T(x) \neq 0$] the formalism developed here should be slightly completed. This will also be done in the other publication.

All the above remains valid for the theories in any space dimension $D \geq 3$. The case $D = 2$ is exceptional. The expansion of the type (3.20) has the form

$$T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) = [C_T] + [T_{\mu\nu}] + \cdots\,, \quad D = 2\,, \tag{3.22}$$

where C_T is the central charge. The fields P_s^T, $s = 2, 3, \ldots$, exist for any C_T, the case of $C_T = 0$ included, and represent the superpositions of secondary fields introduced in Ref. 3.

4. The Green Functions $\langle P_s^T \varphi T_{\mu\nu}\rangle$ and the Anomalous Ward Identities for the Secondary Fields

Consider the Green functions

$$G^s_{\mu\nu}(x_1x_2x_3) = \langle P_s^T(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle\,, \tag{4.1}$$

where the dimensions of the fields φ and P_s^T are d and $d_s = d + s$. In order to find its coordinate dependence, let us consider the conformally invariant expression for the Green function of the field P_s^l for $l \neq d + s$, and then take the limit $l = d + s$. The general conformally invariant expression has the form

$$\begin{aligned}\langle P^l_{\mu_1\cdots\mu_s}(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = A_1 {C'}^l_{1\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) + A_2 {C'}^l_{2\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3)\\ + A_3 {C'}^l_{3\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3)\,,\end{aligned} \tag{4.2}$$

where A_1, A_2, A_3 are arbitrary constants,

$${C'}^l_{1\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) = \lambda^{x_3}_{\mu\nu}(x_1x_2)\lambda^{x_1}_{\mu_1\cdots\mu_s}(x_3x_2)\Delta^{ld}_T(x_1x_2x_3)\,, \tag{4.3}$$

$$\begin{aligned}{C'}^l_{2\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) = \frac{1}{x^2_{13}}\Bigg\{\sum_{k=1}^{s}\lambda^{x_1}_{\mu_1\cdots\hat{\mu}_k\cdots\mu_s}(x_3x_2)\Bigg[g_{\mu\mu_k}(x_{13})\lambda^{x_3}_{\nu}(x_1x_2)\\ + g_{\nu\mu_k}(x_{13})\lambda^{x_3}_{\mu}(x_1x_2) + \frac{2}{D}\delta_{\mu\nu}\frac{x^2_{12}}{x^2_{13}}\lambda^{x_1}_{\mu_k}(x_2x_3)\Bigg]\\ - \text{traces in } \mu_1\cdots\mu_s\Bigg\}\Delta^{ld}_T(x_1x_2x_3)\,,\end{aligned} \tag{4.4}$$

$$\begin{aligned}{C'}^l_{3\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) = \frac{1}{(x^2_{13})^2}\Bigg(\sum_{k,r=1}^{s}\lambda^{x_1}_{\mu_1\cdots\hat{\mu}_k\cdots\hat{\mu}_r\cdots\mu_s}(x_3x_2)\\ \times\left[g_{\mu_k\mu}(x_{13})g_{\mu_r\nu}(x_{13}) - \frac{1}{D}\delta_{\mu\nu}\delta_{\mu_r\mu_k}\right]\\ - \text{traces in } \mu_1\cdots\mu_s\Bigg)\Delta^{ld}_T(x_1x_2x_3)\,,\end{aligned} \tag{4.5}$$

$$\Delta^{ld}_T(x_1x_2x_3) = (x^2_{12})^{-\frac{l+d-s-D+2}{2}}(x^2_{13})^{-\frac{l-d-s+D-2}{2}}(x^2_{23})^{-\frac{d+s-l+D-2}{2}}\,.$$

As in the case of the current, each term of (4.2) is singular in the limit

$$l \to d^s_T = d + s\,. \tag{4.6}$$

To prove this, one can perform the contractions in indices μ, μ_k or ν, μ_k, $k = 1, \ldots, s$, in (4.2), or calculate the derivative by $\partial_\nu^{x_3}$. As a result of cumbersome calculations one obtains

$$\partial_\nu^{x_3}\langle P^l_{\mu_1\cdots\mu_s}(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = \Bigg\{A\lambda^{x_1}_{\mu_1\cdots\mu_s}(x_3x_2)\lambda^{x_3}_\mu(x_1x_2) + B\frac{1}{x_{13}^2}\sum_{k=1}^{s}[g_{\mu\mu_k} \times (x_{13})\lambda^{x_1}_{\mu_1\cdots\hat{\mu}_k\cdots\mu_s}(x_3x_2) - \text{traces in } \mu_1\cdots\mu_s]\Bigg\} \times (x_{12}^2)^{-\frac{l+d-s-D}{2}}(x_{13}^2)^{-\frac{l-d-s+D}{2}}(x_{23}^2)^{-\frac{d+s-l+D}{2}}, \tag{4.7}$$

where

$$\begin{aligned} A = A(l,s) &= \frac{1}{D}[(D-1)(l-d)+s]A_1 - s\frac{D-2}{D}(D-l+d+s)A_2\,, \\ B = B(l,s) &= -\frac{1}{D}A_1 + \left(l-d-\frac{2s}{D}\right)A_2 + (s-1)(l-d-s-D+2)A_3\,. \end{aligned} \tag{4.8}$$

The expression (4.7) contains factors

$$(x_{13}^2)^{-\frac{D+\varepsilon}{2}}(x_{23}^2)^{-\frac{D-\varepsilon}{2}}, \qquad \varepsilon = l-d-s\,, \tag{4.9}$$

singular in the limit

$$\varepsilon \to 0\,.$$

Setting

$$A_1 = D(D-2)A_3\,, \qquad A_2 = DA_3 \tag{4.10}$$

we get

$$A(l,s)|_{\varepsilon\to 0} \sim B(l,s)|_{\varepsilon\to 0} \sim \varepsilon\,.$$

If the condition (4.10) is satisfied, the r.h.s. contains the ambiguity $0\times\infty$,

$$\varepsilon(x_{13}^2)^{-\frac{D+\varepsilon}{2}}(x_{23}^2)^{-\frac{D+\varepsilon}{2}},$$

which is revealed to produce quasilocal terms. Though each term of the sum (4.2) has poles

$$\text{res}_{l=d+s}C'^{l}_{i\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) \neq 0\,,$$

their combination

$$D(D-2)C'^{l}_{1\mu\nu,\mu_1\cdots\mu_s} + DC'^{l}_{2\mu\nu,\mu_1\cdots\mu_s} + C'^{l}_{3\mu\nu,\mu_1\cdots\mu_s} \tag{4.11}$$

is finite. Indeed, up to the terms $\sim \varepsilon$ the latter may be put into a form of the function explicitly regular at the points (4.6):

$$\begin{aligned}
&\langle P_s^{d+s}(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle_{\rm reg} \\
&= \langle P_s^{d+s+\varepsilon}(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle \\
&= g_T^s\big\{(x_{23}^2)^{-\frac{D-2-\varepsilon}{2}} R_{\mu\nu}(\partial^{x_3})[(x_{13}^2)^{-\frac{D-2+\varepsilon}{2}}\lambda^{x_1}_{\mu_1\cdots\mu_s}(x_3x_2)] + O(\varepsilon)\big\}(x_{12}^2)^{-d+\frac{D-2}{2}}\,,
\end{aligned} \tag{4.12}$$

where g_T^s is a constant,

$$\begin{aligned}
R_{\mu\nu}(\partial^x) &= \left[\overleftrightarrow{\partial}^x{}_\mu \overleftrightarrow{\partial}^x{}_\nu - \tfrac{2}{D-2}(\overrightarrow{\partial}^x{}_\mu \overleftarrow{\partial}^x{}_\nu + \overrightarrow{\partial}^x{}_\nu \overleftarrow{\partial}^x{}_\mu) - \text{trace}\right], \\
\overleftrightarrow{\partial}_\mu &= \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu\,.
\end{aligned} \tag{4.13}$$

Let us define the Green functions (4.1) as the limit of the expression (4.12):

$$\langle P_s^T(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = \lim_{\varepsilon\to 0}\langle P_s^{d+s}(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle_{\rm reg}\,. \tag{4.14}$$

To resolve ambiguities $0\times\infty$ due to terms $\sim O(\varepsilon)$ is performed in the same manner as for the Green functions of the current in Ref. 10. Thus, for the Green function (4.1) we get

$$\begin{aligned}
\langle P_s^T(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = g_T^s(x_{12}^2)^{-\frac{D-2}{2}}\bigg\{(x_{23}^2)^{-\frac{D-2}{2}}\bigg[&\overleftrightarrow{\partial}_\mu^{x_3}\overleftrightarrow{\partial}_\nu^{x_3} \\
&- \frac{2}{D-2}\Big(\overleftarrow{\partial}_\mu^{x_3}\overrightarrow{\partial}_\nu^{x_3} + \overleftarrow{\partial}_\nu^{x_3}\overrightarrow{\partial}_\mu^{x_3}\Big) - \text{trace}\bigg] \\
&\times (x_{13}^2)^{-\frac{D-2}{2}}\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2)\bigg\}\langle\varphi(x_1)\varphi(x_2)\rangle + \cdots\,,
\end{aligned} \tag{4.15}$$

where g_T^s is the coupling constant, and the dots stand for quasilocal terms. In general, these terms depend on two parameters; see Ref. 10. For $s=1$ they read

$$f_{1T}^{s=1}\left[\delta_{\mu\mu_1}\left(\partial_\nu^{x_3} + 2D\frac{(x_{12})_\nu}{x_{12}^2}\right)\delta(x_{13}) - \text{trace in } \mu,\nu\right](x_{12}^2)^{-d} + (\mu\leftrightarrow\nu)\,.$$

The Green functions (4.15) satisfy the Ward identities

$$\partial_\nu^{x_3}\langle P^{d+s}_{\mu_1\cdots\mu_s}(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = H^T_{s,s}(\partial^{x_3},\delta(x_{13}),\partial^{x_1})\langle\varphi(x_1)\varphi(x_2)\rangle\,. \tag{4.16}$$

Due to the complexity of the operator $H^T_{s,s}$ we present its explicit form only for $s=1$ (and for $s=2$ in Ref. 10):

$$\begin{aligned}
H^T_{1,1} \sim{}& \frac{D-2}{D}\partial_{\mu_1}^{x_3}\partial_\mu^{x_3}\delta(x_{13}) + \delta_{\mu\mu_1}\Box_{x_3}\delta(x_{13}) \\
&- \frac{D}{d}\left[\partial_{\mu_1}^{x_3}\delta(x_{13})\partial_\mu^{x_1} + \delta_{\mu\mu_1}\partial_\rho^{x_3}\delta(x_{13})\partial_\rho^{x_1} - \frac{2}{D}\partial_\mu^{x_3}\delta(x_{13})\partial_{\mu_1}^{x_1}\right].
\end{aligned} \tag{4.17}$$

The secondary fields P_s^T have anomalous commutators with the energy–momentum tensor,

$$\delta(x^0 - y^0)[T_{0\mu}, P_s(y)] = \delta^{(D)}(x-y)\partial_\mu P_s(y) + \cdots , \tag{4.18}$$

where the dots stand for contributions of fields

$$P_{s'}(x) , \qquad s' = 0, 1, \ldots, s-1 ; \tag{4.19}$$

see Ref. 2 for more details. Correspondingly, the Green functions of the fields P_s

$$G^{P_s}_{\mu\nu,\mu_1\cdots\mu_s}(x, x_1, \ldots, x_m) = \langle T_{\mu\nu}(x) P_s(x_1)\varphi_2(x_2)\cdots\varphi_m(x_m)\rangle \tag{4.20}$$

also satisfy anomalous Ward identities:

$$\begin{aligned}\partial^x_\nu G^{P_s}_{\mu\nu,\mu_1\cdots\mu_s}(x, x_1\cdots x_m) = -\Bigg[&\sum_{k=1}^m \delta(x-x_k)\partial^{x_k}_\mu - \sum_{k=2}^m \frac{d_k}{D}\partial^x_\mu \delta(x-x_k) \\ &+ H^T_{s,0}(\partial^x, \delta(x-x_1), \partial^{x_1})\Bigg]\langle P_s(x_1)\varphi_2(x_2)\cdots\varphi_m(x_m)\rangle \\ &+ \sum_{k=1}^{s-1} H^T_{s,k}(\partial^x, \delta(x-x_1), \partial^{x_1})\langle P_{s-k}(x_1) \\ &\times \varphi_2(x_2)\cdots\varphi_m(x_m)\rangle ,\end{aligned} \tag{4.21}$$

where $H^T_{s,k}$ are the differential operators, made up of the terms

$$(\partial^x)^{k+1-r}\delta(x-x_1)(\partial^{x_1})^r , \qquad r = 0, 1, \ldots, k . \tag{4.22}$$

The explicit form of these operators can be found, analyzing Ward identities for the Green functions

$$\langle T_{\mu\nu}(x) P_s(x_1) P_{s'}(x_2)\rangle , \qquad s' = 0, 1, \ldots, s-1 , \tag{4.23}$$

in a similar way as was done for the functions (4.1) and their Ward identities. The expression for $H^T_{1,1}$ is given by (4.17). In the general case these expressions have very cumbersome, until the proper variables and notations are found.

Let us return to the discussion of conformal partial wave expansions (3.1). As already mentioned before, the fields P_s^T correspond to the poles of one of the terms, while the other term cannot have any physical poles. The reason lies in the properties of the functions $C^\sigma_{1\mu\nu}$ and $C^\sigma_{2\mu\nu}$. General expressions for both functions have the forms (4.2). There are three different linearly independent superpositions of the type (4.2). In the limit $l = d + s$ each function (4.3)–(4.5) is nonintegrable. We have shown that there exists a definite superposition which is regular in this limit. Choose for the function $C^\sigma_{2\mu\nu}$ a superposition (4.2) coincident (up to a factor) in the limit $l = d + s$ with the function (4.15):

$$\lim_{l=d+s} C^\sigma_{2\mu\nu}(x_1x_2x_3) = N_s\langle P_s^T(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle . \tag{4.24}$$

For the function $C^{\sigma}_{1\mu\nu}$ one can choose another independent superposition, for example

$$C^{\sigma}_{1\mu\nu}(x_1x_2x_3) = B_1(\sigma)C'^{\sigma}_{1\mu\nu}(x_1x_2x_3) + B_2(\sigma)C'^{\sigma}_{2\mu\nu}(x_1x_2x_3)\,, \tag{4.25}$$

where $B_1(\sigma)$ and $B_2(\sigma)$ are arbitrary functions of the variables l, s. Finally, for the third independent superposition one can choose the transversal function $C^{\mathrm{tr},\sigma}_{\mu\nu}$:

$$C^{\mathrm{tr},\sigma}_{\mu\nu} = A^{\mathrm{tr}}_1(\sigma)C'^{\sigma}_{1\mu\nu} + A^{\mathrm{tr}}_2(\sigma)C'^{\sigma}_{2\mu\nu} + A^{\mathrm{tr}}_3(\sigma)C'^{\sigma}_{3\mu\nu}\,, \tag{4.26}$$

where $A^{\mathrm{tr}}_i(\sigma)$, $i = 1, 2, 3$, are the functions satisfying the equations [see (4.8)]

$$\begin{gathered}\frac{1}{D}[(D-1)(l-d)+s]A^{\mathrm{tr}}_1(\sigma) - s\frac{D-2}{D}(D-l+d+s)A^{\mathrm{tr}}_2(\sigma) = 0\,,\\ -\frac{1}{D}A^{\mathrm{tr}}_1(\sigma) + \left(l-d-\frac{2s}{D}\right)A^{\mathrm{tr}}_2(\sigma) + (s-1)(l-d-s-D+2)A^{\mathrm{tr}}_3(\sigma) = 0\,.\end{gathered} \tag{4.27}$$

The function (4.26) enters the third term (3.2) prohibited by Eqs. (2.44) and has nothing to do with the expansion (3.1). Note that at the points $l = d + s$ it differs from the function $C^{\sigma}_{2\mu\nu}$ by quasilocal terms.

Note that the function $C^{\sigma}_{1\mu\nu}$ is singular at points $l = d + s$, so the first term under the sign $\sum_\sigma$ in (3.1) has poles at $\sigma = \sigma_s$. However, the residues

$$\operatorname*{res}_{\sigma=\sigma_s} \int dy_1\, dy_2\, C^{\sigma}_{1\mu\nu}(xx_1y_1)D^{-1}_{\sigma}(y_{12})G^{T}_{1\sigma}(y_2x_2\cdots x_m)$$

do not correspond to any physical fields since

$$\operatorname*{res}_{\sigma=\sigma_s} C^{\sigma}_{1\mu\nu}(x_1x_2x_3) = \text{quasilocal terms}\,. \tag{4.28}$$

The other poles, i.e. those different from $\sigma = \sigma_s$, cannot appear in the first term of the expansion (3.1) either. Indeed, suppose that some field $\Phi_i = \Phi^{s_i}_{l_0}$ with $l_0 \neq d + s_i$ has appeared from the second term. The Ward identity for the function $\langle\Phi_i\varphi T_{\mu\nu}\rangle$ has the form

$$\partial^{x_3}_{\mu}\langle\Phi_i(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = \{-\delta(x_{13})\partial^{x_1}_{\nu} - \delta(x_{23})\partial^{x_2}_{\nu} + \cdots\}\langle\Phi_i(x_1)\varphi(x_2)\rangle = 0\,, \tag{4.29}$$

because the conformal fields satisfy the orthogonality condition

$$\langle\Phi_i(x_1)\varphi(x_2)\rangle = 0\,, \qquad \text{if } l_0 \neq d\,,\ s_i = 0\,.$$

Hence it follows that the Green function $\langle\Phi_i\varphi T_{\mu\nu}\rangle$ either is transverse or vanishes. However, the transversal functions cannot appear due to Eq. (2.44), and thus

$$\langle\Phi_i(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = 0\,, \qquad \text{if } l_i \neq d\,,\ s_i \neq 0\,. \tag{4.30}$$

The fields P^T_s constitute the exception to this rule, because their Green function satisfies the anomalous Ward identities whose right hand sides contain the terms $\sim \langle\varphi(x_1)\varphi(x_2)\rangle$ instead of the "usual" terms taken into account in (4.29). The latter is possible only for the fields of dimensions $l_i = d + s$.

So the integrand in the first term of (3.1) contains only kinematic poles of the type (4.28), while the physical poles are exhausted by the fields P_s^T.

5. Solution of Ward Identities in Two-Dimensional Field Theory

Two-dimensional space is specific by the property that both the current and the energy–momentum tensor are irreducible fields. When $D = 2$, there is no problem in the decoupling of Euclidean transversal field $\tilde{T}^{\rm tr}_{\mu\nu}(x)$, just because this field is zero. Gravitational interaction in this case is trivial and has no influence on the dynamics of matter fields. The representation of conformal group,[e] given by the transformation law

$$T_{\mu\nu}(x) \xrightarrow{R} T'_{\mu\nu}(x) = \frac{1}{(x^2)^2} g_{\mu\rho}(x) g_{\nu\sigma}(x) T_{\rho\sigma}(Rx)\,, \tag{5.1}$$

is irreducible. The energy–momentum tensor, being the traceless symmetric tensor, has two independent components:

$$T_{11} + T_{22} = 0\,, \qquad T_{12} = T_{21}\,. \tag{5.2}$$

The transversality condition

$$\partial_\mu \tilde{T}^{\rm tr}_{\mu\nu}(x) = 0$$

is equivalent to a pair of equations on these components, having the unique solution

$$\tilde{T}^{\rm tr}_{\mu\nu}(x) = 0\,. \tag{5.3}$$

The projection operator introduced in Sec. 2 to utilize the decoupling of the subspace $M_T^{\rm tr}$ also vanishes for $D = 2$, while the longitudinal projector $P_{\mu\nu,\rho\sigma}{}^{\rm long}(\frac{\partial}{\partial x})$ is unity:

$$P^{\rm tr}_{\mu\nu,\rho\sigma}\left(\frac{\partial}{\partial x}\right) = 0\,, \qquad P^{\rm long}_{\mu\nu,\rho\sigma}\left(\frac{\partial}{\partial x}\right) = I_{\mu\nu,\rho\sigma}\,, \qquad \text{for } D = 2\,. \tag{5.4}$$

Thus any traceless symmetric tensor $T_{\mu\nu}(x)$ is longitudinal,

$$T_{\mu\nu}(x) = P^{\rm long}_{\mu\nu,\rho\sigma}\left(\frac{\partial}{\partial x}\right) T_{\rho\sigma}(x) = T^{\rm long}_{\mu\nu}(x)\,,$$

and may be represented in the form

$$T_{\mu\nu}(x) = \partial_\mu \tilde{T}_\nu(x) + \partial_\nu \tilde{T}_\mu(x) - \delta_{\mu\nu}\partial_\lambda \tilde{T}_\lambda(x)\,, \tag{5.5}$$

$$\tilde{T}_\mu = \frac{\partial_\nu}{\Box} T_{\mu\nu}(x) = \frac{1}{\Box} T_\mu(x)\,, \tag{5.6}$$

where $T_\mu(x) = \partial_\nu T_{\mu\nu}(x)$ is the conformal vector of scale dimension $D + 1$. Thus the irreducible representation, given by the transformation law (5.1), is the analog of the representation $\tilde{Q}_T$, which corresponds for $D > 2$ to the models where the gravitation is neglected.

[e] The six-parameter conformal group is assumed.

From the above it is clear that the Green functions

$$\langle T_{\mu\nu}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,, \qquad \langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle \tag{5.7}$$

are uniquely determined by Ward identities. For the case of $D > 2$ this property is proved for the conformal theories satisfying the condition (2.44), which fixes the realization of the representation $\tilde{Q}_T$ and simultaneously drops the gravitational interaction. We have already mentioned the similarity between such theories and two-dimensional models. In the following sections we expand this analogy to a greater extent. For this reason, in the present section we keep the component form of Ward identities (though the complex variables are more useful).

The conformally invariant solution to Ward identities is given by Eqs. (5.5) and (5.6). Consider the Ward identities (2.2) for the Green functions (5.7) for $D = 2$:

$$\begin{aligned}\partial_\nu^x\langle T_{\mu\nu}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle = -\left[\sum_{k=1}^m \delta(x-x_k)\partial_\mu^{x_k} - \frac{1}{2}\partial_\mu^x\sum_{k=1}^m d_k\delta(x-x_k)\right]\\ \times\langle\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,,\end{aligned} \tag{5.8}$$

where d_k are scale dimensions of the scalar fields φ_k, $k = 1,\ldots,m$. The r.h.s. represents the Green functions for the vector $T_\mu(x)$:

$$\langle T_\mu(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,.$$

Using Eqs. (5.5) and (5.6), and the relation

$$\frac{1}{\Box}\delta(x) = -\frac{1}{4\pi}\ln x^2\,, \tag{5.9}$$

we find that

$$\begin{aligned}&\langle T_{\mu\nu}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\\ &\quad= \frac{1}{2\pi}\Bigg\{\sum_{k=1}^m\frac{1}{(x-x_k)^2}[(x-x_k)_\mu\partial_\nu^{x_k} + (x-x_k)_\nu\partial_\mu^{x_k} - \delta_{\mu\nu}(x-x_k)_\lambda\partial_\lambda^{x_k}]\\ &\qquad - \sum_{k=1}^m\frac{d_k}{(x-x_k)^2}g_{\mu\nu}(x-x_k)\Bigg\}\langle\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,.\end{aligned} \tag{5.10}$$

The anomalous Ward identity considered in Ref. 1 takes the form for $D = 2$

$$\begin{aligned}&\partial_\mu^{x_1}\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle\\ &\quad= -\left\{\delta(x_{13})\partial_\nu^{x_3} + \delta(x_{14})\partial_\nu^{x_4} + \delta(x_{12})\partial_\nu^{x_2} - \frac{d}{2}\partial_\nu^{x_1}[\delta(x_{13})+\delta(x_{14})]\right\}\\ &\qquad\times\langle T_{\rho\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle + \partial_\rho^{x_1}\delta(x_{12})\langle T_{\nu\sigma}(x_2)\varphi(x_3)\varphi(x_4)\rangle\\ &\qquad + \partial_\sigma^{x_1}\delta(x_{12})\langle T_{\nu\rho}(x_2)\varphi(x_3)\varphi(x_4)\rangle - \delta_{\rho\sigma}\partial_\lambda^{x_1}\delta(x_{12})\langle T_{\nu\lambda}(x_2)\varphi(x_3)\varphi(x_4)\rangle\\ &\qquad - \frac{1}{24\pi}C\Bigg\{\partial_\nu^{x_1}\left(\partial_\rho^{x_1}\partial_\sigma^{x_1} - \frac{1}{2}\delta_{\rho\sigma}\Box_{x_1}\right)\delta(x_{12})\\ &\qquad - \frac{1}{4}(\delta_{\nu\rho}\partial_\sigma^{x_1} + \delta_{\nu\sigma}\partial_\rho^{x_1} - \delta_{\rho\sigma}\partial_\nu^{x_1})\Box_{x_1}\delta(x_{12})\Bigg\}\langle\varphi(x_3)\varphi(x_4)\rangle\,,\end{aligned} \tag{5.11}$$

where C is the central charge. Its solution may also be easily written using Eqs. (5.5), (5.6) and (5.9).

For the sake of convenience in juxtaposition of some of the D-dimensional theory results with those of known two-dimensional models, let us list several formulas concerning the transition to complex variables for $D = 2$:

$$x^{\pm} = x^1 \pm ix^2 , \qquad \partial_{\pm} = \frac{1}{2}(\partial_1 \mp i\partial_2) .$$

Any traceless symmetric tensor in two-dimensional space has two independent components. Define the complex components of the tensor $V_{\mu_1 \cdots \mu_s}$ by the relations

$$V_{\pm} = 2^{s-2} \left(V_{\underbrace{11\cdots1}_{s}} \mp iV_{\underbrace{11\cdots12}_{s-1}} \right) . \tag{5.12}$$

The contraction of a pair of traceless symmetric tensors V and W has the form

$$2^{s-2} V_{\mu_1 \cdots \mu_s} W_{\mu_1 \cdots \mu_s} = V_+ W_- + V_- W_+ . \tag{5.13}$$

In particular, we will use complex components of the tensor fields P_s, with dimensions $d + s$:

$$P_s^T(x) = P^T_{\mu_1 \cdots \mu_s}(x) .$$

They read

$$P_s^T(x) = P_{\pm}^{d+s}(x) = 2^{s-2} \left(P^{d+s}_{\underbrace{11\cdots1}_{s}} \mp iP^{d+s}_{\underbrace{11\cdots12}_{s-1}} \right) . \tag{5.14}$$

Denote the components of the tensor

$$\lambda_s^{x_1}(x_2x_3) = \lambda^{x_1}_{\mu_1 \cdots \mu_s}(x_2x_3)$$

as $\lambda^{x_1}_{s\pm}(x_2x_3)$. One can show that

$$\lambda^{x_1}_{s\pm}(x_2x_3) = \frac{1}{2^{s-1}} [\lambda^{x_1}_{\pm}(x_2x_3)]^s = \frac{1}{2^{s-1}} \left(\frac{x^{\pm}_{32}}{x^{\pm}_{12} x^{\pm}_{13}} \right)^s , \tag{5.15}$$

where

$$\lambda^{x_1}_{\pm}(x_2x_3) = \lambda^{x_1}_1(x_2x_3) \mp i\lambda^{x_1}_2(x_2x_3) = \frac{x^{\pm}_{32}}{x^{\pm}_{12} x^{\pm}_{13}} . \tag{5.16}$$

The components $(1/x^2)g_{\pm}(x)$ of the tensor $(1/x^2)g_{\mu\nu}(x)$ are

$$\frac{1}{x^2} g_{\pm}(x) = \frac{1}{x^2} [g_{11}(x) \mp ig_{12}(x)] = -\frac{1}{(x^{\pm})^2} . \tag{5.17}$$

Let us rewrite the Ward identities using the complex variables. Introduce $T_{\pm}$ components of the energy–momentum tensor:

$$T_{\pm}(x) = T_{11}(x) \mp iT_{12}(x) . \tag{5.18}$$

The Ward identities have the forms

$$\partial_{\mp}^{x}\langle T_{\pm}(x)\varphi(x_1)\cdots\varphi_m(x_m)\rangle$$
$$= -\left\{\sum_{k=1}^{m}\delta(x-x_k)\partial_{\pm}^{x_k} - \frac{1}{2}\partial_{\pm}^{x}\sum_{k=1}^{m}d_k\delta(x-x_k)\right\}\langle\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,, \quad (5.19)$$

$$\partial_{\mp}^{x_1}\langle T_{\pm}(x_1)T_{\pm}(x_2)\varphi(x_3)\varphi(x_4)\rangle$$
$$= -\left\{\delta(x_{13})\partial_{\pm}^{x_3} + \delta(x_{14})\partial_{\pm}^{x_4} - \frac{d}{2}\partial_{\pm}^{x_1}[\delta(x_{13})+\delta(x_{14})]\right\}$$
$$\times\langle T_{\pm}(x_2)\varphi(x_3)\varphi(x_4)\rangle - [\delta(x_{12})\partial_{\pm}^{x_2} - 2\partial_{\pm}^{x_1}\delta(x_{12})]$$
$$\times\langle T_{\pm}(x_2)\varphi(x_3)\varphi(x_4)\rangle - \frac{C}{12\pi}\partial_{\pm}^{x_1}\partial_{\pm}^{x_1}\partial_{\pm}^{x_1}\delta(x_{12})\langle\varphi(x_3)\varphi(x_4)\rangle\,. \quad (5.20)$$

Equations (5.5) and (5.6) take the forms

$$T_{\pm}(x) = 4\partial_{\pm}\tilde{T}_{\pm}(x)\,, \qquad \Box\tilde{T}_{\pm}(x) = \partial_{\mp}T_{\pm}(x)\,,$$

where

$$\tilde{T}_{\pm}(x) = \frac{1}{2}[\tilde{T}_1(x) \mp i\tilde{T}_2(x)]\,.$$

The solution of the Ward identities (5.19) and (5.20) reads, in complex variables,

$$\langle T_{+}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle$$
$$= \frac{1}{2\pi}\left\{\sum_{k=1}^{m}\frac{d_k}{(x^{\pm}-x_k^{\pm})^2} + \sum_{k=1}^{m}\frac{2}{(x^{\pm}-x_k^{\pm})}\partial_{\pm}^{x_k}\right\}\langle\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,, \quad (5.21)$$

$$\langle T_{\pm}(x_1)T_{\pm}(x_2)\varphi(x_3)\varphi(x_4)\rangle$$
$$= \frac{1}{2\pi}\left\{\frac{4}{(x_{12}^{\pm})^2} + \frac{d}{(x_{13}^{\pm})^2} + \frac{d}{(x_{14}^{\pm})^2} + 2\sum_{k=2}^{4}\frac{1}{x_{1k}^{\pm}}\partial_{\pm}^{x_k}\right\}\langle T_{\pm}(x_2)\varphi(x_3)\varphi(x_4)\rangle$$
$$+ \frac{C}{8\pi^2}\frac{1}{(x_{12}^{\pm})^4}\langle\varphi(x_1)\varphi(x_2)\rangle\,. \quad (5.22)$$

6. The Fields P_s^T in Two-Dimensional Models

Let $\varphi(x)$ be the neutral scalar field of dimension d in two-dimensional space. Consider the secondary fields P_s^T. When $D = 2$, each of the latter fields has a pair of independent components [see (5.14)]:

$$P_s^T(x) = P_{s\pm}(x) = P_{\pm}^{d+s}(x)\,.$$

The operator expansion of the product $T_{\mu\nu}(x_1)\varphi(x_2)$ may be written in the form

$$T_{\pm}(x_1)\varphi(x_2) = [\varphi] + \sum_{s=2}^{\infty}[P_{s\pm}]\,. \quad (6.1)$$

The Euclidean averages of the fields $P_{s\pm}$

$$\langle P_{s\pm}\varphi T\rangle\,, \qquad \langle P_{s\pm}\varphi_1\cdots\varphi_m\rangle\,,$$

where $\varphi_1\cdots\varphi_m$ are two-dimensional scalar fields with dimensions d_m, may be found using the Ward identities for the Green functions

$$G^T_{\pm}(xx_1\cdots x_m) = \langle T_{\pm}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,, \tag{6.2}$$

$$G^T_{\pm\pm}(x_1x_2x_3x_4) = \langle T_{\pm}(x_1)\varphi(x_2)T_{\pm}(x_3)\varphi(x_4)\rangle\,. \tag{6.3}$$

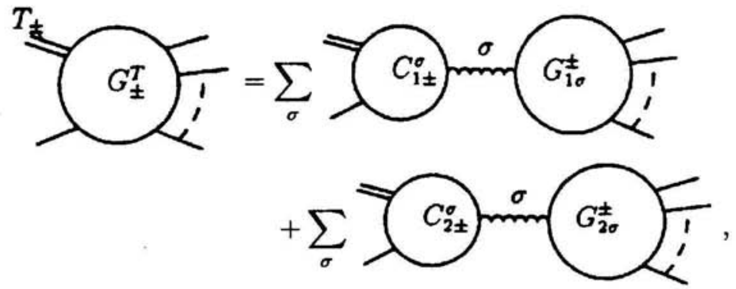
As in the previous sections, it is useful to apply partial wave expansions of the Green functions (6.2) and (6.3). In the general case $D > 2$ each of these expansions includes three terms (see Sec. 3 and Ref. 1), since the functions $C^{\sigma}_{\mu\nu}$ contain three independent structures, (4.3)–(4.5). The $D = 2$ case is an exception. One can show that[f]

$$C'^{\,l}_{2\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) = C'^{\,l}_{3\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) \qquad \text{for } D = 2\,, \tag{6.4}$$

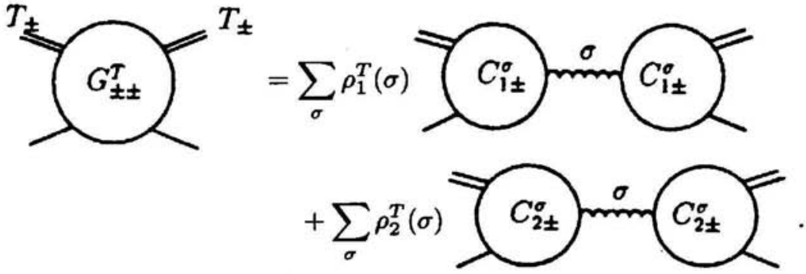
so that only a pair of arbitrary coefficients A_1, A_2 survives out of three A_1, A_2, A_3 entering Eq. (4.2). As a consequence, no transversal function (4.26) exists in two-dimensional space. The r.h.s. of Eq. (4.7) is nonzero for any values of coefficients A_1, A_2:

$$\partial^{x_3}_{\mu} C^{l}_{\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) \neq 0 \quad \text{for } D = 2\,. \tag{6.5}$$

The latter is in an agreement with the structure of representation Q_T for $D = 2$ discussed in Sec. 5. Correspondingly, a partial wave expansion for each of the Green functions (6.2) and (6.3) has only a couple of terms:



(6.6)



(6.7)

[f]To facilitate the proof of this statement, one can use the complex variables $z = x_1 + ix_2$ and the component (5.14).

Here we use the graphical notation and the results of Ref. 1. The transversal term in this case is absent, and the Ward identities determine the kernels of partial wave expansions (6.6) and (6.7) uniquely. Equation (2.44), which expresses the absence of gravitational interaction, also disappears for $D = 2$.

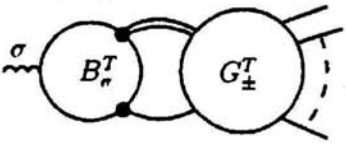
Consider the independent functions (4.3) and (4.4). For the general case of $D > 2$ these functions are singular at the points $l = d + s$. In two-dimensional space the function $C'^{\sigma}_{1\mu\nu}$ is also singular at these points, but $C'^{l}_{2\mu\nu}$ is regular: it is easily seen from (4.4). Therefore the Green functions $\langle P_s \varphi T\rangle$ have the form

$$\langle P_s(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = g_s^T C'^{d+s}_{2\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) + \cdots\,, \qquad D = 2\,. \tag{6.8}$$

The dots stand for the quasilocal term proportional to the residue $\mathrm{res}_{l=d+s}$ $C'^{\sigma}_{1\mu\nu}(x_1x_2x_3)$. As in Sec. 4, one can show that only the kernels $G^T_{2\sigma}$ and $\rho^T_2(\sigma)$ have the poles at $l = d + s$ corresponding to the fields P^T_s. The Green functions of these fields are defined by Eqs. (3.11) at $D = 2$:

$$\langle P_{s\pm}(x_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle$$

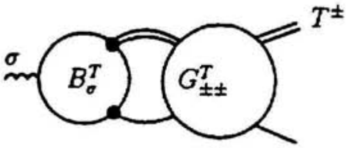
$$= \Lambda^T_s \operatorname*{res}_{l=d+s}$$



$$= \Lambda^T_s \operatorname*{res}_{l=d+s} G^{\pm}_{2\sigma}(x_1\cdots x_m)\,, \tag{6.9}$$

$$\langle P_{s\pm}(x_1)\varphi(x_2)T_\pm(x_3)\rangle$$

$$= \Lambda^T_s \operatorname*{res}_{l=d+s}$$



$$= \Lambda^T_s \left[\operatorname*{res}_{l=d+s} \rho^T_2(\sigma)\right] C'^{d+s}_{2\pm}(x_1x_2x_3) + \cdots\,, \tag{6.10}$$

where the dots stand for quasilocal terms, while the functions B^T_σ have the form

$$B^T_\sigma(x_1x_2x_3) = B_{\mu\nu,\mu_1\cdots\mu_s}(x_1x_2x_3) = \tilde{C}_{1\sigma} - \frac{1}{2s}\tilde{C}_{2\sigma}(x_1x_2x_3)\,. \tag{6.11}$$

This function is derived from the function $\tilde{C}^{\sigma}_{2\mu\nu}$ [see (3.7), (3.8)] after one sets $\tilde{\alpha}_2 = -1/2s$.

In the general $D > 2$ case the function B^T_σ has the form

$$B^{\sigma}_{\mu\nu}(x_1x_2x_3) \sim \langle P^l_s(x_1)\varphi^{D-d}(x_2)h^{\mathrm{long}}_{\mu\nu}(x_3)\rangle$$

$$= \partial^{x_3}_\mu B^\sigma_\nu(x_1x_2x_3) + \partial^{x_3}_\nu B^\sigma_\mu(x_1x_2x_3) - \frac{2}{D}\delta_{\mu\nu}\partial^{x_3}_\rho B^\sigma_\rho(x_1x_2x_3)\,, \tag{6.12}$$

where

$$h_{\mu\nu}(x) = \partial_\mu \tilde{h}_\nu(x) + \partial_\nu \tilde{h}_\mu(x) - \frac{2}{D}\delta_{\mu\nu}\partial_\rho \tilde{h}_\rho(x)\,, \tag{6.13}$$

$$B^\sigma_\nu(x_1x_2x_3) = \langle P^l_s(x_1)\varphi^{D-2}(x_2)\tilde{h}_\nu(x_3)\rangle\,. \tag{6.14}$$

Equation (6.9) may be rewritten as

$$\begin{aligned}
&\langle P_s(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle \\
&= \Lambda^T_s \operatorname*{res}_{l=d+s} \int d^2y_1 d^2y_2 B^\sigma_{\mu\nu}(xy_1y_2)\langle T_{\mu\nu}(y_2)\varphi(y_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle \\
&= -2\Lambda^T_s \operatorname*{res}_{l=d+s} \int d^2y_1 d^2y_2 B^\sigma_\nu(xy_1y_2)\partial^{y_2}_\mu \\
&\quad \times \langle T_{\mu\nu}(y_2)\varphi(y_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,.
\end{aligned} \tag{6.15}$$

After that, to calculate the integral one uses the Ward identity, the way it was done in Refs. 1 and 2.

In the two-dimensional case we shall act by analogy. Technical simplifications characteristic of a two-dimensional space appear when one passes to complex variables (5.12)–(5.17). Each of the functions $B^\sigma_{\mu\nu}$, $\hat{C}_{i\mu\nu}$ has four complex components, two of them in μ,ν indices, and the other two in indices $\mu_1\cdots\mu_s$. Only a pair of them is independent, B^σ_{++}, B^σ_{+-}, $B^\sigma_{-+} = (B^\sigma_{+-})^*$, $B^\sigma_{--} = (B^\sigma_{++})^*$, and similarly for the functions $\tilde{C}^\sigma_{i\mu\nu}$. Equation (6.13) takes the form

$$h_+(x) = 4\partial_+\tilde{h}_+(x)\,, \qquad h_-(x) = 4\partial_-\tilde{h}_-(x)\,.$$

Accordingly, the equation for each of the functions $\tilde{C}^\sigma_{i\mu\nu}$ is written in the form

$$\tilde{C}^\sigma_{+\pm}(x_1x_2x_3) = \partial^{x_3}_\pm \hat{C}^\sigma_{+\pm}(x_1x_2x_3)\,. \tag{6.16}$$

One can show that the function (6.11) has the only independent component[g]

$$B^\sigma_{++}(x_1x_2x_3) = 0\,, \qquad B^\sigma_{+-}(x_1x_2x_3) = \partial^{x_3}_- \tilde{B}^\sigma_{+-}(x_1x_2x_3)\,, \tag{6.17}$$

[g]One gets the following expressions for the functions $\tilde{C}^\sigma_i$:

$$\tilde{C}^\sigma_{1++}(x_1x_2x_3) = \frac{1}{2s}\tilde{C}^\sigma_{2++}(x_1x_2x_3) = \partial^{x_3}_+ \hat{C}^\sigma_{++}(x_1x_2x_3)\,,$$

$$\tilde{C}^\sigma_{1+-}(x_1x_2x_3) = \partial^{x_3}_- \tilde{B}^\sigma_{+-}(x_1x_2x_3)\,, \qquad \tilde{C}^\sigma_{2+-}(x_1x_2x_3) = 0\,,$$

where

$$\hat{C}^\sigma_{++}(x_1x_2x_3) = \frac{1}{2s}\frac{x^+_{12}}{x^+_{13}x^+_{23}}\left(\frac{x^+_{23}}{x^+_{12}x^+_{13}}\right)^s (x^2_{12})^{-\frac{l-d-s+4}{2}}(x^2_{13})^{-\frac{l+d-s-4}{2}}(x^2_{23})^{\frac{l+d-s}{2}}\,.$$

To derive these expressions one uses the identity

$$\frac{1}{x^2_{13}}\sum_{k=1}^{s}[g_{\mu\mu_k}(x_{13})\lambda^{x_1}_{\mu_1\cdots\hat{\mu}_k\cdots\mu_s}(x_3x_2) - \text{traces}] = \partial^{x_3}_\mu \lambda^{x_1}_{\mu_1\cdots\mu_s}(x_3x_2)\,.$$

where

$$\tilde{B}^{\sigma}_{+-}(x_1x_2x_3) \sim \langle P^{\sigma}_{+}(x_1)\varphi^{2-d}(x_2)\tilde{h}_{-}(x_3)\rangle$$
$$\sim \frac{x^{-}_{12}}{x^{-}_{13}x^{-}_{23}}\left(\frac{x^{+}_{23}}{x^{+}_{12}x^{+}_{13}}\right)^{s}(x^2_{12})^{-\frac{l-d-s+4}{2}}(x^2_{13})^{-\frac{l+d-s-4}{2}}(x^2_{23})^{\frac{l+d-s}{2}}. \tag{6.18}$$

In what follows we use the notation

$$B^{l}_{s\pm}(x_1x_2x_3) = \tilde{B}^{\sigma}_{\pm\mp}(x_1x_2x_3) = \tilde{B}^{l}_{s\pm\mp}(x_1x_2x_3).$$

Let us rewrite Eq. (6.15) in the form

$$\langle P_{s\pm}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle$$
$$= \Lambda^{T}_{s}\operatorname*{res}_{l=d+s}\left[\int d^2y_1\,d^2y_2\,B^{\sigma}_{\pm+}(xy_1y_2)\langle T_{-}(y_2)\varphi(y_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\right.$$
$$\left.+\int d^2y_1\,d^2y_2\,B^{\sigma}_{\pm-}(xy_1y_2)\langle T_{+}(y_2)\varphi(y_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\right].$$

Taking into account (6.17) and (6.18), we find for the P_{s+} component that

$$\langle P_{s+}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle$$
$$= \Lambda^{T}_{s}\operatorname*{res}_{l=d+s}\int d^2y_1\,d^2y_2\,B^{\sigma}_{+-}(xy_1y_2)\langle T_{+}(y_2)\varphi(y_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle$$
$$= -\Lambda^{T}_{s}\operatorname*{res}_{l=d+s}\int d^2y_1\,d^2y_2\,B^{l}_{s+}(xy_1y_2)\partial^{y_2}_{-}\langle T_{+}(y_2)\varphi(y_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle. \tag{6.19}$$

The analogous equation holds for the P^{T}_{s-} component.

The integral in the expression (6.19) is calculated using the Ward identities (5.19). Let us demonstrate it on the simplest example, $s=2$. From (6.18) we have

$$B^{l}_{2+}(xy_1y_2) = \left[(\partial^{y_1}_{+})^3[(x-y_1)^2]^{-\frac{l-d}{2}}\right]\frac{(y^{+}_{12})^{\frac{l+d}{2}+1}}{(x^{+}-y^{+}_{2})^{\frac{l+d}{2}-1}}\frac{(y^{-}_{12})^{\frac{l+d}{2}-2}}{(x^{-}-y^{-}_{2})^{\frac{l+d}{2}-2}}.$$

Let us substitute the r.h.s. of the Ward identity (5.19) into (6.19). The terms containing $\delta^{(2)}(y_2-y_1)$ give the vanishing contribution. The other terms contain δ functions $\delta^{(2)}(y_2-x_r)$, $r=1,\ldots,m$, and its derivatives. As the result, the $y^{\pm}_2$ integral is readily calculated. Now find a residue at the pole $l=d+2$ for the derived expression. The latter pole is due to the factor $[(x-y_1)^2]^{-\frac{l-d}{2}}$. One has for $D=2$, $k=0$ (see Ref. 9)

$$\operatorname*{res}_{l=d+2}\frac{1}{[(x-y_1)^2]^{\frac{l-d}{2}}} = 2\pi\delta^{(2)}(x-y_1). \tag{6.20}$$

The remaining y_1 integration is done owing to this δ function. Third derivatives cancel after the similar terms are collected. Finally, we get[13]

$$\langle P_+^{d+2}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle \sim \left\{\frac{3}{2(d+1)}(\partial_+^x)^2 - \sum_{r=1}^{m}\frac{1}{(x^+-x_r^+)}\partial_+^{x_r} - \frac{1}{2}\sum_{r=1}^{m}\frac{d_r}{(x^+-x_r^+)^2}\right\} \times \langle\varphi(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,. \quad (6.21)$$

For the general $s>2$ case one can get[h]

$$\langle P_+^{d+s}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle = \hat{P}_+^s(x,\partial^x)\langle\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,,$$

where

$$\hat{P}_+^s(x,\partial^x) = \frac{1}{2}(s-1)(s+1)(d+s-2)(\partial_+^x)^s - \sum_{k=3}^{s+1}C_k^{s+1}\frac{\Gamma(d+s)}{\Gamma(d+s-k+1)} \times \left\{\sum_{r=1}^{m}(x^+-x_r^+)^{-(k-2)}\left[\partial_+^{x_r} + \frac{1}{2}(k-2)\frac{d_r}{x^+-x_r^+}\right](\partial_+^x)^{s-k+1}\right\}, \quad (6.22)$$

with

$$C_k^{s+1} = \frac{(s+1)!}{k!(s-k+1)!}\,.$$

The analogous result may be obtained for the $P_-^{d+s}(x)$ component.

Consider Eq. (6.10). Let us rewrite it in the form

$$\langle P_{s+}(x_1)\varphi(x_2)T_+(x_3)\rangle = -\Lambda_s^T\int d^2x_4d^2x_5B_{s+}^l(x_1x_4x_5)\partial_-^{x_5} \times\langle T_+(x_5)\varphi(x_4)\varphi(x_2)T_+(x_3)\rangle\,. \quad (6.23)$$

The integral is taken with the help of the Ward identity (5.20) together with the relation (6.20); see footnote h. The result is

$$\langle P_{s+}(x_1)\varphi(x_2)T_+(x_3)\rangle = g_s^T(d,C)\left(\frac{x_{12}^+}{x_{13}^+x_{23}^+}\right)^2\left(\frac{x_{23}^+}{x_{12}^+x_{13}^+}\right)^s\frac{1}{(x_{12}^2)^d}\,,\quad s\geq 2\,, \quad (6.24)$$

where C is the central charge,

$$g_s^T(d,C) \sim \left\{\frac{C}{12} + \frac{d(d-1)(d-2)}{(d+s)(d+s-1)s(s-1)}\left[\frac{d+s-1}{d+s-2} - \frac{1}{s+1}\right] + (-1)^{s+1}\Gamma(s-1)\frac{\Gamma(d+1)}{\Gamma(d+s+1)}\left[1-\frac{1}{4}(s+1)d(d+s-2)\right]\right\}. \quad (6.25)$$

[h]More detailed calculations will be presented elsewhere (in collaboration with V. N. Zaikin). To carry out these calculations it is sufficient to apply the formulas of App. A.

In this calculation we have used the fact that due to (5.21) we get

$$\langle\varphi(x_1)\varphi(x_2)T_+(x_3)\rangle = -\frac{d}{2\pi}\left(\frac{x_{12}^+}{x_{13}^+x_{23}^+}\right)^2 \langle\varphi(x_1)\varphi(x_2)\rangle\,, \tag{6.26}$$

where

$$\langle\varphi(x_1)\varphi(x_2)\rangle = (x_{12})^{-d}\,.$$

As in the general $D > 2$ case, the commutator of any field $P_{s\pm}$ for $s \geq 2$ with the energy–momentum tensor components $T_\pm$ includes anomalous terms

$$\begin{aligned}[T_+(x_1), P_{s+}(x_2)] &= i\delta(x_1^+ - x_2^+)\partial_+^{x_2}P_{s+}(x_2)\\ &\quad + \sum_{s'=2}^{s-1}\sum_{k=0}^{s-s'} \alpha_{s'}^k(\partial_+^{x_1})^{s-s'-k+1}\delta(x_1^+ - x_2^+)(\partial_+^{x_2})^k P_{s'+}(x_2)\\ &\quad + \sum_{k=0}^{s}\alpha_0^k(\partial_+^{x_1})^{s-k+1}\delta(x_1^+ - x_2^+)(\partial_+^{x_2})^k\varphi(x_2)\,, \end{aligned} \tag{6.27}$$

where $\alpha_{s'}^k$ are definite coefficients calculated from Ward identities. They can be expressed through the coupling constants (the normalization factors) of the Green functions

$$\begin{aligned}\langle P_{s+}(x_1)P_{s'+}(x_2)T_+(x_3)\rangle &= \Lambda_s^T \operatorname*{res}_{l=d+s}\int d^2x_4 d^2x_5 B_{+-}^\sigma(x_1x_4x_5)\\ &\quad \times \langle T_+(x_5)\varphi(x_4)P_{s'+}(x_2)T_+(x_3)\rangle\,. \end{aligned} \tag{6.28}$$

Correspondingly, the Ward identities for the Green functions also contain anomalous contributions related to the fields $\varphi(x)$ and $P_{s'}(x)$, $2 \leq s' \leq s-1$. For example, in the simplest case, $s = 2$, one has

$$\begin{aligned}&\partial_-^x\langle T_+(x)P_{2+}(x_1)\varphi(x_2)\cdots\varphi(x_m)\rangle\\ &\quad = -\left[\sum_{k=1}^{m}\delta^{(2)}(x-x_k)\partial_+^{x_k} - \frac{d+4}{2}\partial_+^x\delta^{(2)}(x-x_1)\right.\\ &\qquad \left. - \frac{d}{2}\sum_{k=2}^{m}\partial_+^x\delta^{(2)}(x-x_k)\right]\langle P_{2+}(x_1)\varphi(x_2)\cdots\varphi(x_m)\rangle\\ &\qquad + f(\partial_+^x)^3\delta^{(2)}(x-x_1)\langle\varphi(x_1)\varphi(x_2)\cdots\varphi(x_m)\rangle\,, \end{aligned} \tag{6.29}$$

where f is the constant (its evaluation is done in the next section). Here the anomalous contribution arises owing to the anomalous contribution of the field $\varphi(x)$ to the commutator $[T_+, P_{2+}]$. In the general case, an anomalous contribution of any field $P_{s'}$ in (6.27) leads to the appearance of nonvanishing Green functions

$$\langle P_s(x_1)P_{s'}(x_2)T(x_3)\rangle \quad \text{for } s' \neq s\,, \tag{6.30}$$

which satisfy anomalous Ward identities, for example

$$\partial_-^{x_3}\langle P_{2+}(x_1)\varphi(x_2)T_+(x_3)\rangle = f\partial_+^{x_3}\delta(x_{13})\langle\varphi(x_1)\varphi(x_2)\rangle\,. \tag{6.31}$$

The normalization factors of the Green functions (6.30) may be found using Eqs. (6.28). These equations, as well as the Ward identities for the Green functions

$$\langle T_+\varphi\varphi T_+\rangle\,, \qquad \langle T_+\varphi P_{s'}T_+\rangle\,, \qquad \langle T_+P_sP_{s'}\rangle\,, \tag{6.32}$$

allow one to find the coefficients before the anomalous terms in Ward identities. An example of such calculation is given in the next section and also in Ref. 2 for $D > 2$. These coefficients are expressed through the dimension of the field φ and the central charge.

In principle, the relations (6.27) define the transformation properties of P_s fields with respect to infinite-dimensional conformal group. They are discussed (in a slightly different form) in App. B.

The secondary fields of descendant generations arise in the operator product expansions

$$P_{s_1\pm}(x_1)T_\pm(x_2) = \sum_s [P^{s_1}_{s\pm}]\,, \quad P^{s_2}_{s_1\pm}(x_1)T_\pm(x_2) = \sum_s [P^{s_1 s_2}_{s\pm}]\,, \quad \text{etc.} \tag{6.33}$$

The Green functions of these fields may be derived from the equations analogous to (6.19), if one substitutes $P_{s_1}(x)$ in place of $\varphi(x)$. An example of such calculations is given in the next section. A complete family of secondary fields includes the fields of all generations

$$P_{s\pm}\,, \qquad P^{s_1}_{s\pm}\,, \qquad P^{s_1 s_2}_{s\pm}, \ldots \tag{6.34}$$

Each solvable model is given by the equation

$$Q_{s\pm}(x) = 0\,, \tag{6.35}$$

where $Q_s(x)$ is the primary field introduced for $D \geq 2$ in the work.[2] An important point is that for $D = 2$ the field P_1 does not appear between P_s. Hence, in the second generation there are no fields $P^{s_1}_s$ with $s = 2, 3$. Thus for the simplest models we have

$$Q_2(x) = P_2(x) \qquad \text{for } D = 2\,. \tag{6.36}$$

The models with the field Q_s given by its most general expression[2] arise for $s \geq 4$. In the next section we consider one of such models as an example. Thus, the simplest two-dimensional models are defined by the equations

$$P_2(x) = 0\,, \quad \text{and} \quad P_3(x) = 0\,.$$

7. Models in Two-Dimensional Space

7.1. *Models defined by fields Q_2 and Q_3*

Consider the simplest model defined by the equation

$$Q_2(x) = P_2^T(x) = 0\,. \tag{7.1}$$

According to (6.21), its Green functions satisfy the equations[13,14] (see also Ref. 10)

$$\left\{\frac{3}{2(d+1)}(\partial_+^x)^2 - \sum_{k=1}^{m}\frac{1}{(x^+ - x_k^+)}\partial_+^{x_k} - \frac{1}{2}\sum_{k=1}^{m}\frac{d_k}{(x^+ - x_k^+)^2}\right\} \times \langle\varphi(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle = 0\,, \qquad (7.2)$$

and in x^- variable by analogy.

Suppose that there exists a pair of fundamental neutral fields

$$\varphi(x)\,, \qquad \chi(x) \qquad (7.3)$$

with dimensions d, Δ. Consider Eqs. (7.1) for three-point Green functions

$$\langle P_2(x_1)\varphi(x_2)\chi(x_3)\rangle = 0\,, \qquad \langle P_{2\pm}(x_1)\varphi(x_2)T_\pm(x_3)\rangle = 0\,. \qquad (7.4)$$

The first of these equations reads, according to (7.2),

$$\left\{\frac{3}{2(d+1)}(\partial_\pm^{x_1})^2 - \frac{1}{x_{12}^\pm}\partial_\pm^{x_2} - \frac{1}{x_{13}^\pm}\partial_\pm^{x_3} - \frac{1}{2}\left[\frac{d}{(x_{12}^\pm)^2} + \frac{\Delta}{(x_{13}^\pm)^2}\right]\right\} \times \langle\varphi(x_1)\varphi(x_2)\chi(x_3)\rangle = 0\,. \qquad (7.5)$$

After the substitution of the known coordinate dependence of the invariant function $\langle\varphi\varphi\chi\rangle$ one finds[13,10] that

$$d = \frac{3}{8}\Delta - \frac{1}{4}\,. \qquad (7.6)$$

The second equation in (7.4) gives, according to (6.24) and (6.25) for $s = 2$,

$$C = \frac{d(5-4d)}{d+1}\,. \qquad (7.7)$$

Now consider differential equations for the Green functions

$$G_1(x_1x_2x_3x_4) = \langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle\,,$$
$$G_2(x_1x_2x_3x_4) = \langle\varphi(x_1)\chi(x_2)\varphi(x_3)\chi(x_4)\rangle\,.$$

From (7.2) one gets

$$\left\{\frac{3}{2(d+1)}(\partial_\pm^{x_1})^2 - \sum_{k=2}^{4}\frac{1}{x_{1k}^\pm}\partial_\pm^{x_k} - \frac{d}{2}\sum_{k=2}^{4}\frac{1}{(x_{1k}^\pm)^2}\right\}G_1(x_1x_2x_3x_4) = 0\,, \quad (7.8)$$

$$\left\{\frac{3}{2(d+1)}(\partial_\pm^{x_1})^2 - \sum_{k=2}^{4}\frac{1}{x_{1k}^\pm}\partial_\pm^{x_k} - \frac{d}{2}\frac{1}{(x_{13}^\pm)^2} - \frac{\Delta}{2}\left[\frac{1}{(x_{12}^\pm)^2} + \frac{1}{(x_{14}^\pm)^2}\right]\right\} \times G_2(x_1x_2x_3x_4) = 0\,. \qquad (7.9)$$

The equations coincide with well-known results of two-dimensional theory based on infinite-dimensional symmetry; see, for example, Refs. 3 and 15. This fact is commented on more thoroughly in the next section. Note that in our case these results were obtained based on the requirement of six-dimensional conformal symmetry and Eq. (7.1) on the states of dynamical sector. The solutions of Ising and Potts models are, under certain additional conditions discussed below, defined by Eqs. (7.6)–(7.9). The relation (7.7) coincides with the Kac formula[16,17] [see (B.51)] with $N = 2$. The general solution of Eqs. (7.9) is found in Refs. 13 and 14.

Let us consider the second model, which is defined by the equation

$$Q_{3\pm}(x) = P_{3\pm}(x) = 0\,. \tag{7.10}$$

The equations on the Green functions of this model are derived from (6.21) for $s = 3$:

$$\begin{aligned}\Bigg\{&\frac{4}{d+2}(\partial_\pm^x)^3 - 4\sum_{k=1}^{m}\frac{1}{(x^\pm - x_k^\pm)}\left[\partial_\pm^{x_k} + \frac{d_k}{2}\frac{1}{(x^\pm - x_k^\pm)}\right]\partial_\pm^x \\ &- d\sum_{k=1}^{m}\frac{1}{(x^\pm - x_k^\pm)^2}\left[\partial_\pm^{x_k} + \frac{d_k}{(x^\pm - x_k^\pm)}\right]\Bigg\}\langle\varphi(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle = 0\,.\end{aligned} \tag{7.11}$$

The equations on the three-point Green functions

$$\langle P_3^T(x_1)\varphi(x_2)\chi(x_3)\rangle = 0\,, \qquad \langle P_{3\pm}(x_1)\varphi(x_2)T_\pm(x_3)\rangle = 0 \tag{7.12}$$

give two relations between the parameters

$$d, \Delta, C\,.$$

The first from Eqs. (7.12) leads to

$$d = \frac{1}{3}(\Delta - 2)\,. \tag{7.13}$$

The second from Eqs. (7.12) gives, according to (6.24) and (6.25) at $s = 3$,

$$C = -\frac{3d^2 - 14d + 8}{2(d+2)}\,, \tag{7.14}$$

which coincides with the Kac formula[16,17] [see (B.51)] for $N = 3$. In the next section we demonstrate that Eq. (7.11) possesses an infinite-dimensional symmetry.

One can show that in this model the operator expansion

$$\varphi(x_1)\chi(x_2) = [\varphi] + \cdots$$

also holds, provided that the dimensions d, Δ satisfy Eq. (7.13).

7.2. *Ising model*

In the language of the approach we developed in Ref. 1, the solution of the Ising model is given by a pair of equations in the dynamical sector:

$$P^{d+2}(x) = 0\,, \qquad P^{\Delta+2}(x) = 0\,, \tag{7.15}$$

where P^{d+2} and $P^{\Delta+2}$ are the fields which appear in operator product expansions

$$T(x_1)\varphi(x_2)\,, \qquad T(x_1)\chi(x_2)\,. \tag{7.16}$$

The fields φ and χ play the role of fundamental fields of the model (usually named after σ, ε). Formally, Eq. (7.15) may be derived from the model $L = \lambda\varphi^4$, after completing its definition.[1] Fundamental fields of this model satisfy the Ising model field algebra

$$\varphi\chi = [\varphi] + \cdots\,, \qquad \varphi\varphi = [I] + [\chi] + \cdots$$

The first from Eqs. (7.15) has already been studied; see (7.6)–(7.9). Let us consider the second one. One has for three-point functions

$$\langle P^{\Delta+2}(x_1)\varphi(x_2)\varphi(x_3)\rangle = 0\,, \tag{7.17}$$

$$\langle P_{\pm}^{\Delta+2}(x_1)\chi(x_2)T_{\pm}(x_3)\rangle = 0\,. \tag{7.18}$$

Equation (7.17) gives

$$\left\{\frac{3}{2(\Delta+1)}(\partial_{\pm}^{x_1})^2 - \frac{1}{x_{12}^{\pm}}\partial_{\pm}^{x_1} - \frac{1}{x_{13}^{\pm}}\partial_{\pm}^{x_1} - \frac{d}{2}\left[\frac{1}{(x_{12}^{\pm})^2} + \frac{1}{(x_{13}^{\pm})^2}\right]\right\} \times \langle\chi(x_1)\varphi(x_2)\varphi(x_3)\rangle = 0\,.$$

It follows that

$$d = \frac{1}{4}\frac{\Delta(2-\Delta)}{\Delta+1}\,. \tag{7.19}$$

Together with (7.6) this equation gives

$$d = \frac{1}{8}\,, \qquad \Delta = 1\,. \tag{7.20}$$

From (7.18) one finds, on account of (6.25) at $s = 2$, that

$$C = \frac{\Delta(5-4\Delta)}{\Delta+1}\,. \tag{7.21}$$

For the values of dimensions (7.20) one has

$$C = \frac{d(5-4d)}{d+1} = \frac{\Delta(5-4\Delta)}{\Delta+1} = \frac{1}{2}\,. \tag{7.22}$$

The equations on four-point Green functions of the Ising model are deduced from (7.8) and (7.9) after the substitution of dimensions (7.20) into these equations. For example,

$$\left\{\frac{4}{3}(\partial_{\pm}^{x_1})^2 - \sum_{k=2}^{4}\frac{1}{(x_1^{\pm}-x_k^{\pm})}\partial_{\pm}^{x_k} - \frac{1}{16}\sum_{k=2}^{4}\frac{1}{(x_1^{\pm}-x_k^{\pm})^2}\right\}$$

$$\times\,\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = 0\,. \tag{7.23}$$

The values of dimensions and the central charge, as well as the equations for higher correlators in the Ising model, were found in Ref. 3 on the basis of infinite-dimensional conformal symmetry. Here the same results are shown to follow from Eqs. (7.15) and the symmetry condition under the six-dimensional conformal group. As already mentioned above, an infinite-dimensional symmetry was also realized in this approach. We, however, aimed at the new formulation of the model — the one that would admit a straightforward generalization to three-dimensional space, regardless of whether a $D = 3$ analog of the Virasoro algebra is known or not.

Note that, as opposed to other examples we discuss here, the Ising model is defined by the two Eqs. (7.15) on the states of the dynamical sector. In the approach developed in Ref. 3, the so-called "minimal conformal theories" fall into this class. The latter are specific by the property that they include a finite number of primary fields, which can be treated as fundamental fields. The Ising model is the simplest example of such a kind. The other example is the Potts model, discussed in Ref. 15. Its correlation functions (of the energy operator) also satisfy Eq. (7.9) for the special value of the central charge.

In the approach presented here, minimal theories are selected by two equations [of the (7.15) kind] on the states of the dynamical sector. Below we consider a more complex example of such an equation (for $s = 4$). These equations have evident generalizations to the D-dimensional space. At this point, instead of the c-number central charge, the equations on the correlation functions of the field $P^{D-2}(x)$ — the operator analog of the central charge — should appear. We think that the three-dimensional Ising model belongs to this class of theories.

7.3. *A model defined by two generations of secondary fields*[i]

Let $P_{2\pm}$, $P_{4\pm}$ and $\tilde{P}_{4\pm}$ be the secondary fields contributing to the operator product expansions of $T\varphi$ and TP_2:

$$T_{\pm}(x_1)\varphi(x_2) = [\varphi] + [P_{2\pm}] + [P_{3\pm}] + [P_{4\pm}] + \cdots\,, \tag{7.24}$$

$$T_{\pm}(x_1)P_{2\pm}(x_2) = [\varphi] + [P_{2\pm}] + [\tilde{P}_{4\pm}] + \cdots\,. \tag{7.25}$$

The fields P_2 and P_4, $\tilde{P}_4$ have dimensions

$$d_2 = d + 2\,, \qquad d_4 = d + 4\,.$$

[i]The analog of this model for $D \geq 4$ (in the case of the fields P_s^j) is considered in Ref. 2.

Consider the model defined by the equations

$$Q_{4\pm}(x) = 0\,, \tag{7.26}$$

where

$$Q_{4\pm}(x) = P_{4\pm}(x) + \alpha\tilde{P}_{4\pm}(x)\,, \tag{7.27}$$

α being the free parameter.

There are three free parameters in the model

$$d, \quad C, \quad \alpha\,. \tag{7.28}$$

The consistency conditions of the model

$$\langle Q_4(x_1)\varphi(x_2)T(x_3)\rangle = \langle Q_4(x_1)P_2(x_2)T(x_3)\rangle = 0\,,$$

or, more explicitly,

$$\langle P_{4\pm}(x_1)\varphi(x_2)T_\pm(x_3)\rangle + \alpha\langle\tilde{P}_{4\pm}(x_1)\varphi(x_2)T_\pm(x_3)\rangle = 0\,, \tag{7.29}$$

$$\langle P_{4\pm}(x_1)P_{2\pm}(x_2)T_\pm(x_3)\rangle + \alpha\langle\tilde{P}_{4\pm}(x_1)P_{2\pm}(x_2)T_\pm(x_3)\rangle = 0\,, \tag{7.30}$$

are equivalent to a pair of algebraic equations on the parameters (7.28). Eliminating the parameter α, we get the relation between central charge and dimension.

The Green functions of the fields P_2 and P_4 are determined by Eqs. (6.23) with $s = 2, 4$. Let us write them in the form

$$\begin{aligned}\langle P_{2+}(x_1)\varphi(x_2)T_+(x_3)\rangle &= -2\Lambda_2 \operatorname*{res}_{l=d+2} \int d^2x_4 d^2x_5 B_2^l(x_1x_4x_5)\partial_-^{x_5} \\ &\quad \times \langle T_+(x_5)\varphi(x_4)\varphi(x_2)T_+(x_3)\rangle\,,\end{aligned} \tag{7.31}$$

$$\begin{aligned}\langle P_{4+}(x_1)\varphi(x_2)T_+(x_3)\rangle &= -2\Lambda_4 \operatorname*{res}_{l=d+4} \int d^2x_4 d^2x_5 B_4^l(x_1x_4x_5)\partial_-^{x_5} \\ &\quad \times \langle T_+(x_5)\varphi(x_4)\varphi(x_2)T_+(x_3)\rangle\,,\end{aligned} \tag{7.32}$$

$$\begin{aligned}\langle P_{2+}(x_1)P_{2+}(x_2)T_+(x_3)\rangle &= -2\Lambda_2 \operatorname*{res}_{l=d+2} \int d^2x_4 d^2x_5 B_2^l(x_1x_4x_5)\partial_-^{x_5} \\ &\quad \times \langle T_+(x_5)\varphi(x_4)P_{2+}(x_2)T_+(x_3)\rangle\,,\end{aligned} \tag{7.33}$$

$$\begin{aligned}\langle P_{4+}(x_1)P_{2+}(x_2)T_+(x_3)\rangle &= -2\Lambda_4 \operatorname*{res}_{l=d+4} \int d^2x_4 d^2x_5 B_4^l(x_1x_4x_5)\partial_-^{x_5} \\ &\quad \times \langle T_+(x_5)\varphi(x_4)P_{2+}(x_2)T_+(x_3)\rangle\,,\end{aligned} \tag{7.34}$$

where

$$B_2^l(x_1x_2x_3) \sim \left(\frac{x_{12}^-}{x_{13}^-x_{23}^-}\right)\left(\frac{x_{23}^+}{x_{12}^+x_{13}^+}\right)^2 (x_{12}^2)^{-\frac{l-d+2}{2}}(x_{13}^2)^{-\frac{l+d-6}{2}}(x_{23}^2)^{\frac{l+d-2}{2}}\,, \tag{7.35}$$

$$B_4^l(x_1x_2x_3) \sim \left(\frac{x_{12}^-}{x_{13}^-x_{23}^-}\right)\left(\frac{x_{23}^+}{x_{12}^+x_{13}^+}\right)^4 (x_{12}^2)^{-\frac{l-d}{2}}(x_{13}^2)^{-\frac{l+d-8}{2}}(x_{23}^2)^{\frac{l+d-4}{2}}\,. \tag{7.36}$$

When calculating the integrals in (7.31)–(7.34) we use the Ward identities (5.20) and

$$\partial_-^{x_4}\langle P_{2+}(x_1)\varphi(x_2)T_+(x_3)T_+(x_4)\rangle$$
$$= -\Big\{\delta^{(2)}(x_{14})\partial_+^{x_1} + \delta^{(2)}(x_{24})\partial_+^{x_2} + \delta^{(2)}(x_{34})\partial_+^{x_3}$$
$$- \frac{d+4}{2}\partial_+^{x_4}\delta^{(2)}(x_{14}) - \frac{d}{2}\partial_+^{x_4}\delta^{(2)}(x_{24}) - 2\partial_+^{x_4}\delta^{(2)}(x_{34})\Big\}$$
$$\times \langle P_{2+}(x_1)\varphi(x_2)T_+(x_3)\rangle + f(\partial_+^{x_4})^3\delta^{(2)}(x_{34})\langle\varphi(x_1)\varphi(x_2)T_+(x_3)\rangle\,, \quad (7.37)$$

where f is a constant.

The Green functions of the field $\tilde{P}_4$ are determined by the equations

$$\langle\tilde{P}_{4+}(x_1)\varphi(x_2)T_+(x_3)\rangle = -2\tilde{\Lambda}_4 \operatorname*{res}_{l=d+4}\int d^2x_4 d^2x_5 \tilde{B}_4^l(x_1x_4x_5)\partial_-^{x_5}$$
$$\times \langle T_+(x_5)P_{2+}(x_4)\varphi(x_2)T_+(x_3)\rangle\,, \quad (7.38)$$
$$\langle\tilde{P}_{4+}(x_1)P_{2+}(x_2)T_+(x_3)\rangle = -2\tilde{\Lambda}_4 \operatorname*{res}_{l=d+4}\int d^2x_4 d^2x_5 \tilde{B}_4^l(x_1x_4x_5)\partial_-^{x_5}$$
$$\times \langle T_+(x_5)P_{2+}(x_4)P_{2+}(x_2)T_+(x_3)\rangle\,, \quad (7.39)$$

where

$$\tilde{B}_4(x_1x_2x_3) \sim \frac{x_{12}^-}{x_{13}^-x_{23}^-}\left(\frac{x_{13}^-}{x_{12}^-x_{23}^-}\right)^2\left(\frac{x_{23}^+}{x_{12}^+x_{13}^+}\right)^4$$
$$\times (x_{12}^2)^{-\frac{l-d-2}{2}}(x_{13}^2)^{-\frac{l+d-2}{2}}(x_{23}^2)^{\frac{l+d+2}{2}}\,. \quad (7.40)$$

When calculating the integrals in (7.38) and (7.39) we use the Ward identities (7.37) and

$$\partial_-^{x_4}\langle P_{2+}(x_1)P_{2+}(x_2)T_+(x_3)T_+(x_4)\rangle$$
$$= -\Big\{\delta(x_{14})\partial_+^{x_1} + \delta(x_{24})\partial_+^{x_2} + \delta(x_{34})\partial_+^{x_3}$$
$$- \frac{d+4}{2}\partial_+^{x_4}[\delta(x_{14}) + \delta(x_{24})] - 2\partial_+^{x_4}\delta(x_{34})\Big\}$$
$$\times \langle P_{2+}(x_1)P_{2+}(x_2)T_+(x_3)\rangle + f[(\partial_+^{x_4})^3\delta(x_{41})\langle\varphi(x_1)P_{2+}(x_2)T_+(x_3)\rangle$$
$$+ (\partial_+^{x_4})^3\delta(x_{24})\langle P_{2+}(x_1)\varphi(x_2)T_+(x_3)\rangle]$$
$$- \frac{1}{12\pi}C(\partial_+^{x_4})^3\delta(x_{34})\langle P_{2+}(x_1)P_{2+}(x_2)\rangle\,. \quad (7.41)$$

Invariant three-point functions entering the Ward identities read

$$\langle\varphi(x_1)\varphi(x_2)\rangle = (x_{12}^2)^{-d},$$

$$\langle\varphi(x_1)\varphi(x_2)T_+(x_3)\rangle = -\frac{d}{2\pi}\left(\frac{x_{12}^+}{x_{13}^+x_{23}^+}\right)^2\frac{1}{(x_{12}^2)^d}, \tag{7.42}$$

$$\langle P_{2+}(x_1)\varphi(x_2)T_+(x_3)\rangle = -\frac{6f}{\pi}\frac{1}{(x_{13}^+)^4}\frac{1}{(x_{12}^2)^d}, \tag{7.43}$$

$$\langle P_{2+}(x_1)P_{2+}(x_2)T_+(x_3)\rangle = -\frac{d+4}{2\pi}g_2\left(\frac{x_{12}^+}{x_{13}^+x_{23}^+}\right)^2\left(\frac{x_{13}^+}{x_{12}^+x_{23}^+}\right)^2 \times\left(\frac{x_{23}^+}{x_{12}^+x_{13}^+}\right)^2\frac{1}{(x_{12}^2)^d}, \tag{7.44}$$

where g_2 is a constant.

Equations (7.31)–(7.44) contain auxiliary parameters

$$\Lambda_2, \Lambda_4, \tilde{\Lambda}_4, f, g_2\,.$$

Parameters Λ_2, f and g_2 are evaluated by the comparison of the expressions (7.43) and (7.44) with the result of calculation of the integrals (7.31) and (7.33). Parameters $\Lambda_4, \tilde{\Lambda}_4$ may be calculated analogously. In fact, these values are redundant, since they are factored in the combination $\alpha\tilde{\Lambda}_4/\Lambda_4$ in Eqs. (7.29) and (7.30). Eliminating this parameter from the pair of algebraic equations, we obtain two solutions for the central charge[j]

$$C = C^{(1)} = 1-4d\,, \qquad C = C^{(2)} = \frac{1}{5}\frac{(d-2)(33-4d)}{d+3}\,. \tag{7.45}$$

The same values are found using the Kac formula (B.51) for $N=4$.

Differential equations for higher Green functions

$$\langle\varphi(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle \tag{7.46}$$

could be derived from the equation

$$\langle Q_{4\pm}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle = 0\,,$$

or

$$\operatorname*{res}_{l=d+4}\int d^2y_1\,d^2y_2\,B_4^l(xy_1y_2)\partial_-^{y_2}\langle T_+(y_2)\varphi(y_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle + \alpha\frac{\tilde{\Lambda}_4}{\Lambda_4}\operatorname*{res}_{l=d+4}\int d^2y_1\,d^2y_2\,\tilde{B}_4^l(xy_1y_2)\partial_-^{y_2} \times\langle T_+(y_2)P_{2+}(y_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle = 0\,, \tag{7.47}$$

[j] To calculate them it is sufficient to apply the formulas of App. A. More detailed calculations will be presented elsewhere (in collaboration with V. N. Zaikin). The examples of such calculations are found in Ref. 2.

where

$$\langle T_+(y)P_{2+}(x)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle = -\Lambda_2 \operatorname*{res}_{l=d+2} \int d^2y_1\, d^2y_2\, B_2^l(xy_1y_2)\partial^{y_2} \times \langle T_+(y)T_+(y_2)\varphi(y_1)\varphi_1(x_1)\cdots\varphi_m(x_m)\rangle\,. \tag{7.48}$$

The integrals are easily calculated, as explained in the previous section. One can show that Eq. (7.47) reduces to a differential equation of the fourth order[k] for the Green function (7.46).

Acknowledgments

The authors would like to thank Dr. David Edward for his valuable and encouraging support. This work was supported in part by NATO Grant No. 931717 and RBRF 96-02-18966. E. S. F. would like to thank Prof. M. Virasoro and the International Centre for Theoretical Physics, Trieste, for hospitality.

Appendix A. Integral Relations in Two-Dimensional Space

$$\int d^2x_4\Big\{\langle P_+^{s_1,l_1}(x_1)P_-^{s,\bar\delta}(x_4)P_-^{1,-1}(x_2)\rangle\partial_-^{x_2} + \frac{l_2+s_2}{2}\partial_-^{x_2}\langle P_+^{s_1,l_1}(x_1)P_-^{s,\bar\delta}(x_4) \times P_-^{1,-1}(x_2)\rangle\Big\}\langle P_+^{s,\delta}(x_4)P_+^{s_2,l_2}(x_2)P_+^{s_3,l_3}(x_3)\rangle\,,$$

$$= (-1)^{s_1+1}\pi\frac{\Gamma\left(\frac{\delta-s+s_1-l_1}{2}\right)\Gamma\left(\frac{l_1+s_1-l_2-s_2+l_3+s_3}{2}\right)\Gamma\left(\frac{2-\delta+l-l_3+s_3+l_2-s_3}{2}\right)}{\Gamma\left(\frac{2+l_1+s_1-\delta-s}{2}\right)\Gamma\left(\frac{l_2-s_2+s_1-l_1-l_3+s_3+2}{2}\right)\Gamma\left(\frac{\delta+s+l_3+s_3-l_2-s_2}{2}\right)} \times\left[\frac{(l_2+s_2+l_3+s_3-\delta-s)(2+l_2+s_2-l_3-s_3-\delta-s)}{2(l_1+s_1-\delta-s+2)} + \frac{(l_2+s_2)(l_1+s_1+\delta+s-4)}{4}\right]\langle P_+^{s_1,l_1}(x_1)P_+^{s_2,l_2}(x_2)P_+^{s_3,l_3}(x_3)\rangle\,, \tag{A.1}$$

$$\int d^2x_4(\partial_-^{x_2})^3\langle P_+^{s_1,l_1}(x_1)P_-^{s,\bar\delta}(x_4)P_-^{1,-1}(x_2)\rangle\langle P_+^{s;\delta}(x_4)P_+^{s_2-2,l_2-2}(x_2)P_+^{s_3,l_3}(x_3)\rangle$$

$$= \frac{1}{8}(-1)^{s_1+1}\pi(l_1+s_1+l+s)(l_1+s_1+\delta+s-2)(l_1+s_1+\delta+s-4) \times\frac{\Gamma\left(\frac{\delta-s-l_1+s_1}{2}\right)\Gamma\left(\frac{l_1+s-l_2-s_2+l_3+s_3}{2}\right)\Gamma\left(\frac{2-\delta+s+l_2-s_2-l_3+s_3}{2}\right)}{\Gamma\left(\frac{l_1+s_1-\delta-s-2}{2}\right)\Gamma\left(\frac{2-l_1+s_1+l_2-s_2-l_3+s_3}{2}\right)\Gamma\left(\frac{4+\delta+s-l_2-s_2+l_3+s_3}{2}\right)} \times\langle P_+^{s_1,l_1}(x_1)P_+^{s_2,l_2}(x_2)P_+^{s_3,l_3}(x_3)\rangle\,. \tag{A.2}$$

[k]Fifth derivatives also appear during intermediate stages of calculations, canceling out after the similar terms are collected.

These formulas may be derived from the integral relations presented in Refs. 10 and 11 (see the appendices), if one passes to the variables $x^{\pm} = x_1 \pm ix_2$ in the case $D = 2$.

Appendix B. Infinite-Dimensional Symmetry of Two-Dimensional Conformal Theories

In this appendix we present a comparative analysis of the results obtained above with the ones given by the method of the solution of $D = 2$ models proposed in Refs. 3 and 4. The latter formalism is based on the explicit application of the symmetry under the infinite-dimensional group of conformal (analytic) transformations of two-dimensional complex space

$$z \to f(z)\,, \qquad \bar{z} \to \bar{f}(\bar{z})\,. \tag{B.1}$$

Let us start with a brief review of the method. By assumption, the fundamental (primary[3]) fields transform by the law

$$\varphi(z,\bar{z}) \to \varphi'(z,\bar{z}) = \left(\frac{df}{dz}\right)^{\delta}\left(\frac{d\bar{f}}{d\bar{z}}\right)^{\bar{\delta}}\varphi\left(f(z),\bar{f}(\bar{z})\right)\,, \tag{B.2}$$

where $\delta, \bar{\delta}$ are the parameters which play the role of scale dimensions for each group $\mathrm{SL}(2,R)$ in (1.5). They are related with the ordinary dimension and "spin" of two-dimensional space by the formulas

$$d = \delta + \bar{\delta}\,, \qquad s = \delta - \bar{\delta}\,. \tag{B.3}$$

It is supposed that $\delta > 0$, $\bar{\delta} > 0$.

Each primary field is accompanied by a set of secondary fields

$$L_{-k_1}\varphi\,, \qquad L_{-k_1}L_{-k_2}\varphi, \ldots\,, \qquad k_i \geq -1\,, \tag{B.4}$$

where L_{-k} are the generators of the algebra (B.19). Their transformation properties differ from (B.2). The set of all fields

$$\varphi^{(-k_1,-k_2,\ldots,-k_n)} = L_{-k_1}L_{-k_2}\cdots L_{-k_n}\varphi \tag{B.5}$$

with $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$ forms an infinite family of secondary fields generated by the primary field φ. One can produce an analogous family of secondary fields, acting with generators $\bar{L}_n$ on the field φ. Below we show that the set of states

$$\varphi^{(-k_1,-k_2,\ldots,-k_n)}(x)|0\rangle \tag{B.6}$$

yields a new realization of dynamical sector H of the theory; see (1.1) for $D = 2$.

The dimension of the secondary field (B.5) is found from the equation

$$L_0\varphi^{(-k_1,-k_2,\ldots,-k_n)} = \delta^{(-k_1,-k_2,\ldots,-k_n)}\varphi^{(-k_1,-k_2,\ldots,-k_n)}$$

to be

$$\delta^{(-k_1,-k_2,\dots,-k_n)} = \delta + k_1 + \cdots + k_n = \delta + N\,. \tag{B.7}$$

It is convenient to classify secondary fields by values of the levels they belong to:

$$N = \sum_{i=1}^{n} k_i\,. \tag{B.8}$$

The level N comprises the fields of dimension (B.7) with N fixed and $k_1,\dots,k_n$ running through all possible values.

From the set of secondary fields of the level N one can construct a definite superposition

$$Q_N = \sum_{\{k_i\}} \alpha^{(-k_1,-k_2,\dots,-k_n)} \varphi^{(-k_1,-k_2,\dots,-k_n)} \tag{B.9}$$

where

$$k_1 + k_2 + \cdots + k_n = N\,, \qquad 1 \le n \le N\,, \tag{B.10}$$

which has the transformation properties of the primary field. The field Q_N is related to a definite vector (null vector) in the representation space of the Virasoro algebra. This vector should be set to zero in irreducible representations.[16,17] The coefficients $\alpha^{(-k_1,-k_2,\dots,-k_n)}$ are chosen from the requirement for Q_N to be a primary field. Then the equation

$$Q_N = 0 \tag{B.11}$$

turns out to be covariant with respect to an infinite-dimensional conformal group (see below). Each of the equations (B.11) defines an exactly solvable model.

From the mathematical viewpoint the equations (B.11) single out irreducible representations of the Virasoro algebra in a special class of indecomposable representations.[l] The corresponding state space is the space of irreducible representations of the Virasoro algebra.

Below we show that the fields P_s^T represent definite superpositions of the fields (B.5) of the level $N = s$, and the superpositions Q_N coincide with the fields Q_s^T. Let us discuss the above statements more comprehensively. To proceed, consider the infinitesimal form of transformations (B.1)

$$z \to z + \varepsilon(z)\,, \qquad \bar{z} \to \bar{z} + \bar{\varepsilon}(z)\,. \tag{B.12}$$

From (B.2) one has

$$\Delta\varphi(z,\bar{z}) = \left[\varepsilon(z)\partial_z + \varepsilon'(z)\delta + \bar{\varepsilon}(\bar{z})\partial_{\bar{z}} + \bar{\varepsilon}'(\bar{z})\bar{\delta}\right] \varphi'(z,\bar{z})\,, \tag{B.13}$$

[l]Such representations correspond to a definite (infinite) set[16,17] of scale dimensions of the primary field for a given value of the central charge.

where $\varepsilon'(z) = \frac{d\varepsilon}{dz}, \bar{\varepsilon}'(\bar{z}) = \frac{d\bar{\varepsilon}}{d\bar{z}}, \partial_z = \frac{\partial}{\partial z}, \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}$. The Ward identities take the form[3] (the factor $1/2\pi i$ was introduced for usefulness)

$$\frac{1}{2\pi i}\oint_C dz\varepsilon(z)\langle T(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle$$
$$= \sum_{k=1}^{m}[\varepsilon(z_k)\partial_{z_k} + \delta_k\varepsilon'(z_k)]\langle\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\,. \tag{B.14}$$

The analogous relation involving the $\bar{z}$ coordinate is also valid. Here and in what follows we omit the $\bar{z}, \bar{z}_k$ coordinate dependence. We get from (B.14)

$$\langle T(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle$$
$$= \sum_{k=1}^{m}\left[\frac{\delta_k}{(z-z_k)^2} + \frac{1}{(z-z_k)}\partial_{z_k}\right]\langle\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\,, \tag{B.15}$$

where δ_k, $k = 1,\ldots,m$, is the dimension of the field φ_k. This expression coincides (up to a factor) with the solution of the Ward identities found in Sec. 5. The change of the normalization is caused by the change of the definition of the components $T_\pm$, T, $\bar{T}$; see (5.18) and (B.14). For simplicity, let us set from here on

$$\varphi_1 = \varphi_2 = \cdots = \varphi_m = \varphi\,.$$

Consider the operator product expansion

$$T(z)\varphi(z_1)\,. \tag{B.16}$$

It is useful to represent the $T(z)$ component in terms of Laurent expansion:

$$T(z) = \sum_{n=-\infty}^{\infty}\frac{L_n(z_q)}{(z-z_1)^{n+2}}\,, \tag{B.17}$$

where $L(z)$ are the generators of the Virasoro algebra

$$L_n(z) = \frac{1}{2\pi i}\oint_{C_z} dz'(z'-z)^{n+1}T(z')\,. \tag{B.18}$$

They satisfy the commutation relations

$$[L_n(z), L_m(z)] = (n-m)L_{n+m}(z) + \frac{C}{12}(n^3-n)\delta_{n+m,0}\,. \tag{B.19}$$

Thus we get

$$T(z)\varphi(z_1) = \sum_{n=-\infty}^{\infty}\frac{1}{(z-z_1)^{n+2}}L_n(z_1)\varphi(z_1)\,. \tag{B.20}$$

From (B.15) we have

$$L_n(z)\varphi(z) = 0\,, \quad n \geq 1\,, \tag{B.21}$$

$$L_0(z)\varphi(z) = \delta\varphi(z)\,, \tag{B.22}$$

$$L_{-1}(z)\varphi(z) = \partial_z\varphi(z)\,. \tag{B.23}$$

The following terms of the expansion produce secondary fields:

$$\varphi^{(-k)}(z) = L_{-k}(z)\varphi(z)\,, \qquad k \geq 2\,. \tag{B.24}$$

The Green functions of the fields $\varphi^{(-k)}$ may be found, expanding (B.15) into powers of $z_1 - z$. Let us represent (B.15) in the form

$$\begin{aligned}\langle T(z)\varphi(z_1)\varphi(z_2)\cdots\varphi(z_m)\rangle &= \left[\frac{\delta}{(z-z_1)^2} + \frac{1}{(z-z_1)}\partial_{z_1}\right]\langle\varphi(z_1)\cdots\varphi(z_m)\rangle \\ &\quad + \sum_{k=2}^{\infty}(z-z_1)^{-2+k}\langle\varphi^{(-k)}(z_1)\varphi(z_2)\cdots\varphi(z_m)\rangle\,.\end{aligned} \tag{B.25}$$

The operator product expansion (B.16) may be rewritten as

$$\begin{aligned}T(z)\varphi(z_1) &= \sum_{k=0}^{\infty}(z-z_1)^{-2+k}\varphi^{(-k)}(z_1) \\ &= \frac{\delta}{(z-z_1)^2}\varphi(z_1) + \tfrac{1}{(z-z_1)}\partial_{z_1}\varphi(z_1) \\ &\quad + \sum_{k=2}^{\infty}(z-z_1)^{-2+k}\varphi^{(-k)}(z_1)\,,\end{aligned} \tag{B.26}$$

$$\varphi^{(0)}(z) = L_0(z)\varphi(z) = \delta\varphi(z)\,, \qquad \varphi^{(-1)}(z) = \partial_z\varphi(z)\,.$$

In the general case one has

$$\varphi^{(-k)}(z) = L_{-k}(z)\varphi(z) = \frac{1}{2\pi i}\oint_{C_z} dz'(z'-z)^{-k+1}T(z')\varphi(z)\,, \tag{B.27}$$

where the contour C_z surrounds the point z.

The Green functions of the field $\varphi^{(-k)}$ are given by the equation

$$\begin{aligned}&\langle\varphi^{(-k)}(z_1)\varphi_2(z_2)\cdots\varphi_m(z_m)\rangle \\ &\quad = \frac{1}{2\pi i}\oint_{C_{z_1}}(z-z_1)^{-k+1}\langle T(z)\varphi(z_1)\varphi_2(z_2)\cdots\varphi_m(z_m)\rangle\,.\end{aligned}$$

Substituting (B.15) into this equation, one finds[3] that

$$\langle\varphi^{(-k)}(z_1)\varphi_2(z_2)\cdots\varphi_m(z_m)\rangle = \hat{L}_{-k}(z,\partial_z)\langle\varphi(z_1)\varphi_2(z_2)\cdots\varphi_m(z_m)\rangle\,, \tag{B.28}$$

where

$$\hat{L}_{-k}(z,\partial_z) = \sum_{r=2}^{m}\left[\frac{(k-1)\delta_r}{(z_1-z_r)^k} + \frac{1}{(z_1-z_r)^{k-1}}\partial_{z_r}\right]. \tag{B.29}$$

These equations are analogous to Eqs. (6.22); see below.

Let us compare the expansion (B.26) with the expansion derived above:

$$\begin{aligned} T(z)\varphi(z_1) &= [\varphi] + \sum_{s=2}^{\infty}[P_s] \\ &= \frac{\delta}{(z-z_1)^2}\varphi(z_1) + \frac{1}{z-z_1}\partial_{z_1}\varphi(z_1) + \cdots \\ &\quad + \{P_2(z_1)+\cdots\} + \{P_3(z_1)+\cdots\} + \cdots, \end{aligned} \tag{B.30}$$

where $\{P_s + \cdots\}$ stands for the contribution of the field P_s and sum of all its derivatives. Each derivative enters with the definite coefficient which can be found from the expansion of Q functions (see Ref. 1) for $D = 2$ in complex variables. Comparing the expansions (B.26) and (B.30), it is possible to find the relation between the sets of secondary fields[m]

$$\{\varphi^{(-k)}\} \qquad \text{and} \qquad \{P_s\}. \tag{B.31}$$

The equivalence of the two sets becomes apparent from the derivation of these expansions. Indeed, each of them results from the expansion of the product $T\varphi$ based on the solution of Ward identities. The only difference is in the fact that the fields $\varphi^{(-k)}$ are noncovariant coefficients of the power expansion (B.26), while the fields P_s represent its SL$(2,C)$ covariant[n] combinations. The SL$(2,C)$ generators are

$$L_{-1}, L_0, L_1,$$

and the covariance conditions for the fields P_s read

$$L_{-1}(z)P_s(z) = \partial_z P_s(z), \quad L_0(z)P_s(z) = (\delta+s)P_s(z), \tag{B.32}$$

$$L_1(z)P_s(z) = 0. \tag{B.33}$$

These conditions allow one to establish a straightforward link between the fields P_s and $\varphi^{(-k)}$ (up to a normalization). Let us set

$$P_s(z) = [L_{-s} + \alpha_1 L_{-1}L_{-s+1} + \alpha_2 L_{-1}^2 L_{-s+2} + \cdots + \alpha_{s-1}(L_{-1})^s]\varphi(z). \tag{B.34}$$

Here we have used the fact that the generator L_{-1} coincides with the derivative, and all the terms have the same dimension, $\delta + s$. Hence the conditions (B.32) are

[m] With each field P_s, the set $\{P_s\}$ includes all its derivatives.

[n] Let us recall that the SL$(2,R)$ group is the conformal group of one-dimensional space R^1 with the coordinate x^+ or x^-, the P_s fields transforming as conformal fields with respect to the SL$(2,R)$ group. The transformation law is preserved in the transition to a complex coordinate and the SL$(2,C)$ group.

satisfied identically, while the condition (B.33) leads to algebraic equations on the coefficients α_i, $1 \le i \le s-1$. In particular,

$$P_2(z) = \left[L_{-2} - \frac{3}{2}\frac{1}{(2\delta+1)}(L_{-1})^2\right]\varphi(z)\,, \tag{B.35}$$

$$P_3(z) = \left[L_{-3} - \frac{2}{\delta+2}L_{-1}L_{-2} + \frac{1}{(\delta+1)(\delta+2)}(L_{-1})^3\right]\varphi(z)\,, \tag{B.36}$$

$$P_4(z) = \left[L_{-4} - \frac{5}{2(\delta+3)}L_{-1}L_{-3} + \frac{5}{(\delta+3)(2\delta+5)}(L_{-1})^2L_{-2} - \frac{15}{4(\delta+3)(2\delta+3)(2\delta+5)}(L_{-1})^4\right]\varphi(z)\,. \tag{B.37}$$

Passing to the fields (B.24) and using the fact that the action of L_{-1} is just a differentiation, we find that

$$P_2(z) = \varphi^{(-2)}(z) - \frac{3}{2}\frac{1}{(2\delta+1)}\partial_z^2\varphi(z)\,, \tag{B.38}$$

$$P_3(z) = \varphi^{(-3)}(z) - \frac{2}{\delta+2}\partial_z\varphi^{(-2)}(z) + \frac{1}{(\delta+1)(\delta+2)}\partial_z^3\varphi(z)\,, \tag{B.39}$$

$$P_4(z) = \varphi^{(-4)}(z) - \frac{5}{2(\delta+3)}\partial_z\varphi^{(-3)}(z) + \frac{5}{(\delta+3)(2\delta+5)}\partial_z^2\varphi^{(-2)}(z) - \frac{15}{4(\delta+3)(2\delta+3)(2\delta+5)}\partial_z^4\varphi(z)\,. \tag{B.40}$$

Note that for $D = 2$ no covariant field P_1 exists, since $\varphi^{(-1)} = \partial_z\varphi$. When $D > 2$, such a field may be constructed, its existence depending on the choice of anomalous terms in the Ward identities; see (3.15).

The operator equations (B.38)–(B.40), or (B.34) in the general case, are meant to represent the equations for Green functions. In particular, for $s = 2$ one has from (B.38)

$$\langle P_2(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle = \langle\varphi^{(-2)}(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle - \frac{3}{2}\frac{1}{(2\delta+1)}\partial_z^2\langle\varphi(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\,. \tag{B.41}$$

Employing (B.28) and (B.29) one finds that

$$\langle P_2(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle \sim \left\{\partial_z^2 - \frac{2}{3}(2\delta+1)\sum_{r=1}^{m}\left[\frac{\delta_r}{(z-z_r)^2} + \frac{1}{(z-z_r)}\partial_{z_r}\right]\right\} \times \langle\varphi(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\,. \tag{B.42}$$

This coincides with the expression (7.2), derived from the general definition of the fields P_s. Analogously, Green functions of P_s with $s > 2$ may be expressed through the Green functions of the fields $\varphi^{(-k)}$ with $k \le s$ and their derivatives.

The transformation laws of the fields $\varphi^{(-k)}$ and P_s are determined by the Ward identities, and possess similar structures. Consider the infinitesimal transformations (B.13). The transformation law of the fields P_s was the subject of the discussion in Sec. 6. This law manifests itself either in terms of the commutator (6.27) or in terms of Ward identities of the type (6.29). The transformation laws of the fields $\varphi^{(-k)}$ may be found from the expansion (B.26) after taking the variation Δ_ε of both sides of the equality. One easily shows[3] that

$$\begin{aligned}\Delta_\varepsilon\varphi^{(-k)}(z) &= \varepsilon(z)\partial_z\varphi^{(-k)} + (\delta+k)\varepsilon'(z)\varphi^{(-k)}(z)\\ &\quad+\sum_{l=1}^{k}\frac{k+l}{(l+1)!}\left[\frac{d^{l+1}}{dz^{l+1}}\varepsilon(z)\right]\varphi^{(l-k)}(z)\\ &\quad+\frac{C}{12}\frac{1}{(k-2)!}\left[\frac{d^{k+1}}{dz^{k+1}}\varepsilon(z)\right]\varphi(z)\,. \end{aligned} \tag{B.43}$$

A common feature of transformation laws of the fields P_s and $\varphi^{(-k)}$ is in the following: a variation of each P_s is expressed through φ and $P_{s'}$ with $s' < s$; on the other side, a variation of each $\varphi^{(-k)}$ is expressed through the fields φ and $\varphi^{(-k')}$ with $k' < k$. One can easily express the variation $\Delta_\varepsilon P_s$ through the variations of the fields $\Delta_\varepsilon\varphi^{(-k)}$.

Thus, the two sets of fields (B.31) beget equivalent bases in the subspace of states

$$T_\pm(x_1)\varphi(x_2)|0\rangle\,. \tag{B.44}$$

The significant fact is that this subspace is completely determined by Ward identities with no regard to whether the Virasoro algebra is explicitly used. The D-dimensional analog of the above subspace was dealt with in Refs. 1 and 2 and the first part of this work.

Let us inspect the next generation of secondary fields, which arise in the operator product expansions

$$T(x_1)P_s(x_2) \quad \text{and} \quad T(x_1)\varphi^{(-s)}(x_2)\,. \tag{B.45}$$

They define the basis of subspace of dynamical sector, corresponding to the states

$$T(x_1)T(x_2)\varphi(x_3)|0\rangle\,. \tag{B.46}$$

The solution of the Ward identity for the Green functions $\langle TT\varphi_1\cdots\varphi_m\rangle$ has the form[3]

$$\begin{aligned}\frac{1}{2\pi i}\oint_C &\varepsilon(z')\langle T(z')T(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\\ &=\frac{C}{12}\varepsilon'''(z)\langle\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\\ &\quad+[\varepsilon(z)\partial_z+2\varepsilon'(z)]\langle T(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\\ &\quad+\sum_{k=1}^{m}[\varepsilon(z_k)\partial_{z_k}+\delta_k\varepsilon'(z_k)]\langle T(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\,.\end{aligned}$$

Due to the arbitrariness of $\varepsilon(z)$ we have

$$\begin{aligned}
&\langle T(z')T(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle \\
&\quad = \frac{C}{2}\frac{1}{(z-z')^4}\langle\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle \\
&\qquad + \left[\frac{2}{(z'-z)^2}+\frac{1}{(z'-z)}\partial_z\right]\langle T(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle \\
&\qquad + \sum_{k=1}^{m}\left[\frac{\delta_k}{(z-z_k)^2}+\frac{1}{(z-z_k)}\partial_{z_k}\right]\langle T(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\,, \qquad \text{(B.47)}
\end{aligned}$$

which is just the same as (5.22). The expression (B.47) determines the decomposition of the states (B.46) into the basis of secondary fields $\varphi^{(-k_1)}$, $\varphi^{(-k_1,-k_2)}$, where

$$\varphi^{(-k_1,-k_2)}(z) = L_{-k_1}L_{-k_2}\varphi(z)\,, \qquad k_1 \leq k_2\,. \qquad \text{(B.48)}$$

On the other hand, the states (B.46) may be decomposed into the basis of fields P_s, $P_{s_1}^{s_2}$. As above, one can establish the relation between the fields $\varphi^{(-k)}$, $\varphi^{(-k_1,-k_2)}$ and the fields P_s, $P_{s_1}^{s_2}$, comparing expansions of the operator products (B.45) and using the equalities of the type (B.38)–(B.40). Secondary fields of higher generations may be treated in a similar fashion. A complete family of secondary fields (B.6) begets the space of representation of the Virasoro algebra. Conversely, a complete family of covariant secondary fields (6.34) in the approach presented above defines the basis of the dynamical sector of the Hilbert space (both in the two-dimensional and in the D-dimensional case).

Consider a field Q_N, defined by (B.9). It represents a superposition of the secondary fields (B.5) of a given level N. The weight factors in this superposition are defined by the requirement of the primary field transformation properties [see (B.21)–(B.23)]

$$L_n Q_N = 0\,, \qquad n \geq 1\,, \qquad \text{(B.49)}$$

$$L_0 Q_N = (\delta + N)Q_N\,. \qquad \text{(B.50)}$$

These condition hold only for definite values of dimension δ, namely those given by the Kac formula[16,17]

$$\delta = \delta_{n,m} = \frac{(\alpha_- n + \alpha_+ m)^2 - (\alpha_- + \alpha_+)^2}{4}\,, \qquad \text{(B.51)}$$

where

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2+1}\,, \qquad \alpha_0^2 = \frac{1-C}{24}\,,$$

where C is the central charge, and the numbers n, m are related to the level number

$$N = n \times m\,.$$

A space of irreducible representation for such values of dimension is picked up with the condition $Q_N|0\rangle = 0$, or

$$Q_N(z, \bar{z}) = 0 .$$

Owing to (B.49), this equation is covariant with respect to the infinite group of conformal transformations.

It is clear that the field Q_N may be represented as a superposition of covariant secondary fields P_s, $P_{s_1}^{s_2}, \ldots$ of different generations. The latter is discussed below in more detail.

For the sake of transparency, let us consider some simplest examples. Let $N = 2$. Then to the second level a pair of secondary fields

$$\varphi^{(-2)} = L_{-2}\varphi , \qquad \varphi^{(-1,-1)} = (L_{-1})^2\varphi = \partial_z^2\varphi \tag{B.52}$$

belong. Construct their superposition, following Refs. 3 and 4,

$$Q_2 = [L_{-2} + \gamma(L_{-1})^2]\varphi ,$$

and examine the conditions (B.49) for $n = 1, 2$:

$$L_1 Q_2 = L_2 Q_2 = 0 . \tag{B.53}$$

The first one fixes up the coefficient γ. Consequently [see (B.33) and (B.38)] the field Q_2 coincides with the SL$(2, C)$-covariant secondary field P_2:

$$Q_2(z) = P_2(z) , \qquad \gamma = -\frac{3}{2(2\delta + 1)} . \tag{B.54}$$

The second condition in (B.53) relates the dimension with the central charge

$$C = \frac{2\delta(5 - 8\delta)}{2\delta + 1} . \tag{B.55}$$

The conditions (B.49) for $n > 2$

$$L_n Q_2 = 0 , \qquad n \geq 3 \tag{B.56}$$

are the consequences of the conditions (B.53):

$$L_3 Q_2 = [L_2, L_1]Q_2 = L_2 L_1 Q_2 - L_1 L_2 Q_2 = 0 , \quad \text{etc.}$$

Consider the differential equations for higher Green functions. They follow from Eq. (B.11) for $N = 2$:

$$\langle Q_2(z)\varphi_1(z_1) \cdots \varphi_m(z_m)\rangle = 0 .$$

Employing (B.54), we have

$$\langle L_{-2}\varphi(z)\varphi_1(z_1) \cdots \varphi_m(z_m)\rangle - \frac{3}{2(2\delta + 1)}\partial_z^2\langle\varphi(z)\varphi_1(z_1) \cdots \varphi_m(z_m)\rangle = 0 .$$

Here the first term is given by Eq. (B.28) for $k=2$:

$$\begin{aligned}
&\langle L_{-2}\varphi(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle \\
&\quad = \frac{1}{2\pi i}\oint_{C_z} dz'(z'-z)^{-1}\langle T(z')\varphi(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle \\
&\quad = \sum_{l=1}^{m}\left(\frac{\delta_l}{(z-z_l)^2}+\frac{1}{z-z_l}\partial_{z_l}\right)\langle\varphi(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle\,.
\end{aligned}$$

As a result, we get

$$\begin{aligned}
&\frac{3}{2(2\delta+1)}\partial_z^2\langle\varphi(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle \\
&\qquad -\sum_{l=1}^{m}\left(\frac{\delta_l}{(z-z_l)^2}+\frac{1}{z-z_l}\partial_{z_l}\right)\langle\varphi(z)\varphi_1(z_1)\cdots\varphi_m(z_m)\rangle = 0\,. \qquad \text{(B.57)}
\end{aligned}$$

Equation (B.11) for the third level is treated analogously. Let us put $N=3$. There are three secondary fields of the third level:

$$\varphi^{(-3)} = L_{-3}\varphi\,, \qquad \varphi^{(-1,-2)}\varphi\,, \qquad \varphi^{(-1,-1,-1)} = (L_{-1})^3\varphi\,. \qquad \text{(B.58)}$$

Make up the superposition Q_3 of these fields, satisfying

$$L_1Q_3 = L_2Q_3 = 0\,. \qquad \text{(B.59)}$$

The first of these equations gives [see (B.36) and (B.39)]

$$Q_3 = P_3 = \left[L_{-3} - \frac{2}{\delta+2}L_{-1}L_{-2} + \frac{1}{(\delta+1)(\delta+2)}(L_{-1})^3\right]\varphi\,. \qquad \text{(B.60)}$$

The second equation leads to the relation between the dimension and the central charge

$$C = -\frac{3\delta^2-7\delta+2}{\delta+1}\,. \qquad \text{(B.61)}$$

The Q_N fields of the succeeding levels are constructed in the same manner.

One easily understands that these results could be obtained using the covariant secondary fields P_s from the very beginning. Indeed, the expressions (B.55) and (B.57) coincide° with the results of the previous section [see (7.7) and (7.2)] and Eq. (B.61) coincides with (7.14). Each field Q_s with $s=N$ is a superposition of the covariant secondary fields (6.34) of dimension $d+s$ which fall into different generations. The coefficients before these fields in the superposition Q_s are calculated from Eq. (6.35).

It is readily seen that the latter guarantees the covariance of the equations $Q_s=0$ under the infinite parametric conformal group. The fields Q_s transform

°The dimension d is related to the parameter δ by $d=\delta+\bar{\delta}$; see (B.3). For the scalar fields one should set $\delta=\bar{\delta}=d/2$.

similarly to primary fields. Indeed, the variation $\Delta_\varepsilon P_s$ includes two types of terms: "ordinary" terms containing the P_s field together with its derivative, and anomalous terms, which contain the fields $P_{s'}$ for $s' < s$, the field φ, and their derivatives. The factors weighting the fields (6.34) in the superposition Q_s should be chosen so as to ensure the cancellation of anomalous contributions to the variation $\Delta_\varepsilon Q_s$. In this case, the field Q_s transforms as a primary field and satisfies the conditions (B.49). Let us present several examples.

1. *The field* Q_2. It is given by the covariant combination of the fields (B.52) and coincides with P_2. So, the first equation in (B.53) is satisfied identically. The second equation is equivalent to the requirement of cancellation of anomalous terms in the total contribution of the fields (B.52) to the variation $\Delta_\varepsilon P_2$. Indeed, this variation contains two types of terms. The ordinary term includes the field P_2 and its derivative; the anomalous term includes derivatives of the field φ. Let us recall that the anomalous term results from the commutator $[P_2, T]$. The existence of the anomalous term is equivalent to the existence of the nonzero Green function

$$\langle P_2 \varphi T \rangle ,$$

which appears in the Ward identities. Hence, the absence of the anomalous contribution to the variation $\Delta_\varepsilon P_s$ is equivalent to the equation

$$\langle P_2 \varphi T \rangle = 0 . \tag{B.62}$$

The latter guarantees the contributions of the anomalous type to cancel after the substitution into the sum Q_s, and fixes the value of the central charge. Thus Eq. (B.62) is an $\mathrm{SL}(2, C)$-covariant version of the second equation in (B.59). Together with (B.56), the condition (B.62) ensures the covariance of the equation

$$Q_2 = P_2 = 0$$

with respect to infinite-dimensional conformal transformations.

2. *The field* Q_3. Being the $\mathrm{SL}(2, C)$-covariant combination of the fields (B.58), it coincides with P_3. The first of the conditions (B.59) holds identically. Consider the variation $\Delta_\varepsilon P_3$. Besides ordinary terms, i.e. the P_3 field and its derivative, it has two types of anomalous contributions. The contributions of the first type include the terms due to the field φ, while those of the second type are related to the field P_2. The equation

$$\langle P_3 \varphi T \rangle = 0 \tag{B.63}$$

guarantees the cancellation of the first type contributions and fixes the value of the central charge. Upon this, the second type contributions cancel by themselves, with no special constraints involved. Thus Eq. (B.63) is an $\mathrm{SL}(2, C)$-covariant version of the second equation in (B.59). It ensures conformal covariance of the equation

$$Q_3 = P_3 = 0 .$$

3. *The field* Q_4. The situation becomes more complicated in the case of models for subsequent levels $N \geq 4$. One can show that

$$\langle P_s \varphi T\rangle \neq 0 \quad \text{for} \quad s \geq 4\,. \tag{B.64}$$

Let $s = N = 4$. Among the secondary fields of the fourth level

$$(L_{-2})^2\varphi\,, \quad L_{-4}\varphi\,, \quad L_{-1}L_{-3}\varphi\,, \quad (L_{-1})^2 L_{-2}\varphi\,, \quad (L_{-1})^4\varphi \tag{B.65}$$

one has the second generation field

$$\varphi^{(-2,-2)} = (L_{-2})^2\varphi\,.$$

There exists a pair of independent combinations,

$$P_4 = L_{-4}\varphi + \cdots\,, \qquad \tilde{P}_4 = (L_{-2})^2\varphi + \cdots\,. \tag{B.66}$$

Both of them satisfy

$$L_1 P_4 = L_1 \tilde{P}_4 = 0\,. \tag{B.67}$$

The field $\tilde{P}_4$ belongs to the second generation of SL$(2,C)$-covariant secondary fields. From the results of Sec. 6 it follows that the variations

$$\Delta_\varepsilon P_4\,, \qquad \Delta_\varepsilon \tilde{P}_4 \tag{B.68}$$

include several types of anomanous terms. Let us consider those containing the fields φ and P_2. The terms related to φ lead to the appearance of the nonzero Green functions

$$\langle P_4 \varphi T\rangle\,, \qquad \langle \tilde{P}_4 \varphi T\rangle\,.$$

To the anomalous terms related to P_2 the nonzero Green functions

$$\langle P_4 P_2 T\rangle\,, \qquad \langle \tilde{P}_4 P_2 T\rangle$$

correspond. Consider the combination

$$Q_4 = P_4 + \alpha \tilde{P}_4\,. \tag{B.69}$$

The requirement of cancellation of total anomalous contributions to the variation $\Delta_\varepsilon Q_4$ is equivalent to a pair of equations,

$$\langle Q_4 \varphi T\rangle = \langle P_4 \varphi T\rangle + \alpha \langle \tilde{P}_4 \varphi T\rangle = 0\,, \tag{B.70}$$

$$\langle Q_4 P_2 T\rangle = \langle P_4 P_2 T\rangle + \alpha \langle \tilde{P}_4 P_2 T\rangle = 0\,, \tag{B.71}$$

which fix the coefficient α and the central charge. These calculations have been done in the previous section; see (7.45). One sees that the values of the central charge found there, and those imposed by the Kac formula, actually do coincide. Indeed, setting $N = 4$ in (B.51), we get two solutions:

$$\delta = \delta_{2,2} = 3\alpha_0^2 = \frac{1}{8}(1 - C),$$
$$\delta = \delta_{1,4} = \frac{1}{4}\left(30\alpha_0^2 + 30\alpha_0\sqrt{\alpha_0^2 + 1} + 9\right),$$
$$\alpha_0^2 = \frac{1 - C}{24},$$

or

$$C = 1 - 8\delta,$$
$$C = \frac{2(\delta - 1)(33 - 8\delta)}{5(2\delta + 3)}.$$

Passing to the dimension $d = 2\delta$ we get (7.45).

Thus, Eqs. (B.70) and (B.71) represent an $\mathrm{SL}(2, C)$-covariant form of the condition

$$L_2 Q_4 = 0.$$

This leads, on account of (B.67), to the covariance of the equation

$$Q_4 = 0$$

under the infinite dimensional group of conformal transformations.

Equations (B.11) for the arbitrary level may be treated similarly. For any N the field Q_N is representable as a superposition of $\mathrm{SL}(2, C)$-covariant secondary fields (6.34). Under that, one can rewrite the conditions (B.49) in the form of equations of the type (B.70), (B.71).

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