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Functional Techniques in Physics.

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Introduction.

The purpose of this paper is to collect some results obtained by FRADKIN *et al.* (see the references) by using functional techniques.

The general technique is given and some applications are made to quantum field theory; we note however that this method has been fruitfully applied also to other fields, such as many-body problems and statistical physics.

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It is noteworthy that this approach allows the introduction of a modified perturbation theory.

Some of the results collected here appear for the first time in English.

1. – Functional equations for the S-matrix [1].

As is well known, every problem of quantum field theory can be reduced to the determination of Green's functions for interacting particles (bosons or fermions). The knowledge of the Green's function allows not only the solution of problems involving particle collisions, but also the determination of bound states of a system, which are represented by the poles of Green's functions or of their analytic continuations. That is, the problem reduces to the finding of the propagators $\tau(x_1 \dots x_n) = T\langle\varphi(x_1) \dots \varphi(x_n)\rangle$, where $\varphi(x)$ are the field operators in Heisenberg's picture, and T is the usual time-ordering operator.

The equations for the operators, in Heisenberg's picture, give an infinite system of coupled equations for the propagators; it is possible to reduce this infinite system to a closed system of one or two functional equations.

To obtain this [2] an additional interaction between Bose and Fermi fields and external sources is introduced in the Lagrangian

$$L = L_0 + L_{\text{int}} + L_{\text{ext}} , \tag{1.1}$$

$$\left\{ \begin{aligned} L_{\text{int}} &= j(x)\varphi(x) , \\ L_{\text{ext}} &= I(x)\varphi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) , \end{aligned} \right. \tag{1.2}$$

where

$$j(x) = \frac{i}{2} g_0 \,\text{Sp}\, \gamma[\bar{\psi}(x)\psi(x) - \psi(x)\bar{\psi}(x)] ; \tag{1.3}$$

$\eta(x)$ is a Fermi-field source which anticommutes with $\bar{\eta}$, ψ and $\bar{\psi}$, and commutes with the Bose operators; L_0 is the Lagrangian of free fields, g_0 the coupling constant and $I(x)$ is the source of the Bose field; γ indicates the generic interaction.

As is well known the functional equations for the S-matrix may be written as follows:

$$\left\{ \begin{aligned} &\left(\gamma_\mu \partial_\mu + m_0 - g_0 \gamma \frac{\delta}{\delta I(x)}\right) \frac{\delta S}{\delta \bar{\eta}} = i\eta S , \\ &(-\Box + k_0^2) \frac{\delta S}{\delta I} = iIS - \text{Sp}\, g_0 \gamma \frac{\delta^2 S}{\delta\bar{\eta}(x)\,\delta\eta(x)} , \\ &\frac{\delta S}{\delta \eta}\left(-\gamma_\mu \partial_\mu + m_0 - g_0 \gamma \frac{\delta}{\delta I}\right) = i\bar{\eta} S . \end{aligned} \right. \tag{1.4}$$

As in what follows we will be interested in the vacuum expectation values of T ordered operator products, it will be sufficient to solve the system of functional equations for the matrix element $Z=\langle 0|S|0\rangle$ of the vacuum-vacuum transition which is the generating function of all Green's functions. By eqs. (1.4) Z will satisfy the relationship [1, 5, 6]

$$\left\{\begin{aligned}&\left(\gamma_\mu\partial_\mu+m_0-g_0\gamma\frac{\delta}{\delta I}\right)\frac{\delta Z}{\delta\bar\eta}=i\eta Z\,,\\&(-\Box+k_0^2)\frac{\delta Z}{\delta I}=iIZ-g_0\,\mathrm{Sp}\,\gamma\frac{\delta^2 Z}{\delta\bar\eta(x)\,\delta\eta(x)}\,,\\&\frac{\delta Z}{\delta\eta}\left(-\gamma_\mu\partial_\mu+m_0-g_0\gamma\frac{\delta}{\delta I}\right)=i\bar\eta Z\,,\end{aligned}\right.\tag{1.5}$$

with boundary conditions:

1) $\delta Z/\delta\eta=\delta Z/\delta\bar\eta=\delta Z/\delta I=0$ for $I=\eta=\bar\eta=0$.

2) The functional derivatives of Z for $\eta,\bar\eta$ must satisfy the spectral representation of Heisenber's operators.

Equations (1.5) give a conservation law of the current in the presence of sources:

$$\mathrm{Sp}\,\gamma_\mu\partial_\mu\frac{\delta^2 Z}{\delta\bar\eta(x)\,\delta\eta(x)}=i\eta\frac{\delta Z}{\delta\eta}-i\bar\eta\frac{\delta Z}{\delta\bar\eta}\,.\tag{1.6}$$

It can be shown that a simple relationship between the derivatives of the functional Z and the S-matrix for $\eta=I=0$ exists. Such a relationship can be expressed in terms of operators φ_{in} and Ψ_{in} in the normal form. The derivatives in (1.8) operate on the derivatives of the renormalized field sources

$$S_{\eta=I=0}=:\!\exp[\Omega]\!:Z|_{I=\eta=0}\,,\tag{1.7}$$

where

$$\Omega=\int\left[\varphi_{in}(x)(\mu^2-\Box)\frac{\delta}{\delta I(x)}+\overline{\Psi}_{in}(x)(\gamma_\mu\partial_\mu+m_0)\frac{\delta}{\delta\bar\eta'(x)}+\right.\\\left.+\frac{\delta}{\delta\eta'(x)}(-\gamma_\mu\partial_\mu+m_0)\Psi_{in}(x)\right]\mathrm{d}^4x\,.\tag{1.8}$$

In particular, for vector coupling $\gamma\to\gamma_\mu$ (in quantum electrodynamics $k_0^2=0$) from eq. (1.5) we obtain

$$(-\Box+k_0^2)\partial_\mu\frac{\delta Z}{\delta\mathscr{I}_\mu(x)}=i\partial_\mu\mathscr{I}_\mu Z+ie\left(\bar\eta(x)\frac{\delta Z}{\delta\bar\eta(x)}-\frac{\delta Z}{\delta\eta}\eta(x)\right).\tag{1.9}$$

Having taken the functional derivative of (1.9) with respect to the η, $\bar{\eta}$, $\mathscr{I}$ and setting $\eta = \bar{\eta} = \mathscr{I} = 0$, we can obtain all the modified Ward relations, which are obtained in [4].

2. – Solutions of functional equations for the S-matrix [1, 2, 5-11].

2·1. *Solutions in integral form* [1]. – To solve the eqs. (1.5) it is convenient to pass to the momentum representation:

$$\text{(2.1)} \quad \begin{cases} (i\gamma_\mu p_\mu + m)\dfrac{\delta Z}{\delta\eta(p)} - \dfrac{g}{(2\pi)^2}\gamma\displaystyle\int\dfrac{\delta^2 Z}{\delta I(p-k)\,\delta\bar{\eta}(k)}\,\mathrm{d}^4k = i\eta(p)Z\,, \\ \dfrac{\delta Z}{\delta\bar{\eta}(p)}(i\gamma_\mu p_\mu + m) - \dfrac{g}{(2\pi)^2}\displaystyle\int\dfrac{\delta^2 Z}{\delta I(k-p)\,\delta\eta(k)}\,\mathrm{d}^4k\gamma = i\bar{\eta}(p)Z\,, \\ (k^2+\mu^2)\dfrac{\delta Z}{\delta I(-k)} = iI(k)Z - \dfrac{g}{(2\pi)^2}\mathrm{Sp}\displaystyle\int\gamma\,\dfrac{\delta^2 Z}{\delta\bar{\eta}(p)\,\delta\eta(q)}\,\delta(p-k-q)\,\mathrm{d}^4p\,\mathrm{d}^4q\,. \end{cases}$$

To solve this system of equations we seek a solution of the form [5]

$$\text{(2.2)} \qquad Z = c\int\exp\left[i\int I(k)\,a(k)\,\mathrm{d}^4k\right]R(\eta, \bar{\eta}, a(k_1)\,\dots\,a(k_n))\prod_k \mathrm{d}a(k)\,,$$

where c is a constant fixed by the initial conditions. Equation (2.1) therefore gives a system of equations for R:

$$\text{(2.3)} \quad \begin{cases} \displaystyle\int\left\{(i\gamma_\mu p_\mu + m)\,\delta(p-k) - \dfrac{ig}{(2\pi)^2}\gamma a(p-k)\right\}\dfrac{\delta R}{\delta\bar{\eta}(k)}\,\mathrm{d}^4k = i\eta(p)R\,, \\ \dfrac{\delta R}{\delta\eta(p)}(i\gamma_\mu p_\mu + m) - \dfrac{ig}{(2\pi)^2}\displaystyle\int a(k-p)\,\dfrac{\delta R}{\delta\eta(k)}\,\gamma\,\mathrm{d}^4k = i\bar{\eta}(p)R\,, \\ i(k^2+\mu^2)\,a(-k)\,R = -\dfrac{\delta R}{\delta a(k)} - \dfrac{g}{(2\pi)^2}\mathrm{Sp}\displaystyle\int\gamma\,\dfrac{\delta^2 R}{\delta\eta(p)\,\delta\bar{\eta}(q)}\,\delta(p-q-k)\,\mathrm{d}^4p\,\mathrm{d}^4q, \end{cases}$$

where

$$a(k) = \frac{1}{(2\pi)^2}\int\exp\left[-ikx\right]a(x)\,\mathrm{d}^4x\,.$$

The solution of the first two equations can be represented in the form

$$\text{(2.4)} \qquad \begin{cases} \dfrac{\delta R}{\delta\bar{\eta}(p)} = i\displaystyle\int G(p, k|a)\,\eta(k)\,\mathrm{d}^4k R\,, \\ \dfrac{\delta R}{\delta\eta(p)} = i\displaystyle\int\bar{\eta}(k)\,G(k, p|a)\,\mathrm{d}^4k R\,, \end{cases}$$

where $G(k, p|a)$ indicates the Green's function of the nucleon in an external field. $G(k, p|a)$ satisfies the equation

$$(2.5)\qquad (i\gamma_\mu p_\mu + m)G(p, k|ga) - \frac{ig}{(2\pi)^2}\gamma\int a(p-k')G(k', k|a)\,\mathrm{d}^4k' = \delta(p-k)\,.$$

From (2.4) we have

$$(2.6)\qquad R = R_1(a)\exp\left[i\int\bar{\eta}(p)G(p, k|a)\eta(k)\,\mathrm{d}^4p\,\mathrm{d}^4k\right],$$

where R_1 depends only on the external field a and satisfies the equation

$$(2.7)\quad i(k^2+\mu^2)a(-k)R_1 = -\frac{\delta R_1}{\delta a(k)} - \frac{ig}{(2\pi)^2}\,\mathrm{Sp}\,\gamma\int G(p, q|ga)\,\delta(p-q-k)\,\mathrm{d}^4p\,\mathrm{d}^4q\,R_1\,.$$

Equation (2.7) is easily solved giving

$$(2.8)\qquad R_1 = \exp\left[\int\left\{-\frac{i}{2}\,a(k)(k^2+\mu^2)a(-k) - \right.\right.$$
$$\left.\left. - \frac{i}{(2\pi)^2}\,\mathrm{Sp}\,\gamma\int\int_0^g \mathrm{d}\lambda\, G(p, q|\lambda a)\,a(k)\,\delta(p-q-k)\,\mathrm{d}^4p\,\mathrm{d}^4q\right\}\mathrm{d}^4k\right]$$

and therefore for Z we find

$$(2.9)\qquad c^{-1}Z = \int\exp\left[\int\left[iI(k)a(k) - \frac{i}{2}a(k)(k^2+\mu^2)\,a(-k) + \right.\right.$$
$$+ i\int\bar{\eta}(p)\,G(p, k|ga)\,\eta(p)\,\mathrm{d}^4p - \frac{i}{(2\pi)^2}\,\mathrm{Sp}\,\gamma\int\int_0^g\mathrm{d}\lambda\,G(p, q|\lambda a)\,\cdot$$
$$\left.\left.\cdot\, a(k)\,\delta(p-q-k)\,\mathrm{d}^4p\,\mathrm{d}^4q\right]\mathrm{d}^4k\right]\prod_k \mathrm{d}a(k)\,,$$

where $1/c$ is the value of the right-hand side of (2.9) for $\eta = \bar{\eta} = I = 0$.

2·2. *Operator solution of the problem of interaction of two fields* [13]. – Let us now look for the operator solution of the functional equations (1.5). In the case of free fields ($g = 0$) the solution has the form

$$(2.10)\quad Z(g=0) = Z_0 = \exp\left[i\int\left\{\bar{\eta}(x)\,S_F(x-y)\,\eta(y) + \frac{I(x)}{2}\,D_F(x-y)\,I(y)\right\}\mathrm{d}^4x\,\mathrm{d}^4y\right],$$

where

$$(2.11)\qquad \begin{cases} S_F(x-y) = \dfrac{1}{(2\pi)^4}\displaystyle\int \dfrac{\exp[ip(x-y)]}{\gamma_\mu p_\mu + m}\,\mathrm{d}^4p\,, \\ D_F(x-y) = \dfrac{1}{(2\pi)^4}\displaystyle\int \dfrac{\exp[ip(x-y)]}{p^2+k^2}\,\mathrm{d}^4p\,. \end{cases}$$

If $g \neq 0$, let us look for a solution of the form

$$(2.12)\qquad Z = AZ_0\,,$$

where A is an operator depending on the functional derivatives $\delta/\delta I$, $\delta/\delta\eta$, $\delta/\delta\bar\eta$. Substituting eqs. (2.12) in (1.5) we obtain the following equations for A:

$$(2.13)\qquad \begin{cases} [A, \bar\eta(x)] = \quad iAg\,\dfrac{\delta}{\delta I}\,\dfrac{\delta}{\delta\eta}\,\gamma\,, \\ [A, \eta(x)] = -iAg\gamma\,\dfrac{\delta}{\delta I}\,\dfrac{\delta}{\delta\bar\eta}\,, \\ [A, I(x)] = \quad iAg\,\mathrm{Sp}\,\gamma\,\dfrac{\delta^2}{\delta\eta\,\delta\bar\eta}\,, \end{cases}$$

from which

$$(2.14)\qquad A = \exp\left[\int\left\{ig\,\frac{\delta}{\delta\eta_\alpha(x)}\,\gamma_{\alpha\beta}\,\frac{\delta}{\delta\bar\eta_\beta(x)}\,\frac{\delta}{\delta I(x)}\,\mathrm{d}^4x\right\}\right];$$

substituting in (2.12) we therefore obtain for Z

$$(2.15)\qquad cZ = \exp\left[ig\int\frac{\delta}{\delta\eta(x)}\,\gamma\,\frac{\delta}{\delta\bar\eta(x)}\,\frac{\delta}{\delta I(x)}\,\mathrm{d}^4x\right]\cdot \\ \cdot\exp\left[i\left\{\int\bar\eta(x)S_F(x-y)\eta(y) + \frac{I(x)D_F(x-y)I(y)}{2}\right\}\mathrm{d}^4y\,\mathrm{d}^4x\right].$$

Generally, for an arbitrary interaction the operator A is of the kind

$$(2.16)\qquad A = \exp[iL_{\mathrm{int}}]\,,$$

where the fields ψ, $\bar\psi$, φ in the interaction Lagrangian are substituted respectively by $-i(\delta/\delta\bar\eta(x))$, $i(\delta/\delta\eta(x))$, $-i(\delta/\delta I(z))$.

In (2.15) it is possible to carry out the functional differentiation with respect to the variables of one of the interacting fields. Moreover, employing the properties of the displacement operator $I(x)$ we obtain

$$(2.17)\qquad cZ = \exp\left[i\int\left\{\left(I(x) + ig\,\frac{\delta}{\delta\eta_\alpha(x)}\,\gamma_{\alpha\beta}\,\frac{\delta}{\delta\bar{\eta}_\beta(x)}\right)\frac{D_F(x-y)}{2}\cdot\right.\right.$$
$$\left.\left.\cdot\left(I(y) + ig\,\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\,\gamma_{\alpha\beta}\,\frac{\delta}{\delta\bar{\eta}_\beta(x)}\right)\right\}\mathrm{d}^4x\,\mathrm{d}^4y\right]\exp\left[i\int\bar{\eta}(x)\,S_F(x-y)\,\eta(y)\,\mathrm{d}^4x\,\mathrm{d}^4y\right].$$

Eliminating the functional derivatives with respect to the variables η and $\bar{\eta}$, we obtain

$$(2.18)\qquad Z = \frac{1}{c}\,R\left(\eta, \bar{\eta}, \frac{\delta}{\delta I}\right)\exp\left[i\int I(x)\,\frac{D_F(x-y)}{2}\,I(y)\,\mathrm{d}^4x\,\mathrm{d}^4y\right].$$

Substituting (2.18) in the first and third of (1.5) we obtain

$$(2.19)\qquad \begin{cases} \dfrac{1}{R}\,\dfrac{\delta R}{\delta\bar{\eta}(x)} = \quad i\displaystyle\int G\left(x, y\,\middle|\,\frac{\delta}{\delta I}\right)\eta(y)\,\mathrm{d}^4y\,, \\ \dfrac{1}{R}\,\dfrac{\delta R}{\delta\eta(y)} = -i\displaystyle\int\bar{\eta}(x)\,G\left(x, y\,\middle|\,\frac{\delta}{\delta I}\right)\mathrm{d}^4x\,, \end{cases}$$

where $G(x, y|\delta/\delta I)$ is the Green's function of an electron in an external field $a(x)$; here the substitution $a(x)\to\delta/\delta I$ is effected. $G(x, y|a)$ satisfies the equation

$$(2.20)\qquad [\gamma_\mu\,\partial_\mu + m - ig\gamma a(x)]G(x, y|a) = \delta(x-y)\,.$$

Equations (2.19) can be solved, giving

$$(2.21)\qquad R = \exp\left[i\int\bar{\eta}(x)\,G\left(x, y\,\middle|\,\frac{\delta}{\delta I}\right)\eta(y)\,\mathrm{d}^4x\,\mathrm{d}^4y\,R_1\left(\frac{\delta}{\delta I}\right)\right].$$

The operator R_1 depends only on the operator $\delta/\delta I$ and is determined by the second equation of (1.5):

$$(2.22)\qquad [R, I] =$$
$$= -g\,\mathrm{Sp}\,\gamma\left[G\left(x, x\,\middle|\,\frac{\delta}{\delta I}\right) - i\int G\left(x, y\,\middle|\,\frac{\delta}{\delta I}\right)\eta(y)\,\mathrm{d}^4y\int\bar{\eta}(y)\,G\left(y, x\,\middle|\,\frac{\delta}{\delta I}\right)\mathrm{d}^4y\right]R\,,$$

from which for (2.21)

$$(2.23)\qquad [R_1, I] = -R_1\,\mathrm{Sp}\,\gamma G\left(x, x\,\middle|\,g\,\frac{\delta}{\delta I}\right)$$

and therefore

$$(2.24)\qquad R_1=\exp\left[-\int\int_0^g \mathrm{d}g'\,\mathrm{Sp}\,\gamma G\left(x,x\,\middle|\,g'\frac{\delta}{\delta I}\right)\frac{\delta}{\delta I}\right]\mathrm{d}^4x\,.$$

Finally, Z results from the equation

$$(2.25)\qquad cZ=\exp\left[i\int\bar\eta(x)\,G\left(x,y\,\middle|\,g\frac{\delta}{\delta I}\right)\eta(y)\,\mathrm{d}^4x\,\mathrm{d}^4y\right]-$$
$$-\int\int_0^g \mathrm{Sp}\,\gamma G\left(x,x\,\middle|\,g'\frac{\delta}{\delta I}\right)\mathrm{d}g'\frac{\delta}{\delta I}\,\mathrm{d}^4x\,\exp\left[\int iI(x)\,\frac{D_F(x,y)}{2}\,I(y)\,\mathrm{d}^4y\,\mathrm{d}^4x\right].$$

Equations (1.7) and (2.15) now allow us to write the operational solution of $S(I,\eta)$ in the presence of sources:

$$(2.26)\qquad S(I,\eta)=:\exp:[\Omega]|_{I=\eta=0}=Z[I(x)+Z_3^{-\frac12}(\mu^2-\partial_\mu^2)\varphi_{\mathrm{in}}(x),\bar\eta(x)+$$
$$+Z_2^{-\frac12}\overline{\Psi}_{\mathrm{in}}(x)(-\gamma_\mu\overrightarrow{\partial}_\mu+m),\eta(x)+Z_2^{-\frac12}(\gamma_\mu\overleftarrow{\partial}_\mu+m)\Psi_{\mathrm{in}}(x)]\,.$$

The ordinary S-matrix is obtained from (2.26) when $I=\eta=\bar\eta=0$. It is possible to develop the integral solution in a powers series of g^2 and all Feynman's diagrams are reobtained. In conclusion, it is possible, by means of this functional method, not only to add up the Feynman's diagrams, but also to attempt the construction of methods differing from the usual perturbation theory.

In the application of such a method we find essentially two mathematical difficulties. It is difficult to find a Green's function in an arbitrary external field; integrations can be effected only on restricted classes of functions. Specific problems can, in some cases, help to overcome the first difficulty in the sense that, since Green's functions have a physical interest in definite energy regions, we can try to find good approximations for the Green's functions in the external field, and from these obtain asymptotic behaviours of physical quantities in the desired region.

2·3. *Dependence of Green's functions on the longitudinal part of the electromagnetic field* [15]. – Let us consider the interaction between electrons and the electromagnetic field with arbitrary gauges [3, 15, 16]. Let us decompose the field A into transverse and longitudinal components:

$$(2.27)\qquad A_\mu=A_\mu^t+A_\mu^l=A_\mu^t+\frac{\partial\varphi}{\partial x_\mu},$$

where

$$\varphi = \int \frac{ik_\nu A_\nu(k)}{k^2} \exp[ik_\mu x_\mu] \, d^4k . \tag{2.28}$$

The interaction Lagrangian is written

$$\int I_\mu A_\mu \, d^4x = \int I^t_\mu(x) A^t_\mu(x) \, d^4x + \int \frac{\partial \varphi}{\partial x_\mu} I^l_\mu d^4x = \int I^t_\mu(x) A^t_\mu(x) \, d^4x - \int \varphi J \, d^4x , \tag{2.29}$$

where

$$J = \frac{\partial I^l_\mu(x)}{\partial x_\mu} .$$

It is easy to verify the following relationship:

$$\begin{aligned} \int \{ I_\mu(x) D_{\mu\nu}(x-y) I_\mu(y) \} d^4x \, d^4y = \\ = \int \{ I^t_\mu(x) D^t_{\mu\nu}(x-y) I^t_\nu(y) + I^l_\mu(x) D^l_{\mu\nu}(x-y) I^t_\nu(y) \} d^4x \, d^4y = \\ = \int \{ I^t_\mu D^t_{\mu\nu}(x-y) I^t_\nu(y) + J(x) \Delta^l(x-y) J(y) \} d^4x \, d^4y , \end{aligned} \tag{2.30}$$

where

$$D^l_{\mu\nu}(x-y) = \frac{\partial^2}{\partial x_\mu \partial y_\nu} \Delta^l(x-y) = \int k_\mu k_\nu \Delta^l(k) \exp[ik(x-y)] \, d^4k \tag{2.31}$$

for the free field $\Delta^l = 1/(k^4 - i\varepsilon k^2)$.

The operational solution in closed form for the generating function is

$$\begin{aligned} cZ = \exp\Bigg[\iint \Bigg\{ -\int_0^e de' \, \mathrm{Sp} \, \gamma_\mu G\left(x, x \,\Big|\, e' \frac{\delta}{\delta I}\right) \frac{\delta}{\delta I_\mu(x)} + \\ + i \int \bar\eta(x) G\left(x, y \,\Big|\, \frac{e\delta}{\delta I}\right) \eta(y) \, d^4y \Bigg\} d^4x \Bigg] \cdot \\ \cdot \exp\left[\frac{i}{2} \int \left\{ I^t_\mu(x) D^t_{\mu\nu}(x-y) I^t_\nu(y) + I^l_\mu(x) D^l_{\mu\nu}(x-y) I^l_\nu(y) \right\} d^4x \, d^4y \right], \end{aligned} \tag{2.32}$$

where $G(x, y|\delta/\delta y)$ is the Green's function of an electron in an arbitrary external field A_μ, that is $G(x, y|A)$, where $A_\mu = A^t_\mu + A^l_\mu$ with the substitution $iA_\mu \to \delta/\delta I_\mu = \delta/\delta I^t_\mu + \delta/\delta I^l_\mu$. $G(x, y|A)$ satisfies the equation

$$\left[\gamma_\mu \frac{\partial}{\partial x_\mu} + m - ie\gamma_\mu \left(A^t_\mu + \frac{\partial \varphi}{\partial x_\mu} \right) \right] G(x, x'|A) = \delta(x - x') . \tag{2.33}$$

From eq. (2.33) the explicit dependence of $G(xx'|A)$ of the longitudinal field φ is obtained:

$$G(xx'|A) = G(xx'|ieA^t)\exp\left[ie[\varphi(x)-\varphi(x')]\right]. \tag{2.34}$$

Therefore

$$G\left(xx'\left|\frac{\delta}{\delta I}\right.\right) = G\left(xx'\left|e\frac{\delta}{\delta I^t}\right.\right)\exp\left[\frac{\delta}{\delta J(x')}-\frac{\delta}{\delta J(x)}\right]^l, \tag{2.35}$$

which shows that G depends only on the transversal component. Substituting eq. (2.35) in (2.32) and using the properties of the displacement operator, we obtain [15]

$$\begin{aligned}cZ = {}&\exp\left[-\int\int_0^e \mathrm{d}e'\,\mathrm{Sp}\left(\gamma_\mu G\left(xx'\left|e'\frac{\delta}{\delta I^t}\right.\right)\frac{\delta}{\delta I^t_\mu(x)}\right)\mathrm{d}^4x\right]\cdot\\ &\cdot\sum_{k=0}^\infty\frac{i^k}{k!}\int \mathrm{d}x_1\ldots\mathrm{d}x_k\,\mathrm{d}x_1'\ldots\mathrm{d}x_k'\,\bar\eta(x_1)\,G\left(x_1,x_1'\left|\frac{\delta}{\delta I^t}\right.\right)\eta(x_1')\ldots\bar\eta(x_k)\,G\left(x_k,x_k'\left|e\frac{\delta}{\delta I^t}\right.\right)\cdot\\ &\cdot\eta(x_k')\exp\left[\frac{ie}{2}\left\{\sum_{m,n}^k e[\Delta^l(x_m-x_n)+\Delta^l(x_m'-x_n')-\Delta^l(x_m'-x_n)-\Delta^l(x_m-x_n')]+\right.\right.\\ &\left.\left.+2\sum_{m=1}^n\int\left[\frac{\partial\Delta^l(x_m-z)}{\partial z_\mu}-\frac{\partial\Delta^l(x_m'-z)}{\partial z_\mu}\right]I^l_\mu(z)\,\mathrm{d}^4z\right\}\right]\cdot\\ &\qquad\cdot\exp\left[\frac{i}{2}\left\{\int I^t_\mu(x)\,D^t_{\mu\nu}(x-y)\,I^t_\nu(y)+I^l_\mu(x)\,D^l_{\mu\nu}(x-y)\,I^l_\mu(y)\right\}\mathrm{d}^4x\,\mathrm{d}^4y\right].\end{aligned} \tag{2.36}$$

It is therefore possible to render explicit completely the dependence on the longitudinal part of the electromagnetic field, retaining only the functional derivatives of the transversal field. By functional differentiation with respect to external sources, it is then possible to obtain the dependence on the longitudinal field of the corresponding Green's function.

3. – Some solvable models and approximations.

3·1. *Theory of scalar interaction* [8, 17]. – Let us consider the interaction between a neutral scalar meson and a nucleon in a nonrelativistic case. Let us show that in this case we can easily obtain all the Green's functions, taking approximately into account the nucleon recoil. The equation for the Green's function in an external field, disregarding nucleon recoil, has the form

$$\left\{-i\frac{\partial}{\partial t}-m-g\varphi(t)\right\}G(t,t'|\varphi)=\delta(t-t')\,\delta^3(x-x'). \tag{3.1}$$

Equation (3.1) has the solution

$$G(t, t'|\varphi) = S(t-t') \exp\left[ig\int_{t'}^{t}\varphi(s)\,\mathrm{d}s\right], \tag{3.2}$$

where

$$S(t-t') = i\theta(t-t') \exp\left[-im(t-t')\right]\delta^3(x-x'). \tag{3.3}$$

Substituting (3.2) in the operator solution (2.25) we obtain the general solution for the S-matrix:

$$\frac{\langle S\rangle}{c} = \sum_{k=0}^{\infty}\frac{i^k}{k!}\int_0^{\infty}\mathrm{d}t_n\mathrm{d}t'_n\bar{\eta}(t_n)S(t_n-t'_n)\eta(t'_n)\exp\left[\frac{i}{2}\iint\left\{\left[I(t,x)+\sum_{m=1}^{k}g\int_{t'_m}^{t_m}\delta(t-s)\,\mathrm{d}s\right]\cdot\right.\right.$$
$$\left.\left.\cdot D(t-t',0)\left[I(t',x')+\sum_{m=1}^{k}g\int_{t'_m}^{t_m}\delta(t'-s)\,\mathrm{d}s\right]\right\}\mathrm{d}x^4\,\mathrm{d}^4x'\right]. \tag{3.4}$$

If we take into account the nucleon recoil, the equation for the Green's function becomes

$$\left\{-i\frac{\partial}{\partial t}-\frac{\nabla^2}{2m}-g\varphi(x)\right\}G(x,y|\varphi)=\delta(x-y). \tag{3.5}$$

In the p-representation (3.5) becomes

$$G(xy)=\frac{1}{(2\pi)^4}\int G(x,p|\varphi)\exp\left[ip(x-y)\right]\mathrm{d}^4p, \tag{3.6}$$

$$\left\{-i\frac{\partial}{\partial t}-\frac{\nabla^2}{2m}-i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}-\omega+\frac{p^2}{2m}-g\varphi(x)\right\}G(x,p)=1. \tag{3.7}$$

From (3.7) we obtain

$$G(x,p)=i\int_0^{\infty}\exp\left[-i\nu\left(\frac{p^2}{2m}-\omega-\frac{\nabla^2}{2m}-i\frac{\partial}{\partial t}-i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}-g\varphi-i\delta\right)\right]\mathrm{d}\nu=$$
$$=i\int_0^{\infty}\exp\left[-i\nu\left(\frac{p^2}{2m}-\omega-i\delta\right)\phi(\nu)\right]\mathrm{d}\nu, \tag{3.8}$$

where $\phi(\nu)$ satisfies the equation

$$i\frac{\partial\phi(\nu)}{\partial\nu}=\left(-\frac{\nabla^2}{2m}+g\varphi(x)+i\frac{\partial}{\partial t}+i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}\right)\phi(\nu). \tag{3.9}$$

Let us look for a solution of the kind $\phi = \exp[F]$ for (3.9), where F satisfies the equation

$$i\frac{\partial F}{\partial \nu} = -\left(\frac{\nabla^2}{2m} + i\frac{\partial}{\partial t} + i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}\right)F - \frac{1}{2m}\left(\frac{\partial F}{\partial x_\mu}\right)^2 - g\varphi(x)\,. \tag{3.10}$$

It is now possible to express F in the form of a series of powers of g

$$F = \sum_{n=1}^{\infty} g^n F_n\,, \tag{3.11}$$

where the coefficients F_n satisfy the equation

$$-i\frac{\partial F_n}{\partial \nu} = \left(\frac{\nabla^2}{2m} + i\frac{\partial}{\partial t} + i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}\right)F_n + \sum_{s=1}^{n-1}\frac{1}{2m}\left(\frac{\partial F_s}{\partial x_\mu}\frac{\partial F_{n-s}}{\partial x_\mu}\right) + \varphi(x)\,\delta_{n1}\,. \tag{3.12}$$

Equation (3.12) supplies the expression for F_1 and a recurrence formula for F_n:

$$F_1 = \frac{i}{(2\pi)^2}\int_0^\nu \varphi(k)\exp\left[ikx - i\left(\frac{\boldsymbol{k}^2}{2m} + \frac{\boldsymbol{p}\cdot\boldsymbol{k}}{m} + k^0\right)\nu'\right]\mathrm{d}\nu'\,\mathrm{d}^4k\,, \tag{3.13}$$

$$F_n = \frac{i}{m}\int_0^\infty \mathrm{d}\nu' \exp\left[i\left(\frac{\nabla^2}{2m} + i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m} + i\frac{\partial}{\partial t}\right)(\nu' - \nu)\right]\sum_{s=1}^{n-1}\left(\frac{\partial F_s(x,\nu)}{\partial x_\mu}\frac{\partial F_{n-s}(x,\nu')}{\partial x_\mu}\right), \tag{3.14}$$

in particular

$$F_2 = \frac{i}{m}\int \mathrm{d}^4k\,\mathrm{d}^4k_1 f(k, k_1, x, \nu)\varphi(k)\varphi(k_1)\,, \tag{3.15}$$

where

$$\begin{aligned} f(k, k_1, x, \nu) &= \\ &= \frac{\boldsymbol{k}\cdot\boldsymbol{k}_1}{(2\pi)^4 m}\exp[i(k + k_1)x]\int_0^\nu \mathrm{d}\nu' \exp\left[i\left[\frac{(\boldsymbol{k}+\boldsymbol{k}_1)^2}{2m} + \frac{\boldsymbol{p}\cdot(\boldsymbol{k}+\boldsymbol{k}_1)}{m} + k_1^0 + k^0\right](\nu' - \nu)\right]\cdot \\ &\quad \cdot\int_0^{\nu'} \mathrm{d}\nu_1\,\mathrm{d}\nu_2 \exp\left[-i\left[\left(\frac{\boldsymbol{k}^2}{2m} + \frac{\boldsymbol{p}\cdot\boldsymbol{k}}{m} + k^0\right)\nu_1 + \left(\frac{\boldsymbol{k}_1^2}{2m} + \frac{\boldsymbol{p}\cdot\boldsymbol{k}_1}{m} + k_1^0\right)\nu_2\right]\right]. \end{aligned}$$

The operator solution for Z gives, if we take $F = gF_1$, the compact expression

$$\begin{aligned} Z = 1 + \sum \frac{i^{2s+1}}{(2\pi)^{3s} s!}\prod_{i=1}^{s}\left[\int \mathrm{d}^3p_i\,\mathrm{d}^4x_i\,\mathrm{d}^4x_i'\,\bar{\eta}(x_i)\theta(t_i - t_i')\cdot\right. \\ \left.\cdot\,\eta(x_i')\exp\left[i(\boldsymbol{x}_i - \boldsymbol{x}_m)\cdot\boldsymbol{p}_i - i(t_i - t_i')\frac{\boldsymbol{p}^2}{2m}\right]\exp[iR]\right], \end{aligned} \tag{3.16}$$

where

$$(3.17)\qquad R=\frac{1}{2}\int\mathrm{d}^4k\left[J(k)+\frac{g}{(2\pi)^2}\sum_{n=1}^{s}\int\limits_0^{t_n-t_n'}\mathrm{d}\xi_n\exp\left[ikx_n-i\left(\frac{k^2}{2m}+\frac{\boldsymbol{p}\cdot\boldsymbol{k}}{m}-k^0\right)\xi_n\right]\right]\cdot$$

$$\cdot D(k)\left[J(-k)+\frac{g}{(2\pi)^2}\sum_{m}^{s}\int\limits_0^{t_m-t_m'}\mathrm{d}\xi_m\exp\left[-ikx_m-i\left(\frac{k^2}{2m}-\frac{\boldsymbol{p}\cdot\boldsymbol{k}}{m}+k^0\right)\xi_m\right]\right]$$

and where

$$D(k)=\frac{1}{k^2+\mu^2-i\varepsilon}\,.$$

If $F=gF_1+g^2F_2$, we obtain for R the expression

$$(3.18)\qquad R=\int\mathrm{d}^4k\,\mathrm{d}^4k_1\left[i\int\mathrm{d}g^2\sum_{n=1}^{s}f(k,k_1,x_m;t_n-t_n')D^s(k,k_1)\right]+$$

$$+\frac{i}{2}\left[J(k)+\frac{g}{(2\pi)^2}\sum_{n=1}^{s}\int\limits_0^{t_n-t_n'}\mathrm{d}\xi\exp\left[ikx_n-i\left(\frac{\boldsymbol{k}^2}{2m}+\frac{\boldsymbol{p}_n\cdot\boldsymbol{k}_1}{m}-k_1^0\right)\xi\right]\right]\cdot$$

$$\cdot D^s(k,k_1)\left[J(k_1)+\frac{g}{(2\pi)^2}\sum_{m=1}^{s}\int\limits_0^{t_m-t_m'}\mathrm{d}\xi\exp\left[ik_1m-i\left(\frac{\boldsymbol{k}_1^2}{2m}+\frac{\boldsymbol{p}_m\cdot\boldsymbol{k}_1}{m}-k_1^0\right)\xi\right]\right],$$

where $D^s(k,k_1)$ is determined by the equation

$$(3.19)\qquad (k^2+\mu^2)D^s(k,k_1)+g^2\sum_{n=1}^{s}\int f(k,k_2;t_n-t_n')D^s(k_2,k_1)\,\mathrm{d}^4k_2=\delta(k+k_1)\,.$$

Solving (3.19) with respect to g^2 in the first approximation and keeping, in the exponent of (3.18), only the terms proportional to g and g^2, we obtain for R the approximate expression

$$(3.20)\qquad R=\int\mathrm{d}^4k\left\{ig^2\sum_{n=1}^{s}f(k,-k,x_n,t_n-t_n')D_0(k)+\right.$$

$$+\frac{i}{2}\left[J(k)+\frac{g}{(2\pi)^2}\sum_{n=1}^{s}\int\limits_0^{t_n-t_n'}\mathrm{d}\xi\exp\left[-ikx_n-i\left(\frac{\boldsymbol{k}^2}{2m}+\frac{\boldsymbol{p}_n\cdot\boldsymbol{k}}{m}-k^0\right)\xi\right]\right]\cdot$$

$$\cdot D(k^0)\left[J(-k)+\frac{g}{(2\pi)^2}\sum_{m=1}^{s}\int\limits_0^{t_m-t_m'}\mathrm{d}\xi\exp\left[-ikx_n-i\left(\frac{\boldsymbol{k}^2}{2m}-\frac{\boldsymbol{p}\cdot\boldsymbol{k}}{m}+k^0\right)\xi\right]\right]+$$

$$\left.+ig^2\sum_{n=1}^{s}\int\limits_0^{t_n-t_n'}\mathrm{d}\xi\exp\left[ikx-\left(\frac{\boldsymbol{k}^2}{2m}+\frac{\boldsymbol{p}_n\cdot\boldsymbol{k}}{m}-k^0\right)\xi\right]\right\}D_0(k)\cdot$$

$$\cdot f(k,k_2,x_n,t_n-t_n')D_0(k_1)J(-k_1)\,\mathrm{d}^4k_1\,,$$

which, for the Green's function, gives the expression

$$(3.21)\qquad G(p,t-t')=\theta(t-t')\exp\left[-i\frac{p^2}{2m}(t-t')\right]\exp\left[-\frac{ig^2}{(2\pi)^2}\int\frac{d^4k}{k^2+\mu^2-i\varepsilon}\right]\cdot$$
$$\cdot\left\{\frac{\exp\left[-\left(\boldsymbol{k}^2/2m+(\boldsymbol{p}\cdot\boldsymbol{k})/m-k^0\right)(t-t')\right]-1}{(\boldsymbol{k}^2/2m+(\boldsymbol{p}\cdot\boldsymbol{k})/m-k^0)^2}+\frac{i(t-t')}{(\boldsymbol{k}^2/2m+(\boldsymbol{p}\cdot\boldsymbol{k})/m-k^0)}\right\}.$$

3·2. *The Green's function in an external field.* – We intend to study the equation for the Green's function in an external scalar field with the same method [17].

$$(3.22)\qquad \left(-\Box+m^2-g\varphi(x)\right)G(x,y|\varphi)=\delta(x-y)\,.$$

In the p-representation, if

$$G(x,y|\varphi)=\frac{1}{(2\pi)^4}\int G(x,p)\exp\left[ip(x-y)\right]dp\,,$$

eq. (3.22) becomes

$$(3.23)\qquad \left[-(\partial_\mu+ip_\mu)^2+m^2-g\varphi(x)\right]G(x,p)=1\,.$$

The formal solution for $G(x,p)$ can be written [18]

$$(3.24)\quad G(x,p)=\frac{1}{-(\partial_\mu+ip_\mu)^2+m^2-g\varphi(x)}=i\int_0^\infty d\nu\exp[-i(p^2+m^2-i\varepsilon)\nu]\,Y(\nu)\,,$$

where $Y(\nu)$ satisfies the equation

$$(3.25)\qquad -i\frac{\partial Y}{\partial\nu}=\left(\partial^2+2ip_\mu\partial_\mu+g\varphi(x)\right)Y\,,$$

with the boundary condition $Y(\nu=0)=1$. As usual, we look for a solution in the form $Y=\exp[F]$, where F satisfies the equation

$$(3.26)\qquad -i\frac{\partial F}{\partial\nu}=(\partial_\mu^2+2ip_\mu\partial_\mu)F+(\partial_\mu F)^2+g\varphi(x)\,.$$

By putting $F=\sum_{n=1}^{\infty}g^nF_n$ the following system of equations is obtained:

$$(3.27)\qquad -i\frac{\partial F_n}{\partial\nu}=\{\partial_\mu^2+2ip_\mu\partial_\mu\}F_n+\sum_{m=1}^{\infty}\partial_\mu F_{n-m}\partial_\mu F_m+\varphi\delta_{n1}$$

which gives a solution for F_1 and recurrence relations for F_n $(n > 1)$ [18,19]:

$$
(3.28)\qquad \left\{ \begin{aligned} F_1 &= \frac{i}{(2\pi)^2}\int_0^\nu \varphi(k)\exp[ikx - i(k^2 - 2pk)\nu']\,\mathrm{d}^4k\,\mathrm{d}\nu', \\ F_n &= i\int_0^\nu \mathrm{d}\nu' \exp[-i(\partial_\mu^2 + 2ip_\mu\partial_\mu)(\nu' - \nu)]\sum_{m=1}^{\infty}\left(\partial_\mu F_m(\nu)\,\partial_\mu F_{n-m}\right). \end{aligned}\right.
$$

It can easily be shown that the approximation F_1 corresponds to the sum of all Feynman's graphs for Green's functions in an external field, the only difference being that in the denominators the correlation terms $k_i k_j$ $(j \neq i)$ are omitted for internal momenta:

$$
\left((p + \sum k_i)^2 + m^2\right) \to (p^2 + m^2 + 2p\sum k_i + \sum k_i^2)\,.
$$

In all other F_n, these omitted terms must be considered.

In particular

$$
(3.29)\qquad F_2 = -\frac{i}{2}\int \mathrm{d}^4k_1\int \mathrm{d}^4k_2\, f(k_1, k_2, x, \nu)\,\varphi(k_1)\,\varphi(k_2)\,,
$$

where

$$
(3.30)\qquad f(k_1, k_2, x, \nu) =
$$

$$
= \frac{2}{(2\pi)^4}(k_1\,k_2)^{i(k_1+k_2)x}\int_0^\nu \mathrm{d}\nu'\,\frac{\left(\exp[-i(k_1^2 - 2pk_1)\nu'] - 1\right)\left(\exp[-i(k_2^2 - 2pk_2)\nu'] - 1\right)}{(k_1^2 - 2pk_1)(k_2^2 - 2pk_2)}\cdot
$$

$$
\cdot\exp[-i[(k_1 + k_2)^2 - 2p(k_1 + k_2)](\nu - \nu')]\,.
$$

The one-particle Green's function to the order of g^2 is therefore

$$
(3.31)\qquad G(p) = i\int_0^\infty \mathrm{d}\nu\,\exp[-i\{(p^2 - m^2 - i\varepsilon)\nu\}] -
$$

$$
-\frac{g^2}{(2\pi)^4}\int \frac{\mathrm{d}^4k}{k^2 + \mu^2 - i\varepsilon}\left[\frac{\exp[-i(k^2 - 2p\cdot k)] - 1}{(k^2 - 2p\cdot k)^2} + \frac{i\nu}{(k^2 - 2p\cdot k)}\right].
$$

Inserting $G(x, p)$ in the generating functional, we can obtain in this approximation all quantum Green's functions. However, it is done more consequently and elegantly in following Section (formula (4.43)), according to which expressions given here correspond to the first approximation of the modified perturbation approach (see formulae (4.43), (4.44) of Sect. **4**).

Following [20, 24], we can obtain the following expression for the scattering amplitude $M(p_3p_4|p_1p_2)$ of two particles:

$$M(p_3p_4|p_1p_2) \approx -\frac{g^2}{(2\pi)^4}\int \mathrm{d}^4x\, \mathscr{D}(x) \exp[-ix(p_1-p_3)]\cdot$$
$$\cdot\int_0^1 \mathrm{d}\lambda \exp\left[\frac{ig}{(2\pi)^4}\int \mathrm{d}^4k\, \frac{1}{k^2+\mu^2-i\varepsilon}\,[A(p_i)\lambda + B]\right] + (p_1 \leftrightarrow p_2)\,,$$

$$A(p_i) = \exp[-ikx]\left[\frac{1}{(k^2+2kp_3)(k^2+2kp_2)} + \frac{1}{(k^2-2kp_1)(k^2+2kp_2)}\right] +$$
$$+ \frac{1}{(k^2+2kp_3)(k^2-2kp_4)} + \frac{1}{(k^2-2kp_1)(k^2-2kp_4)}\,,$$

$$B = \frac{1}{(k^2-2kp_2)(k^2-2kp_4)} + \frac{1}{(k^2-2kp_1)(k^2-2kp_3)}\,.$$

3'3. *Scalar electrodynamics* [18, 20]. – The equation for the Green's function in an external electromagnetic field has the form

$$[-(\partial_\mu - ieA_\mu(x))^2 + m^2]\, G(x, x') = \delta(x-x')\,. \tag{3.32}$$

In the p-space (3.32) becomes

$$[-(ip_\mu - ieA_\mu(x) + \partial_\mu)^2 + m^2]\, G(p, x) = 1\,. \tag{3.33}$$

Equation (3.33) has the formal operator solution

$$G(p, x) = i\int_0^\infty \exp\{i\nu[(ip - ieA_\mu(x) + \partial_\mu)^2 - m^2]\,. \tag{3.34}$$

By applying the method described in the preceding Sections we can obtain the solution

$$G(p, x) = i\int_0^\infty \exp[-i\nu(p^2+m^2) + F]\,, \tag{3.35}$$

where $F = \sum \exp[n]F^n$ and

$$F_1 = -\frac{2}{(2\pi)^2}\int \mathrm{d}^4k \exp[ikx]\, p_\mu A_\mu(k)\, \frac{\exp[-i(k^2+2pk)\nu] - 1}{k^2+2pk}$$

and F_n is determined by the recurrence relations

$$F_n = i\int_0^\nu d\nu' \exp[i(\partial_\mu^2 + 2ip_\mu\partial_\mu)(\nu-\nu')]\cdot \cdot\left\{\left(\frac{\partial F_{n-m}}{\partial x_\mu}\frac{\partial F_m}{\partial x_\mu} + 2iA_\mu(x)\frac{\partial F_{n-1}}{\partial x_\mu}\right) - \delta_{n2}A_\mu^2(x)\right\}. \tag{3.36}$$

For $F = eF_1$, the expression of the generating functional Z is easily obtained:

$$cZ = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2\pi)^{2n}n!}\left[\prod_{s=1}^{n} I(x_s)I(p^s)\exp[ip^s x_s]\,d^4p_s\,d^4x_s\,A(I_\mu)\right], \tag{3.37}$$

where

$$A(I_\mu) = \exp\left[-\left\{-\frac{i}{2}\int I_\mu(x)D_{\mu\nu}(x-y)I_\nu(y)\,d^4x\,d^4y + \sum_s^n \frac{2}{(2\pi)^2}p_\mu^s\int D_{\mu\nu}(k)I_\mu(k)\exp[ikx_s]\frac{\exp[-i(k^2+2p^sk)\nu_s]-1}{k^2+2p^sk}d^4k + \sum_{s,m}^n\frac{2ie^2}{(2\pi)^4}\int\frac{\exp[-i(k^2-2p^sk)\nu_s]-1}{k^2-2p^sk}p_\mu^s D_{\mu\nu}(k)p_\nu^m\frac{\exp[-i(k^2+2p_m\cdot k)\nu_m]-1}{k^2+2p^mk}\cdot \cdot\exp[ik(x_m-x_s')] + i\nu_s\big((p^s)^2+m^2-i\delta\big)\right\}\right], \tag{3.38}$$

where I is the source of the Bose field and I_μ is the source of the electromagnetic one.

If $F = eF_1 + e^2F_2$ the expression for Z becomes

$$cZ = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2\pi)^{2n}n!}\prod_{s=1}^{n}\int[I(x_s)I(p_s)\exp[ip^sx_s]\,d^4p^s\,d^4x_s\,A(I_\mu)],$$

where

$$A(I) = \exp\left[-\left\{-\frac{i}{2}\int I_\mu(x)D_{\mu\nu}(x-y)I_\nu(y)\,d^4x\,d^4y + i\nu_s\big((p^s)^2+m^2-i\delta\big) + \sum_{s=1}^n\frac{2}{(2\pi)^2}\int p_\mu^s D_{\mu\nu}(k)I_\nu(k)\exp[ikx_s]\frac{\exp[-i(k^2+2p^sk)k_s]-1}{k^2+2p^sk}d^4k + \sum_{s\neq m}^n\frac{2ie^2}{(2\pi)^4}\int(p_\mu^sD_{\mu\nu}(k)p_\nu^m)\exp[ik(x_m-x_s)]\cdot \cdot\left[\left(\frac{\exp[-i(k^2-2p^sk)\nu_s]-1}{k^2-2p^sk}\cdot\frac{\exp[-i(k^2+2p^mk)\nu_m]-1}{k^2+2p^mk}\right)d^4k + \sum_{s=1}^n\int\frac{4ie^2}{(2\pi)^4}d^4k\,p_\mu^sD_{\mu\nu}(k)p_\nu^s\left(\frac{\exp[-i(k^2+2p^sk)\nu_s]-1}{(k^2+2p^sk)^2}+\frac{i\nu_s}{k^2+2p^sk}\right)\right]\right\}\right]. \tag{3.39}$$

The one-particle Green's funtion will therefore be written

$$(3.40)\qquad G(p) = i\int_0^\infty \mathrm{d}\nu \exp\left[-i\nu(p^2+m^2-i\delta)+M\nu\right].$$

From (40) we obtain the asymptotic behaviour in the infra-red region (in the transverse gauge)

$$(3.41)\qquad G(p) \simeq \frac{1}{p^2+m^2}\cdot\frac{1}{(p^2/m^2+1)^{(e^2/4\pi^2)}}\left[1+\frac{1}{2}\left(\frac{p^2}{m^2}+1\right)+\ldots\right].$$

3.4. *Asymptotic infra-red behaviour in quantum electrodynamics* [18]. – The Green's function of an electron in an external electromagnetic field satisfies the equation

$$(3.42)\qquad \left(\gamma_\mu\,\partial_\mu+m-ie\gamma_\mu a_\mu(x)\right)G(x,x'|a)=\delta(x-x')\,.$$

In the p-representation $G(x, x')$ becomes

$$(3.43)\qquad G(x,x'|a)=\frac{1}{(2\pi)^4}\int \exp\left[ip(x-x')\right]G(p,x|a)\,\mathrm{d}^4p\,,$$

where $G(p, x|a)$ satisfies the equation

$$(3.44)\qquad \left[i\gamma_\mu p_\mu+m-ie\gamma_\mu a_\mu(x)+\gamma_\mu\nabla_\mu\right]G(p,x|a)=1\,.$$

Let us multiply (3.44) by $-i\gamma_\mu p_\mu+m$, and so obtain

$$(3.45)\qquad \left[p^2+m^2-2ep_\mu a_\mu-2ip_\mu\nabla_\mu+(\gamma_\mu\nabla_\mu-ie\gamma_\mu a_\mu)(i\gamma_\mu p_\mu+m)\right]G(p,x|a)= \\ =-i\gamma_\mu p_\mu+m\,.$$

Since we are interested in the region $-i\gamma_\mu p_\mu\sim m$ (3.45) becomes

$$(3.46)\qquad (p^2+m^2-2ep_\mu a_\mu-2ip_\mu\nabla_\mu)G(p,x|a)=-i\gamma_\mu p_\mu+m\,.$$

With the usual method it is possible to find the general solution of this equation:

$$(3.47)\qquad G(p,x|a)=i(-i\gamma_\mu p_\mu+m)\int_0^\infty \mathrm{d}\nu\exp\left[-i\nu(p^2+m^2-i\varepsilon)+iF(\nu,a)\right],$$

where

$$(3.48)\qquad F(\nu, a) = \frac{1-\exp[-2\nu p_\mu \nabla_\mu]}{2p_\mu\nabla_\mu}[2ep_\mu a_\mu(x)] = \\ = \int_0^\nu d\nu' \exp[-2\nu' p\nabla] 2ep_\mu a_\mu = \int_0^\nu 2ep_\mu a_\mu(x-2\nu' p)\, d\nu' .$$

The function $F(\nu, a)$ has a simple form in the momentum representation:

$$(3.49)\qquad F(\nu, a) = -\frac{1}{(2\pi)^2}\int\left\{ep_\mu a_\mu(k)\left(\frac{\exp[-2i\nu p_\mu k_\mu]-1}{ip_\mu k_\mu}\right)\exp[ik_\mu x_\nu]\right\} d^4k = \\ = \frac{2e}{(2\pi)^2}\int p_\mu a_\mu(k)\int_0^\nu \exp[-2i\nu'(pk)]\, d\nu' \exp[ik_\mu x_\mu]\, d^4k ,$$

where

$$a_\mu(k) = \frac{1}{(2\pi)^2}\int a_\mu(k)\exp[-ik_\mu x_\mu] d^4x .$$

Using the closed operator solution (2.25) for the generating functional, we can determine the asymptotic form of the Green's functions in the infrared region. In order to obtain this, it is sufficient to substitute the expression of the Green's function in an external field in the solution (2.25), after substituting $ia(x)\to \delta/\delta I(x)$ and then carrying out the functional derivative with respect to $\delta/\delta I$ in all terms except the polarization one ($\mathrm{Sp}\,\gamma G$). Now it is possible to obtain the expression for the generating functional Z:

$$(3.50)\qquad cZ = \exp\left[-\left[d^4x\int_0^e de'\ \mathrm{Sp}\,\gamma_\mu G\left(x, x'\middle| e'\frac{\delta}{\delta I(x)}\right)\frac{\delta}{\delta I(x)}\right]\right]\cdot \\ \cdot\sum_{k=0}^\infty \frac{(-1)^k}{k!}\left[\prod_{n,m=1}^k \int\int_0^\infty \bar\eta(p^n)(-i\gamma_\mu p^n_\mu + m)\eta(x_n)\exp[-ip^n x_n]\frac{d^4p^n\, d^4x_n\, d\nu_n\, d\nu_m}{(2\pi)^2}\cdot\right. \\ \cdot\exp\left[i\int\frac{1}{2}I_\mu(x)D_{\mu\nu}(x-y)I_\nu(y)\,d^4x\,d^4y + 2ie\int\int_0^{\nu_n} p^n_\mu D_{\mu\nu}(x_n - 2\nu'_n p^n - y)\cdot\right. \\ \cdot I_\nu(y)\,d^4y\,d\nu'_n + 2ie^2\int_0^{\nu_n} d\nu'_n\int_0^{\nu_m} d\nu'_m\big(p^n_\mu D_{\mu\nu}(x_n - x_m - 2\nu'_n p^n + 2\nu'_m p^m)\big)\, p^m_\nu - \\ \left.\left. - i\nu_n\big((p^n)^2 + m^2 - i\varepsilon\big)\right]\right] .$$

The formula (3.50) gives the asymptotic behaviour of all Green's functions in the infra-red region. To obtain the asymptotic behaviour of a particular Green's function it is sufficient to differentiate a suitable number of times with respect to $i\eta$, $i\bar{\eta}$ and iI and then to put $\eta=\bar{\eta}=I=0$. In particular, the Green's function of an electron in the presence of an external photon field is

$$(3.51)\qquad G(p,p')=i(-i\gamma_\mu p_\mu+m)\int\int_0^\infty \mathrm{d}\nu\exp\left[-i(p^2+m^2-i\varepsilon)\nu\right]\cdot$$

$$\cdot\exp\left[\frac{i}{2}\int I(x')D(x'-y')I(y')\,\mathrm{d}^4x'\,\mathrm{d}^4y'-2ie\int_0^\nu \mathrm{d}\nu' I_\mu(x')D_{\mu\nu}(x'-x-2\nu'p)\,p_\nu\mathrm{d}^4x+\right.$$

$$\left.+\,2ie^2\int_0^\nu \mathrm{d}\nu'\int_0^\nu \mathrm{d}\nu'' p_\mu D_{\mu\nu}[2(\nu''-\nu')p]p_\nu\right]\exp\left[-i(p'-p)x\right]\frac{\mathrm{d}^4x}{(2\pi)^4},$$

where

$$(3.52)\qquad G(p,p')=i\int T(\Psi(x),\overline{\Psi}(x'))\exp\left[-ip'x'+ipx\right]\frac{\mathrm{d}^4x\,\mathrm{d}^4x'}{(2\pi)^8}=$$

$$=-i\,\frac{\delta^2 Z}{\delta\eta(p)\,\delta\bar{\eta}(p')}\bigg|_{\eta=0}.$$

From (3.51), writing $D_{\mu\nu}$ in the momentum representation and then integrating over ν' and ν'', we obtain

$$(3.53)\qquad G(p,p')=\int\exp\left[ix(p-p')\right]\frac{\mathrm{d}^4x}{(2\pi)^4}(-i\gamma_\mu p_\mu+m)\cdot$$

$$\cdot\int_0^\infty \mathrm{d}\nu\exp\left[-i\nu(p^2+m^2-i\varepsilon)\right]\exp\left[\frac{i}{2}\int I_\mu(k)D_{\mu\nu}(k)I_\nu(-k)\,\mathrm{d}^4k+\right.$$

$$+\,e\int\frac{\left(\exp\left[2i\nu(pk)\right]-1\right)}{(2\pi)^2(pk)}p_\mu D_{\mu\nu}(k)I_\nu(k)\exp\left[ikx\right]\mathrm{d}^4k+$$

$$\left.+\,\frac{ie^2}{(2\pi)^4}\int\frac{1-\exp\left[2i\nu(pk)\right]}{(pk)^2}p_\mu D_{\mu\nu}(k)\,p_\nu\,\mathrm{d}^4k\right].$$

In particular, in the case $I=0$, from (3.53) it follows that, as $G(p,p')=$ $=G(p)\delta(p'-p)$,

$$(3.54)\qquad G(p)=i(-i\gamma_\mu p_\mu+m)\int_0^\infty \mathrm{d}\nu\exp\left[-i\nu(p^2+m^2-i\varepsilon)\right]+F(\nu,p),$$

where

$$(3.55)\qquad F(\nu, p) = \frac{ie^2}{(2\pi)^4}\int \frac{1-\exp[2i\nu(pk)]}{(kp)^2}\, p_\mu D_{\mu\nu}(k) p_\nu\, d^4\nu = \\ = -\frac{4ie^2}{(2\pi)^4}\int_0^\nu d\nu' \int_0^{\nu'} d\nu'' p_\mu D_{\mu\nu}(k)\, p_\nu(k) \exp[-2ipk(\nu'-\nu'')]\,.$$

In subsequent calculations an arbitrary longitudinal component will be taken for

$$(3.56)\qquad \begin{cases} D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \dfrac{k_\mu k_\nu}{k^2}\right) d^t + \dfrac{k_\mu k_\nu}{k^2}\, d^l\,, \\ d^t = \dfrac{1}{k^2 - i\varepsilon}\,, \qquad d^l = \dfrac{1}{k^2 - i\varepsilon}\, \Delta^l(k)\,. \end{cases}$$

The integral, with respect to k, which appears in (3.55) can be calculated more simply in the Euclidean space. The integral for large k diverges (ultraviolet divergence). This kind of divergence can be eliminated by means of renormalization. By effecting the regularization by means of a cut-off L on the values of k in the asymptotic region $-p^2 \approx m^2$, we obtain

$$(3.57)\qquad F(\nu, p) \approx \frac{e^2}{8\pi^2}(3-\Delta^l)\log \nu L p \sim \frac{e^2}{8\pi^2}(3-\Delta^l)\log \nu L m\,.$$

Substituting in (3.54), we obtain the following expression for G:

$$(3.58)\qquad \begin{cases} G(p) = \dfrac{Z_2(-i\gamma_\mu p_\mu + m)}{p^2 + m^2}\, \dfrac{1}{|p^2/m^2 + 1|^\alpha}\, f\,, \\ \alpha = \dfrac{e^2}{8\pi^2}(3-\Delta^l), \end{cases}$$

where Z_2 is the renormalization constant and f has the form

$$f = i\int_0^\infty dx\, x^{(e^2/8\pi^2)(3-\Delta^l)} \exp[-ix - \varepsilon x]\,, \qquad p^2 > -m^2\,,$$

$$f = i\int_0^\infty dx\, x^{(e^2/8\pi^2)(3-\Delta^l)} \exp[ix - \varepsilon x]\,, \qquad p^2 < -m^2\,.$$

In the region $p^2 \approx -m^2$ and $e^2 \ll 1$, $f \approx 1$, $Z_2 = (L/m)^{(e^2/8\pi^2)(3-\Delta^l)}$, as is well known.

The poles of the Green's function in the infra-red region are different from the corresponding ones for free particles. That is, instead of $1/(p^2+m^2)$ we find $1/(p^2+m^2)^{1+(e^2.../8\pi^2)(3-\Delta^l)}$.

It is therefore possible to develop a systematic perturbation theory in the parameter $(p^2 + m^2)/m^2$. Besides, the developed method allows the asymptotic form of all Green's functions to be obtained.

With this method the contribution of the soft photons can be taken into account exactly [18] and the double logarithmic asymptotics of various cross-sections for high-energy interaction in quantum electrodynamics is obtained [24].

4. – Operator continual solution for Green's functions in the external field and modified perturbation approach [20].

The use of functional methods allows us to obtain easily the determination of the operator forms of solutions of equations (3.32), (3.22) and (3.42), and the construction of a « modified » perturbation theory, more powerful than the usual perturbation theory. In this Section we will, therefore, construct, with the help of the operator solution, a « modified » perturbation theory for the different Green's functions.

4·1. *Bose case.* – Let us consider the interaction between a neutral vectorial field and a spin-zero Bose field

$$\{-(\partial_\mu - ie\varphi_\mu(x))^2 + \mu^2\}G(x, y|i\varphi) = \delta(x-y)\,. \tag{3.32}$$

Its formal solution is written

$$\begin{aligned} G(x, y|i\varphi) &= \frac{1}{-(\partial_\mu - ie\varphi_\mu(x))^2 + \mu^2 - ie}\,\delta(x-y) = \\ &= i\int_0^\infty \mathrm{d}\nu \exp\left[i[(\partial_\mu - ie\varphi_\mu(x))^2 - \mu^2 + i\varepsilon]\nu\right]\delta(x-y) = \\ &= \frac{i}{(2\pi)^4}\int \mathrm{d}^4p \exp[ip(x-y)]\exp[-i(p^2+\mu^2-i\varepsilon)\nu]\Phi(\nu)\,\mathrm{d}\nu\,, \end{aligned} \tag{4.1}$$

where $\Phi(\nu)$ is defined by the equation

$$-i\,\frac{\partial\Phi}{\partial\nu} = [\partial_\mu^2 + 2ip\partial + 2ep_\mu\varphi_\mu(x) - 2ie\varphi_\mu(x)\,\partial_\mu - e^2\varphi^2(x)]\,\Phi\,. \tag{4.2}$$

To obtain the operator solution of (4.2) it is sufficient to introduce an additional interaction with a source t_μ of generalized momentum Π_μ. Therefore; eq. (3.2) becomes

$$-i\,\frac{\partial\Phi}{\partial\nu} = [\Pi_\mu^2(x) + i\Pi_\mu t_\mu(\nu) + 2p_\mu\varphi_\mu(x) + 2ip_\mu\partial_\mu]\,\Phi\,, \tag{4.3}$$

where $\Pi_\mu(x) = \partial_\mu - ie\varphi_\mu(x)$.

It is possible to show that the solution has the form

$$\Phi = \exp\left[i\int_0^{\nu} \frac{\delta^2}{\delta t_\mu(\xi)\,\delta t_\mu(\xi)}\,\mathrm{d}\xi\right]\Phi_1(t)|_{t=0}\,, \tag{4.4}$$

where $\Phi_1(t)$ satisfies the equation

$$-i\,\frac{\partial\Phi_1(t)}{\partial\nu} = \{2ip\partial + i(\partial_\mu - ie\varphi_\mu(x))\,t_\mu(\nu) + 2p_\mu\varphi_\mu(x)\}\,\Phi_1(t)\,. \tag{4.5}$$

From (4.5) we obtain

$$\Phi_1(t) = \exp\left[i\int_0^{\nu}\mathrm{d}\nu'(2p_\mu + t_\mu(\nu'))\,\varphi_\mu(x - 2p(\nu-\nu') - \int_{\nu'}^{\nu} t(\xi)\,\mathrm{d}\xi)\right], \tag{4.6}$$

so that Φ has the form

$$\Phi = \exp\left[i\int_0^{\nu}\frac{\delta^2}{\delta t_\mu(\xi)\,\delta t_\mu(\xi)}\,\mathrm{d}\xi\right]\cdot \cdot\exp\left[i\left\{e\int_0^{\nu}\mathrm{d}\nu'(2p_\mu + t_\mu(\nu'))\,\varphi_\mu\left(x - 2p(\nu-\nu') - \int_{\nu'}^{\nu} t(\xi)\,\mathrm{d}\xi\right)\right\}\bigg|_{t=0}\right]. \tag{4.7}$$

It is now possible to write (4.7) in the form

$$\begin{cases} \Phi(\nu) = \left\langle \exp\left[2ie\int_0^{\nu} P_\mu(\nu')\,\varphi_\mu(x(\nu'))\,\mathrm{d}\nu'\right]\right\rangle, \\ P_\mu(\nu') = p_\mu + \dfrac{1}{2}\,\dfrac{\delta}{\delta t_\mu(\nu')}\,, \end{cases} \tag{4.8}$$

where

$$x(\nu') = x - 2\int_{\nu'}^{\nu} P_\mu(\xi)\,\mathrm{d}\xi = x - 2p(\nu-\nu') - \int_{\nu'}^{\nu}\frac{\delta}{\delta t(\xi)}\,\mathrm{d}\xi\,.$$

Moreover, for every function of the operator $\delta/\delta t$ the sign $\langle\,\rangle$ is defined by the relationship

$$\left\langle A\left(\frac{\delta}{\delta t}\right)\right\rangle = A\left(\frac{\delta}{\delta t}\right)\exp\left[i\int_0^{\nu} t^2(\xi)\,\mathrm{d}\xi|_{t=0}\right]. \tag{4.9}$$

In conclusion, for a Green's function in an external field we have the following operator solution:

$$G(x, y|i\varphi) = \frac{i}{(2\pi)^4}\int \exp[ip(x-y)]\,\mathrm{d}^4p \int_0^\infty \exp[-i(p^2+\mu^2-i\varepsilon)\nu]\,\mathrm{d}\nu \cdot \cdot\left\langle \exp\left[2ie\int_0^\nu P_\mu(\nu')\varphi_\mu(x(\nu'))\,\mathrm{d}\nu'\right]\right\rangle . \tag{4.10}$$

The operator solution obtained has the advantage of being, in the exponent, a linear function of the external field φ_μ; it is therefore possible to effect easily the functional average for obtaining the expression of Z.

From (4.10) it is possible to obtain the « modified » perturbative expansion of the Green's function $G(x, y|i\varphi)$. This type of expansion corresponds to a perturbation theory in the exponent before the integration on ν. The usual perturbative expansion of G corresponds to the series

$$\left\langle \exp\left[2ie\int_0^\nu P_\mu(\nu')\varphi_\mu(x(\nu'))\,\mathrm{d}\nu'\right]\right\rangle = \sum_{n=0}^\infty \frac{e^n a_n}{n!} . \tag{4.11}$$

The modified perturbation theory corresponds to the representation of the same quantity in the form of a series of the exponent:

$$\left\langle \exp\left[2ie\int_0^\nu P_\mu(\nu')\varphi_\mu(x(\nu'))\,\mathrm{d}\nu'\right]\right\rangle = \exp\left[\sum_{n=1}^\infty e^n b_n\right] . \tag{4.12}$$

From (4.11) we obtain, for the coefficients a_n, the relationship

$$a_n = \left\langle \left(2i\int_0^\nu P_\mu(\nu')\varphi_\mu(x(\nu'))\,\mathrm{d}\nu'\right)^n\right\rangle . \tag{4.13}$$

In particular,

$$a_1 = \frac{2i}{(2\pi)^2}\int \mathrm{d}^4k\, P_\mu\varphi_\mu(k)\exp[ikx]\int_0^\nu \exp[-i(k^2+2pk)\nu']\,\mathrm{d}\nu' , \tag{4.14}$$

$$a_2 = -\frac{2}{(2\pi)^4}\int \mathrm{d}^4k\,\mathrm{d}^4k_1\int_0^\nu \mathrm{d}\nu_1\int_0^\nu \mathrm{d}\nu_2\,\varphi_\mu(k_1)\varphi_\varrho(k_2)\cdot \cdot\exp\left[i\cdot\left\{(k_1+k_2)x - \sum_{n=1}^2 2pk_n(\nu-\nu_n) - \sum_{n,m=1}^2 k_n k_m\left(\nu - \frac{\nu_n+\nu_m}{2} - \frac{|\nu_n-\nu_m|}{2}\right)\right\}\right]\cdot \cdot[2P_\mu P_\varrho + 2k_{2\varrho}P_\mu\Theta(\nu_2-\nu_1) + 2P_\varrho k_{2\mu}\Theta(\nu_1-\nu_2) + i\delta_{\mu\varrho}\cdot\delta(\nu_1-\nu_2)] . \tag{4.15}$$

It is easy to see that the coefficients b_n are connected to a_n by the relationship

$$b_J = \sum_n (-1)^{(\sum_i n_i - 1)} \left(\sum_i n_i - 1\right)! \prod_i \left[\frac{1}{n_i!}\left(\frac{a_i}{i!}\right)^{n_i}\right], \tag{4.16}$$

where the first sum is extended to all positive entire numbers (including zero) such that $\sum_i i n_i = J$. As usual, from the determination of the Green's function it is possible to obtain the expression for the generating functional Z:

$$\left\{\begin{aligned} cZ = \exp\Bigg[-\Bigg\{&\frac{1}{(2\pi)^2}\iint \mathrm{d}^4x\,\mathrm{d}^4p\,\eta(x)\exp[ipx]\eta(p)\cdot\\ &\cdot\int_0^\infty \mathrm{d}\nu\exp[-i(p^2+\mu^2-i\varepsilon)\nu]\Big\langle\exp\Big[2e\int_0^\nu \mathrm{d}\nu' P_\mu(\nu')\frac{\delta}{\delta I_\mu(x(\nu'))}\Big]\Big\rangle\Bigg\}\Bigg]\cdot\\ &\cdot\exp\Bigg[\frac{1}{(2\pi)^4}\Bigg\{\iiint \mathrm{d}^4x\,\mathrm{d}^4p\int_{s_0}^\infty\frac{\mathrm{d}\nu}{\nu}\exp[-i(p^2+\mu^2-i\varepsilon)\nu]\cdot\\ &\cdot\Big[\Big\langle\exp\Big[2e\int_0^\nu \mathrm{d}\nu' P_\mu(\nu')\frac{\delta}{\delta I_\mu(x(\nu'))}\Big]\Big\rangle-1\Big]\Bigg\}\Bigg]\cdot\\ &\qquad\cdot\exp\Big[\frac{i}{2}\iint \mathrm{d}^4x\,\mathrm{d}^4y\,I_\mu(x)\bar{D}_{\mu\varrho}(x-y)I_\varrho(y)\Big],\\ \bar{D}_{\mu\varrho}(k) = \Big(\delta_{\mu\varrho}&-\frac{k_\mu k_\varrho}{k^2}\Big)\,i\int_{\alpha_0}^\infty\exp[-i\alpha(k^2+\lambda^2-i\varepsilon)]\,\mathrm{d}\alpha\,. \end{aligned}\right. \tag{4.17}$$

The introduction of two parameters s_0 and $\alpha_0 \neq 0$ eliminates divergences in the theory. It is even possible to show that for s_0 and $\alpha_0 \to 0$ all the divergences are incorporated in the arbitrary constants of the theory. For this reason, the renormalized quantitites do not depend on these. In (4.17) it is possible to effect the functional differentiation explicitly with respect to I_μ. Thus we obtain for Z

$$\begin{aligned} cZ = \sum_{n=0}^\infty \frac{(-1)^n}{n!(2\pi)^{2n}}\prod_{s=1}^n &\int \mathrm{d}^4p^{(s)}\,\mathrm{d}^4x_s\,\eta^*(x_s)\exp[ip^{(s)}x_s]\eta(p^{(s)})\cdot\\ &\cdot\int_0^\infty \mathrm{d}\nu_s\exp[-i((p^{(s)})^2+\mu^2-i\varepsilon)\nu_s]\Big\{1+\sum_{r=1}^\infty\frac{i^r}{r!(2\pi)^4}\prod_{s=1}^r\int \mathrm{d}^4x'_s\,\mathrm{d}^4p_1^{(s)}\cdot\\ &\cdot\int_{s_0\to 0}^\infty\frac{\mathrm{d}\nu'_s}{\nu'_s}\exp[-i((p_1^{(s)})^2+\mu^2-i\varepsilon)\nu'_s]\Big\}\cdot\exp\Big[iA_{nr}\Big(\frac{\delta}{\delta t};\ \frac{\delta}{\delta t_1}\Big)\Big]\cdot\\ &\qquad\cdot\exp\Big[i\Big\{\sum_{s=1}^n\int_0^{\nu_s}t_s^2(\xi)\,\mathrm{d}\xi+\sum_{s=1}^n\int_0^{\nu'_s}\mathrm{d}\xi\,t_{1s}^2(\xi)\Big\}\Big]\Big|_{t_s=t_{1s}=0}, \end{aligned} \tag{4.18}$$

where

$$(4.19)\qquad A_{nr}\left(\frac{\delta}{\delta t};\ \frac{\delta}{\delta t_1}\right)=\frac{1}{2}\int I_\mu(x)\,\bar{D}_{\mu\varrho}(x-y)\,I_\varrho(y)\,\mathrm{d}^4x\,\mathrm{d}^4y+$$

$$+\,2e\sum_{s=1}^{n}\int_0^{\nu_s}\mathrm{d}\eta\int\mathrm{d}^4y\,I_\mu(y)\,\bar{D}_{\mu\varrho}\big(y-x_s(\eta)\big)\,P^{(s)}_\varrho(\eta)\,+$$

$$+\,2e\sum_{s=1}^{r}\int_0^{\nu'_s}\mathrm{d}\xi'\int\mathrm{d}^4y\,I_\mu(y)\,\bar{D}_{\mu\varrho}\big(y-x'_s(\xi')\big)\,P^{(s)}_{1\varrho}(\xi)\,+$$

$$+\,2e^2\sum_{s,s'=1}^{n}\int_0^{\nu_s}\mathrm{d}\xi\int_0^{\nu'_s}\mathrm{d}\xi'\,P^{(s)}_\mu(\xi)\,\bar{D}_{\mu\varrho}\big(x_{s'}(\xi)-x_{s'}(\xi')\big)\,P^{(s')}_\varrho(\xi')\,+$$

$$+\,2e^2\sum_{s,s'=1}^{r}\int_0^{\nu'_s}\mathrm{d}\xi\int_0^{\nu'_s}\mathrm{d}\xi'\,P^{(s)}_{1\mu}(\xi)\,\bar{D}_{\mu\varrho}\big(x'_s(\xi)-x'_{s'}(\xi')\big)\,P^{(s')}_{1\varrho}(\xi')\,+$$

$$+\,4e^2\sum_{s=1}^{n}\sum_{s'=1}^{r}\int_0^{\nu_s}\mathrm{d}\xi\int_0^{\nu'_{s'}}\mathrm{d}\xi'\,P^{(s)}_\mu(\xi)\,\bar{D}_{\mu\varrho}\big(x_s(\xi)-x'_{s'}(\xi')\big)\,P^{(s')}_{1\varrho}(\xi')\,.$$

The expression of Z, if it is possible to disregard the polarization effects, assumes the simpler form

$$(4.20)\qquad \begin{cases} Z=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2\pi)^{2n}}\prod_{s=1}^{n}\int\mathrm{d}^4x_s\,\mathrm{d}^4p^{(s)}\eta^*(x_s)\exp[ip^{(s)}x_s]\eta(p)^{(s)}\cdot\\ \qquad\qquad\cdot\displaystyle\int_0^{\infty}\mathrm{d}v_s\exp\left[-i\big((p^{(s)})^2+\mu^2-i\varepsilon\big)\nu_s\right]\langle\exp[iA_n]\rangle\,,\\ A_n=\displaystyle\frac{1}{2}\int I_\mu(x)\bar{D}_{\mu\varrho}(x-y)\,I_\varrho(y)\,\mathrm{d}^4x\,\mathrm{d}^4y\,+\\ \qquad+\,2e\displaystyle\sum_{s=1}^{n}\int\mathrm{d}^4y\int_0^{\nu_s}\mathrm{d}\xi\,P^{(s)}_\mu(\xi)\,\bar{D}_{\mu\varrho}\big(x_s(\xi)-y\big)\,I_\varrho(y)\,+\\ \qquad+\,2e^2\displaystyle\sum_{s,s'=1}^{n}\int_0^{\nu_s}\mathrm{d}\eta\int_0^{\nu_s}\mathrm{d}\xi'\,P^{(s)}_\mu(\xi)\,\bar{D}_{\mu\varrho}\big(x_s(\xi)-x_{s'}(\xi')\big)\,P^{(s')}_\varrho(\xi')\,,\end{cases}$$

where

$$(4.21)\qquad \left\langle I_n\left(\frac{\delta}{\delta t_1}\cdots\frac{\delta}{\delta t_n}\right)\right\rangle=I_n\left(\frac{\delta}{\delta t_1}\cdots\frac{\delta}{\delta t_n}\right)\exp\left[i\sum_{s=1}^{n}\int_0^{\nu_s}\mathrm{d}\xi\,t_s^2(\xi)\right].$$

From (4.20) it is possible to obtain the modified perturbative expansion of the Green's function; in particular, in the infra-red region in the first approximation

$$\langle \exp[iA_n]\rangle = \exp[i\langle A_n\rangle]$$

and therefore Z assumes the form [18]

$$(4.22)\qquad Z=\sum\frac{(-1)^n}{n!(2\pi)^{2n}}\prod_{s=1}^{n}\int \mathrm{d}^4Z_s\,\mathrm{d}^4p^{(s)}\eta^*(x_s)\exp[ip^{(s)}x_s]\eta(p^{(s)})\cdot \cdot\int_0^\infty \mathrm{d}v_s\exp\left[i\{((p^{(s)})^2+\mu^2-i\varepsilon)\nu_s\}\right]\exp[i\langle A_n\rangle],$$

where

$$(4.23)\qquad \langle A_n\rangle=\frac{1}{2}\int I_\mu(x)\bar{D}_{\mu\varrho}(x-y)I_\varrho(y)\,\mathrm{d}^4x\,\mathrm{d}^4y+ + \sum_{s=1}^{n}\frac{2ie}{(2\pi)^2}\int P_\mu^{(s)}\bar{D}_{\mu\varrho}(k)I_\varrho(k)\exp[ikx_s]\cdot\frac{\exp[-i(k^2+2p^{(s)}k)\nu_s]-1}{k^2+2p^{(s)}k}\,\mathrm{d}^4k- -\sum_{s=1}^{n}\frac{(2e)^2}{(2\pi)^4}\int \mathrm{d}^4k\left[P_\mu^{(s)}\bar{D}_{\mu\varrho}(k)P_\varrho^{(s)}\left(\frac{\exp[-i(k^2+2p^{(s)}k)\nu_s]-1}{(k^2+2p^{(s)}k)^2}+\frac{i\nu_s}{k^2+2p^{(s)}k}\right)- -\frac{i\nu_s}{2^2}\bar{D}_{\mu\mu}(k)\right]-\sum_{s\neq s'=1}^{n}\frac{2e^2}{(2\pi)^2}\int \mathrm{d}^4k\,P_\mu^{(s)}\bar{D}_{\mu\varrho}(k)P_\varrho^{(s')}\cdot \cdot\frac{\left(\exp[-i(k^2+2p^{(s)}k)\nu_s]-1\right)\left(\exp[-i(k^2-2p^{(s')}k)\nu_s]-1\right)}{(k^2+2p^{(s)}k)(k^2-2p^{(s')}k)}.$$

The one-particle Bose Green's function is written thus:

$$(4.24)\qquad G^{(1)}(p)=i\int_0^\infty \mathrm{d}\nu\exp[-i(p^2+\mu^2-i\varepsilon)\nu+e^2a_1(\nu)],$$

where

$$a_1(\nu)=-\frac{2^2i}{(2\pi)^4}\int \mathrm{d}^4k\left[P_\mu\bar{D}_{\mu\varrho}(k)P_\varrho\left\{\frac{\exp[-i(k^2+2pk)\nu]-1}{(k^2+2pk)^2}+\frac{i\nu}{k^2+2pk}\right\}-\frac{i\nu}{4}D_{\mu\mu}(k)\right].$$

The two-particle Green's function is written

$$(4.25)\qquad G(p)=i\int_0^\infty \mathrm{d}\nu_n\exp[-i(p^2+\mu^2-i\varepsilon)\nu]\cdot \cdot\left\langle\exp\left[2ie^2\int_0^\nu \mathrm{d}\nu'\int_0^\nu \mathrm{d}\nu''P_\mu(\nu')\,\bar{D}_{\mu\varrho}\left(2\int_{\nu'}^{\nu}P(\xi)\,\mathrm{d}\xi\right)P_\varrho(\nu'')\right]\right\rangle,$$

where

$$P_\mu(\nu')=P_\mu+\frac{\delta}{2\,\delta t_\mu(\nu')}.$$

Equation (4.25) gives the modified perturbative expansion of $G(p)$:

$$G(p) = i\int_0^\infty \mathrm{d}\nu \exp\left[-i(p^2+\mu^2-i\varepsilon)\nu + \sum_{n=1}^\infty e^{2n} b_n\right]. \tag{4.26}$$

Equation (4.16) gives the relationships

$$a_n = \left\langle \left(2i\int_0^\nu \mathrm{d}\nu' \int_0^\nu \mathrm{d}\nu'' P_\mu(\nu') \bar{D}_{\mu\varrho}\left(2\int_{\nu'}^{\nu'} P(\xi)\,\mathrm{d}\xi\right) P_\varrho(\nu'')^n\right)\right\rangle = \tag{4.27}$$

$$= \frac{(2i)^n}{(2\pi)^{4n}} \prod_{s=1}^n \int \mathrm{d}^4 k_s \int_0^\nu \mathrm{d}\nu'_s \left\{\left(P_\mu + \sum_{s_1=1}^n k_{s,\mu}[\Theta(\nu'_s-\nu''_{s_1}) - \Theta(\nu'_s-\nu'_{s_1})] + \frac{1}{2}\frac{\delta}{\delta t_\mu(\nu'_s)}\right)\bar{D}_{\mu\varrho}(k_s)\cdot\right.$$
$$\cdot\left(P_\varrho + \sum_{s_1=1}^n [\Theta(\nu''_s-\nu''_{s_1}) - \Theta(\nu''_s-\nu'_{s'})]k_{s,\varrho} + \frac{1}{2}\frac{\delta}{\delta t_\varrho(\nu''_s)}\right)\cdot$$
$$\cdot\exp\left[2ipk_s(\nu''_s-\nu'_s) + \frac{i}{2}\sum_{s,s_1=1}^n k_s k_{s_1}\left(|\nu'_s-\nu'_{s_1}| + |\nu''_s-\nu''_{s_1}| - |\nu'_s-\nu''_{s_1}| - |\nu''_{s_1}-\nu''_s|\right)\right]\cdot$$
$$\left.\cdot\exp\left[i\int_0^\nu t^2(\xi)\,\mathrm{d}\xi\right]\Big|_{t=0}\right\}.$$

In particular, from (4.27) (in the transversal gauge)

$$a_1 = \frac{2i}{(2\pi)^4}\int \mathrm{d}^4 k \int_0^\nu \mathrm{d}\nu' \int_0^\nu \mathrm{d}\nu''\cdot \tag{4.28}$$
$$\cdot\left[\exp\left[2ipk(\nu''-\nu') - ik^2|\nu''-\nu'|\right] P_\mu P_\varrho + \frac{1}{2}\, i\delta(\nu'-\nu'')\,\delta_{\mu\varrho}\right] D_{\mu\varrho}(k)\,,$$

$$a_2 = -\frac{4}{(2\pi)^8}\int \mathrm{d}^4 k \int \mathrm{d}^4 k' \int_0^\nu \mathrm{d}\nu' \int_0^\nu \mathrm{d}\nu'' \int_0^\nu \mathrm{d}\nu'_1 \int_0^\nu \mathrm{d}\nu''_1 D_{\mu\varrho}(k) D_{\mu_1\varrho_1}(k')\cdot \tag{4.29}$$
$$\cdot\left\{\left[\left(P_\mu + k'_\mu\left(\Theta(\nu'-\nu''_1) - \Theta(\nu'-\nu'_1)\right)\right)\left(P_\varrho + k'_\varrho\left(\Theta(\nu''-\nu''_1) - \Theta(\nu''-\nu'_1)\right)\right) + \frac{1}{2}\, i\delta(\nu'-\nu'')\delta_{\mu\varrho}\right]\cdot\right.$$
$$\cdot\left[\left(P_{\mu_1} + k_{\mu_1}\left(\Theta(\nu'_1-\nu'') - \Theta(\nu'_1-\nu')\right)\right)\left(P_\varrho + k_{\varrho_1}\left(\Theta(\nu''_1-\nu'') - \Theta(\nu''_1-\nu')\right)\right) + \right.$$
$$\left. + \frac{1}{2}\, i\delta(\nu'_1-\nu''_1)\,\delta_{\mu_1\varrho_1}\right] + \frac{i}{2}\left[c^{\nu'\nu'_1\nu''\nu''_1}_{\mu\mu_1\varrho\varrho_1} + c^{\nu'\nu''_1\nu''\nu'}_{\mu\varrho_1\varrho\mu_1} + c^{\nu''\nu'_1\nu'\nu''_1}_{\varrho\mu_1\mu\varrho_1} + c^{\nu''\nu''_1\nu'\nu'_1}_{\varrho\varrho_1\mu\mu_1}\right] -$$
$$\left. - \frac{1}{4}\,\delta(\nu'-\nu'_1)\,\delta(\nu''-\nu''_1)\,\delta_{\mu\mu_1}\delta_{\varrho\varrho_1} - \frac{1}{4}\,\delta(\nu'-\nu''_1)(\nu''-\nu'_1)\,\delta_{\mu\varrho_1}\delta_{\varrho\mu_1}\right\}\cdot$$
$$\cdot\exp\left[2ipk(\nu''-\nu') + 2ipk'(\nu''_1-\nu'_1) - ik^2|\nu''-\nu'| - \right.$$
$$\left. - i(k')^2|\nu''_1-\nu'_1| + ikk'\left(|\nu'-\nu'_1| + |\nu''-\nu''_1| - |\nu'-\nu''_1| - |\nu''-\nu'_1|\right)\right].$$

For a complete treatment of the vertex part see ref. [20]. Let us now consider the equation (a scalar model with $L_{\text{in}} = g\Psi^2(x)\varphi(x)$)

$$(3.22)\qquad \left(-\Box + m^2 - g\varphi(x)\right) G(x, y|\varphi) = \delta(x-y)\,.$$

Equation (3.22) in the p-space is written [17, 19]

$$(4.30)\qquad [-(\partial_\mu + ip_\mu)^2 + m^2 - g\varphi(x)]\,G(x, p) = 1\,,$$

which admits the formal solution

$$(4.31)\qquad G(x, p) = \frac{1}{-(\partial_\mu + ip_\mu)^2 + m^2 - g\varphi(x)} = i\int_0^\infty \mathrm{d}\nu \exp[-i(p^2 + m^2 - i\varepsilon)\nu]\,Y(\nu),$$

where where $Y(\nu)$ satisfies the equation

$$(4.32)\qquad -i\,\frac{\partial Y}{\partial \nu} = \left(\partial^2 + 2ip_\mu\partial_\mu + g\varphi(x)\right) Y\,.$$

Let us introduce in (4.32) an additional interaction with $t_\mu(\nu)$ the external generator of the operator ∂_μ. Equation (4.32) therefore becomes

$$(4.33)\qquad -i\,\frac{\partial Y(\nu, t)}{\partial \nu} = \left(\partial^2 + 2ip_\mu\partial_\mu + it_\mu(\nu)\,\partial_\mu + g\varphi(x)\right) Y(\nu, t)\,,$$

which has the solution for $Y(\nu)$ in the form

$$(4.34)\qquad Y(\nu) = \exp\left[i\int_0^\nu \frac{\delta^2}{\delta t_\mu(\xi)\,\delta t_\mu(\xi)}\,\mathrm{d}\xi\right] Y_1(\nu, t)\big|_{t=0}\,,$$

where Y_1 satisfies the equation

$$(4.35)\qquad -\frac{\partial Y_1}{\partial \nu} = \left(2p_\mu\partial_\mu + t_\mu(\nu)\,\partial_\mu - ig\varphi(x)\right)Y_1\,.$$

Let us look for the solution for Y_1 of the form

$$(4.36)\qquad Y_1 = \exp\left[-\int_0^\nu (2p_\mu\partial_\mu + t_\mu(\xi)\,\partial_\mu)\,\mathrm{d}\xi\right] Y_2\,.$$

From (4.35) and (4.36) we have

$$(4.37)\qquad -i\,\frac{\partial Y_2}{\partial \nu} = g\varphi\left(x + 2p\nu + \int_0^\nu t_\mu(\xi)\,\mathrm{d}\xi\right) Y_2\,,$$

from which

$$(4.38)\qquad Y_2 = \exp\left[ig\int_0^\nu \varphi\left(x + 2p\nu' + \int_0^\nu t(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\nu'\right],$$

which gives Y in the form

$$(4.39)\quad Y(\nu) = \exp\left[i\int_0^\nu \frac{\delta^2}{\delta t_\mu(\xi)\,\delta t_\mu(\xi)}\,\mathrm{d}\xi\right]\exp\left[ig\int_0^\nu \varphi\left(x - 2p(\nu-\nu') - \int_{\nu'}^\nu t(\xi)\,\mathrm{d}\xi\right)\nu'|_{t=0}\right] =$$

$$= \exp\left[ig\int_0^\nu \varphi\left(x - 2p(\nu-\nu') - \int_{\nu'}^\nu \frac{\delta}{\delta t(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}\nu'\right]\exp\left[i\int_0^\nu t_\mu^2(\xi)\,\mathrm{d}\xi|_{t=0}\right].$$

It is easy to show that (4.38), written as a « continual integral », has the form of a Feynman's integral on the trajectories applied to Green's functions [21]. Equation (4.39) easily gives the solutions (3.28). To obtain these, it is sufficient to turn from (4.39) to the functional equation

$$(4.40)\qquad -i\,\frac{\partial Y(\nu, t)}{\partial\nu} = \left\{\frac{\delta^2}{\delta t_\mu(\nu-\varepsilon)\,\delta t_\mu(\nu-\varepsilon)} + 2ip_\mu\partial_\mu + it_\mu(\nu)\partial_\mu + g\varphi(x)\right\} Y(\nu, t)\,.$$

From (4.39) and (4.40) the following relationships for the coefficients F_n are obtained:

$$(4.41)\qquad F_j = \sum_n (-1)^{(\Sigma n_i - 1)}\left(\sum n_i - 1\right)!\prod\left[\frac{1}{n_i!}\left(\frac{Y_i}{i!}\right)^{n_i}\right],$$

where the sum of n is performed over all positive integers for which

$$\sum_i n_i = n_j\,.$$

Therefore

$$(4.42)\qquad Y_n = (-1)^n\left(\int_0^\nu \mathrm{d}\nu'\varphi(x - 2p(\nu-\nu')) - \int_{\nu'}^\nu \frac{\delta}{\delta t(\xi)}\,\mathrm{d}\xi\right)^n\cdot\exp\left[i\int_0^\nu t^2(\xi)\,\mathrm{d}\xi|_{t=0}\right] =$$

$$= \frac{(-1)^n}{(2\pi)^{2n}}\prod_{m=1}^n\left(\int_0^\nu \mathrm{d}\xi_m\int \mathrm{d}^4k_m\varphi(k_m)\right)\exp\left[i\left\{\sum_{m=1}^n 2pk_m(\nu-\xi_m) + \right.\right.$$

$$\left.\left. - \sum_{m,m_1=1}^n k_m k_{m_1}\left(\nu - \frac{1}{2}\,(\xi_m + \xi_{m_1}) - \frac{1}{2}\,|\xi_m - \xi_m|\right)\right\}\right].$$

The expression of Z for this model (if we disregard the closed-loop diagrams) has the form

$$(4.43)\quad \begin{cases} cZ \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2\pi)^{2n}} \prod_{s=1}^{n} \int \mathrm{d}^4x_s\, \mathrm{d}^4p^s \eta^*(x_s) \exp[ip^s x_s]\eta(p^s)\cdot \\ \qquad\qquad \cdot \int_0^{\infty} \exp[-i\nu_s((p^s)^2+\mu^2-i\varepsilon)]\langle \exp[iA_n]\rangle , \\ A_n(\overline{D}) = \frac{1}{2}\int J(x)\, \overline{D}(x-y)\, J(y)\, \mathrm{d}^4x\, \mathrm{d}^4y + \\ + g\sum_{s=1}^{n}\int \mathrm{d}^4y \int_0^{\nu_s} \mathrm{d}\xi\, \overline{D}(x_s(\xi)-y)\, J(y) + \frac{g^2}{2}\sum_{s,s'=1}^{n}\int_0^{\nu_s}\mathrm{d}\xi \int_0^{\nu'_s}\mathrm{d}\xi' \overline{D}(x_s(\xi)-x_{s'}(\xi'))+ \\ + \frac{1}{(2\pi)^4 i}\int \mathrm{d}^4x\, \mathrm{d}^4p \int_{s_0}^{\infty}\frac{\mathrm{d}\nu}{\nu}\exp[-i\nu(p^2+\mu^2-i\varepsilon)]\cdot \\ \qquad\qquad \cdot\left[\exp\left[ig^2\sum_{s=1}^{n}\int_0^{\nu}\mathrm{d}\xi \int_0^{\nu'_s}\mathrm{d}\xi' \,\overline{D}(x(\xi)-x_s(\xi)-1)\right]\right]. \end{cases}$$

In the first approximation of the modified perturbative expansion we can replace

$$(4.44)\quad \begin{cases} \langle \exp[iA_n(D)]\rangle = \exp iA(\langle \overline{D}\rangle), \\ \langle \overline{D}(x_s(\xi)-x_{s'}(\xi'))\rangle = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, \frac{\exp[ik(x_s-x_{s'})]}{k^2+\mu^2-is}\exp[i\beta], \\ \beta = 2k(p_s(\nu_s-\xi)-p_{s'}(\nu'_s-\xi'))- \\ \qquad -k^2[\nu_s-\xi+\nu'_s-\xi'+2\delta_{ss'}(\nu_s-\max(\xi,\xi'))]. \end{cases}$$

4'2. *Fermi case* [17]. – Let us look again at the eq. (3.42)

$$(3.42)\qquad (\gamma_\mu\partial_\mu + m + ie\gamma_\mu\varphi_\mu(x))\, G(xy|\varphi) = \delta(x-y).$$

Let us look for a solution for G in the form

$$(4.45)\qquad G(x,y) = (-\gamma_\mu\partial_\mu + m + ie\gamma_\mu\varphi_\mu(x))G_1(x,y|\varphi).$$

From (4.45) we have the equation

$$(4.46)\qquad \left[-(\partial_\mu - ie\varphi_\mu(x))^2 + m^2 + e\sigma_{\mu\varrho}\frac{\partial\varphi_\varrho(x)}{\partial x_\mu}\right]G_1(x,y|\varphi) = \delta(x-y),$$

where

$$\sigma_{\mu\varrho} = \tfrac{1}{2} i(\gamma_\mu \gamma_\varrho - \gamma_\varrho \gamma_\mu) . \tag{4.47}$$

In the p-space

$$G_1(x, y|\varphi) = \frac{1}{(2\pi)^4} \int \exp\left[ip(x-y)\right] \mathrm{d}^4 p \, G_1(p, x) . \tag{4.48}$$

Let us look for a solution

$$G_1(p, x) = i \int_0^\infty \exp\left[-i(p^2 + m^2 - i\varepsilon)\nu\right] Y(\nu) \, \mathrm{d}\nu , \tag{4.49}$$

where $Y(\nu)$ satisfies the equation

$$-i \frac{\partial Y}{\partial \nu} = \left[p^2 + (\partial_\mu + ip_\mu - ie\varphi_\mu(x))^2 - e\sigma_{\mu\varrho} \frac{\partial \varphi_\varrho}{\partial x_\mu}\right] Y . \tag{4.50}$$

To find the operator solution for Y, let us introduce an additional interaction by means of $t_\mu(\nu)$, the external generator for the generalized momentum operator, and with the anticommuting generator τ_μ for γ_μ-matrices. Consequently (4.50) becomes

$$-i \frac{\partial Y}{\partial \nu} = \left[p^2 + \Pi^2(x) + it_\mu(\nu)\Pi_\mu(x) + \gamma_\mu \tau_\mu(\nu) - e\sigma_{\mu\varrho} \frac{\partial \varphi_\varrho}{\partial x_\mu}\right] Y , \tag{4.51}$$

where

$$\Pi_\mu(x) = \partial_\mu + ip_\mu - ie\varphi_\mu(x) .$$

Let us look for a solution of the form $Y = A(\delta/\delta\tau)\, Y_1$ for Y, where Y_1 and A are defined respectively by the equations

$$\begin{cases} -i \dfrac{\partial Y_1}{\partial \nu} = \gamma_\mu \tau_\mu(\nu)\, Y_1 , \\[2ex] -i \dfrac{\partial A}{\partial \nu} = \left[p^2 + \Pi^2(x) + it_\mu(\nu) \cancel{\Pi_\mu} ie \dfrac{\partial \varphi_\mu}{\partial x_\varrho} \dfrac{\delta^2}{\delta\tau_\mu(\nu)\, \delta\tau_\varrho(\nu)}\right] A . \end{cases} \tag{4.52}$$

To carry out the calculation, let us introduce, for γ_μ the representation

$$\gamma_\mu = \Gamma_\mu + \frac{\delta}{\delta \Gamma_\mu} , \tag{4.53}$$

where

$$\{\Gamma_\mu, \Gamma_\varrho\}_+ = \{\Gamma_\mu, \tau_\varrho\}_+ = \{\tau_\mu, \tau_\varrho\}_+ = 0 , \qquad \left\{\Gamma_\mu, \frac{\partial}{\delta \Gamma_\nu}\right\} = \delta_{\mu\nu} . \tag{4.54}$$

It is now possible to determine the form of the solution for

$$(4.55)\qquad Y_1 = \exp\left[i\Gamma_\mu\int_0^\nu \tau_\mu(\xi)\,d\xi + \frac{1}{2}\int_0^\nu d\xi_1\,d\xi_2\,\tau_\mu(\xi_1)\varepsilon(\xi_1-\xi_2)\,\tau_\mu(\xi_2)\right]L\left(\frac{\delta}{\delta\Gamma}\right),$$

where

$$(4.56)\qquad \varepsilon(\xi) = \begin{cases} 1, & \xi > 0, \\ -1, & \xi < 0, \end{cases}\qquad L\left(\frac{\delta}{\delta\Gamma}\right) = \exp\left[i\,\frac{\delta}{\delta\Gamma_i}\int_0^\nu \tau_\mu(\xi)\,d\xi\right].$$

The solution of the equation for A is similar to the solution of eq. (3.39), therefore we shall now give the final result for Y:

$$(4.57)\qquad Y = \exp\left[2ie\int_0^\nu P_\mu(\nu')\varphi_\mu(x(\nu'))\,d\nu'\right]\cdot$$
$$\cdot\exp\left[i\int_0^\nu t_\mu^2(\xi)\,d\xi + i\Gamma_\mu\int_0^\nu \tau_\mu(\xi)\,d\xi - \frac{1}{2}\int_0^\nu d\xi_1\,d\xi_2\,\tau_\mu(\xi_1)\,\varepsilon(\xi_1-\xi_2)\,\tau_\mu(\xi_2)\right]L\left(\frac{\delta}{\delta\Gamma}\right)_{t=\tau=0},$$

where

$$(4.58)\qquad \begin{cases} x_\mu(\nu') = x_\mu - 2p_\mu(\nu-\nu') - \displaystyle\int_{\nu'}^\nu d\xi\,\frac{\delta}{\delta t_\mu(\xi)}, \\[2ex] P_\mu(\nu') = p_\mu + \dfrac{1}{2}\,\dfrac{\delta}{\delta t_\mu(\nu')} - \dfrac{i}{2}\,\dfrac{\delta^2}{\delta\tau_\mu(\nu')\,\delta\tau_\varrho(\nu')}\,\dfrac{\delta}{\delta x_\varrho}. \end{cases}$$

Thus it is now possible to find the operator solution for $G(xy)$:

$$(4.59)\qquad G(xy|\varphi) = \frac{1}{(2\pi)^4}\int \exp\left[ip(x-y)\right]d^4p\left[-\gamma_\mu\left(\partial_\mu + ip_\mu - ie\varphi_\mu(x)\right) + m\right]\cdot$$
$$\cdot\int_0^\infty d\nu\,\exp\left[-i(p^2+m^2-i\varepsilon)\,\nu\right]\left\langle\exp\left[2ie\int_0^\nu P_\mu(\nu')\,\varphi_\mu(x(\nu'))\right]\right\rangle,$$

where

$$(4.60)\qquad \left\langle A\left(\frac{\delta}{\delta t},\frac{\delta}{\delta\tau}\right)\right\rangle = A\left(\frac{\delta}{\delta t},\frac{\delta}{\delta\tau}\right)\exp\left[i\int_0^\nu t^2(\xi)\,d\xi + i\Gamma_\mu\int_0^\nu \tau_\mu(\xi)\,d\xi +\right.$$
$$\left.-\frac{1}{2}\int_0^\nu d\xi_1\int_0^\nu d\xi_2\,\tau(\xi_1)\,\tau(\xi_1)\,\varepsilon(\xi_1-\xi_2)\,\tau(\xi_2)\right]L\left(\frac{\delta}{\delta\Gamma}\right).$$

Since τ_μ are anticommuting generators, the expansion of expression

$$\exp\left[i\Gamma_\mu\int_0^\nu \tau_\mu(\xi)\,d\xi\right]$$

contains nonzero terms only of the kind $\approx 1, \Gamma_\mu, \Gamma_\mu\Gamma_\varrho, \Gamma_\mu\Gamma_s\Gamma_{s'}, \Gamma_1\Gamma_2\Gamma_3\Gamma_4$. In our case, as the operator A only contains even derivatives with respect to τ, we can effect in eq. (4.60) the substitution

$$\text{(4.61)}\qquad \exp\left[i\Gamma_\mu\int_0^\nu \tau_\mu\,d\xi\right]L\left(\frac{\delta}{\delta\Gamma}\right)\to 1+\frac{1}{4}(\gamma_\mu\gamma_\varrho-\gamma_\varrho\gamma_\mu)\int_0^\nu d\xi\,d\xi_1\,\tau_\mu(\xi)\,\tau_\varrho(\xi_1)+ \\ +\gamma_5\int_0^\nu\prod_{i=1}^4 d\xi_i\,\tau_4(\xi_4)\,\tau_3(\xi_3)\,\tau_2(\xi_2)\,\tau_1(\xi_1)\,.$$

Thus, by means of introducing the generators τ_μ, we can obtain the solution for Y, written in the normal form with respect to the γ_μ-matrices. As the dependency on τ_μ, $\delta/\delta\tau_\mu$ is of the Gaussian kind, it is easy in (4.59) to carry out functional differentiation.

As usual, let us substitute the expression for the Green's function (4.59) in the operator solution (3.42) (see Sect. 2). After the substitution $i\varphi_\mu(x)\to$ $\to\delta/\delta J_\mu(x)$, the following expression for Z is obtained:

$$\text{(4.62)}\qquad cZ=\sum_n\frac{1}{n!(2\pi)^{2n}}\prod_{s=1}^n\int d^4x_s\,d^4p^s\exp[ip^sx_s]\bar\eta(x_s)\cdot$$

$$\cdot\left[\gamma_\mu^{(s)}\left(\partial_\mu^{(s)}+ip_\mu^{(s)}-\frac{\delta}{\delta J(x_s)}\right)-m\right]\int_0^\nu d\nu_s\exp\left[-i\left((p^{(s)})^2+m^2-i\varepsilon\right)\nu_s\right]\cdot$$

$$\cdot\left\{1+\sum_s^\infty\frac{(-i)^r}{2^r r!(2\pi)^{4r}}\sum_{s=1}^r\int d^4x'_s\,d^4p_1^{(s)}\int_{\varepsilon_0\to0}^\infty\frac{d\nu'_s}{\nu'_s}\exp\left[-\left((p_1^{(s)})^2+m^2\right)\nu'_s\right]\right\}\cdot$$

$$\cdot\exp\left[\frac{1}{2}i\int J_\mu(x)D_{\mu\varrho}(x-y)J(y)+2ie^2\sum_{s,s'}^n\int_0^{\nu_s}d\xi\int_0^{\nu'_s}d\xi'P_\mu^{(s)}(\xi)D_{\mu\varrho}\left(x_s(\xi)-x_{s'}(\xi')\right)P_\varrho^{(s')}(\xi')+\right.$$

$$+2ie^2\sum_{s,s'}^r\int_0^{\nu'_s}d\xi\int_0^{\nu'_{s'}}d\xi'P_{1\mu}^{(s)}(\xi)\,D_{\mu\varrho}\left(x'_s(\xi)-x'_{s'}(\xi')\right)P_{1\varrho}^{(s')}(\xi')+4ie^2\sum_{s=1}^n\sum_{s'=1}^r\int_0^{\nu_s}d\xi\int_0^{\nu'_{s'}}d\xi'P_\mu^{(s)}(\xi)\cdot$$

$$\cdot D_{\mu\varrho}\left(x_s(\xi)-x'_{s'}(\xi')\right)P_{1\varrho}^{(s')}(\xi')+2ie\sum_{s=1}^n\int_0^{\nu_s}d\xi\int d^4y\,J_\mu(y)\,D_{\mu\varrho}\left(y-x_s(\xi)\right)P_\varrho^{(s)}(\xi)+$$

$$+ 2ie \sum_{s=1}^{r} \int_0^{\nu'_s} d\xi \int d^4y\, J_\mu(y)\, D_{\mu\varrho}\big(y - x'_s(\xi)\big)\, P^{(s)}_{1\varrho}(\xi)\Big] \cdot$$

$$\cdot \exp\Big[\sum_{s=1}^{r} i \Big\{ \int_0^{\nu'_s} d\xi \big(t_1^{(s)}(\xi)\big)^2 + \frac{1}{2}\, i \int_0^{\nu'_s} d\xi \int_0^{\nu'_s} d\xi'\, \tau_1(\xi)\, \varepsilon(\xi - \xi')\, \tau_1(\xi') \Big\}\Big|_{\tau_1 = t_1 = 0}\Big] \cdot$$

$$\cdot \exp\Big[\sum_{s=1}^{n} i \Big\{ \int_0^{\nu_s} d\xi \big(\tau^{(s)}(\xi)\big)^2 + \Gamma^{(s)}_\mu \int_0^{\nu_s} \tau^{(s)}_\mu(\xi)\, d\xi +$$

$$+ \frac{1}{2}\, i \int_0^{\nu_s} d\xi \int_0^{\nu_s} d\xi'\, \tau^{(s)}(\xi)\, \varepsilon(\xi - \xi')\, \tau^{(s)}(\xi') \Big\}\, L\Big(\frac{\delta}{\delta \Gamma^s}\Big)\Big|_{t = \tau = 0}\Big]\, \eta(p^{(s)}) ,$$

where

$$(4.63) \qquad \begin{cases} x'_s(\xi) \;= x'_s - 2p_1(\nu'_s - \xi) - \displaystyle\int_0^{\nu'_s} d\xi' \frac{\delta}{\delta t_1(\xi)} , \\[2ex] P^{(s)}_1(\xi) = p^{(s)}_1 + \dfrac{1}{2} \dfrac{\delta}{\delta t_1(\xi)} - \dfrac{1}{2}\, i\, \dfrac{\delta^2}{\delta \tau_1(\xi)\, \delta \tau_1(\xi)}\, \dfrac{\delta}{\partial x'_\varrho} . \end{cases}$$

Disregarding polarization effects, Z assumes the simpler form

$$(4.64) \qquad Z = \sum_n \frac{1}{n!(2\pi)^{2n}} \prod_{s=1}^{n} \int \bar\eta(x_s) \Big[\gamma^{(s)}_\mu\, (\partial_\mu + ip^{(s)}_\mu - e)\, \frac{\delta}{\delta J(x_s)} - m\Big] \cdot$$

$$\cdot \exp\,[ip^{(s)} x_s]\, d^4p^{(s)}\, d^4x_s \int_0^\infty d\nu\, \exp\,[-i((p^{(s)})^2 + m^2 - i\varepsilon)\nu_s]\, \langle \exp\,[A_n] \rangle\, n(p^{(s)})$$

and

$$(4.65) \qquad A_n = \frac{1}{2}\, i \int J(x)\, D(x - y)\, J(y)\, d^4x\, d^4y + 2ie^2 \sum_{s,s_1}^{h} \int_0^{\nu_s} d\nu'_s \int_0^{\nu'_s} d\nu'_{s_1}\, P^{(s)}_\mu(\nu'_s) \cdot$$

$$\cdot D_{\mu\varrho}\big(x_s(\nu'_s) - x_{s'}(\nu'_s)\big)\, P^{(s')}_\varrho + 2ie \sum_{s=1}^{n} \int_0^{\nu_s} d\nu'_s\, P^{(s)}_\mu\, D_{\mu\varrho}\big(x_s(\nu'_s) - y\big)\, J_\varrho(y)\, d^4y ,$$

where

$$(4.66) \qquad \Big\langle F_n\Big(\frac{\delta}{\delta \tau}, \frac{\delta}{\delta t}\Big)\Big\rangle = F\Big(\frac{\delta}{\delta \tau}, \frac{\delta}{\delta t}\Big) \cdot$$

$$\cdot \prod_{i=1}^{n} \exp\Big[i \int_0^{\nu_i} t^{(i)}(\xi)\, d\xi + i\Gamma^{(i)}_\mu \int_0^{\nu_i} d\xi\, \tau^{(i)}_\mu(\xi) - \frac{1}{2} \int_0^{\nu_i} d\xi_1 d\xi_2\, \tau^i(\xi_1)\, \varepsilon(\xi_1 - \xi_2)\, \tau^{(i)}(\xi_2)\Big]\, L\Big(\frac{\delta}{\delta \Gamma}\Big)\Big|_{t = \tau = 0} \cdot$$

Using (4.64) it is possible to construct a « modified » perturbation theory, analogous to that developed in the previous Section. The first approximation of the « modified » perturbation theory corresponds to the sum of all Feynman's graphs, the only difference being that the terms $k_i k_j$ $(i \neq j)$ are absent in the denominators of the propagators. In the following approximations we will take these terms into consideration:

$$\frac{1}{(\sum p_i + \sum k_i)^2 + m^2} = \frac{1}{(\sum p_i)^2 + m^2 + (\sum k_i)^2 + 2\sum p_i k_j}\left\{1 + \sum_{n=1}^{\infty}(-a)^n\right\},$$

where

$$a = \frac{2\sum_{j\neq j_1} k_{j_1} k_j}{(\sum p_i)^2 + m^2 + \sum k_j^2 + 2\sum p_i k_j}.$$

It is easy to see that this « modified » perturbation theory converges well in the infra-red region where the parameter a is small. From (4.65) and (4.66) it is clear that the problem of calculating Z reduces to that of finding the operator $\langle e^A\rangle$. In the conventional perturbation theory, $\langle e^A\rangle$ is defined by the expansion $e^A = \sum a_n/n!$ The « modified » perturbation theory corresponds to the expansion $\exp[A] = \exp[\sum b_n]$. In this case the coefficients b_n are connected to the coefficients a_n by a relationship of the kind (4.16).

In the case of « soft » photons the form of the generating functional is very simple. Limiting ourselves to the first term of the « modified » perturbation theory $(b_1 = a_1)$, we obtain the following form for Z [18]:

$$(4.67)\quad \begin{cases} Z \approx \sum_n \frac{(-1)^n}{(2\pi)^{2n} n!}\prod_{s=1}^{n}\int \bar\eta(x_s)(-i\gamma_\mu^{(s)} p_\mu^{(s)} + m)\eta(p^{(s)})\cdot \\ \qquad\qquad \cdot\exp[ip^{(s)}x_s]\,\mathrm{d}^4p^{(s)}\,\mathrm{d}^4x_s\int_0^\infty \mathrm{d}\nu_s \exp[iA(J)], \\ A(J) = \frac{1}{2}\int J_\mu(x) D_{\mu\varrho}(x-y) J_\varrho(y)\,\mathrm{d}^4x\,\mathrm{d}^4y - \sum_{s=1}^{n}\nu_s\big((p^s)^2 + m^2\big) + \\ + \sum_{s=1}^{n}\frac{2ei}{(2\pi)^2}\int p_\mu^{(s)} D_{\mu\varrho}(k) J_\varrho(k)\exp[ikx_s]\,\frac{\exp[-i(k^2 - 2p^s k)\nu_s] - 1}{k^2 - 2p^s k}\,\mathrm{d}^4k - \\ - \sum_{s=1}^{n}\int\frac{4e^2}{(2\pi)^4} p_\mu^{(s)} D_{\mu\varrho}(k) p_\varrho^{(s)}\left(\frac{\exp[-i(k^2 - 2p^{(s)}k)\nu_s] - 1}{(k^2 - 2p^s k)^2} + \frac{i\nu_s}{k^2 - 2p^s k}\right)\mathrm{d}^4k - \\ - \sum_{s\neq s'}^{n}\frac{2e^2}{(2\pi)^4} p_\mu^{(s)} D_{\mu\varrho}(k) p_\varrho^{(s')}\cdot \\ \qquad \cdot\frac{\big(\exp[-i(k^2 - 2p^{(s)}k)\nu_s] - 1\big)\big(\exp[-i(k^2 + 2p^{(s')}k)\nu_{s'}] - 1\big)\,\mathrm{d}^4k}{(k^2 - 2p^{(s)}k)(k^2 + 2p^{(s')}k)}. \end{cases}$$

From (4.67) we obtain the one-electron Green's function:

$$(4.68)\quad \begin{cases} G(p) = (-i\gamma_\mu p_\mu + m)\, i\int\limits_0^\infty \mathrm{d}\nu \exp\left[-i(p^2+m^2-i\varepsilon)\nu + F(\nu)\right], \\ F(\nu) = -\dfrac{4ie^2}{(2\pi)^4}\displaystyle\int p_\mu D_{\mu\varrho}(k)\, p_\varrho \left[\frac{\exp[-i(k^2-2pk)\nu]-1}{(k^2-2pk)^2} + \frac{i\nu}{k^2-2pk}\right]. \end{cases}$$

From (4.68) we obtain the well-known asymptotic infra-red behaviour [18, 22] for $G(p)$ which, after renormalization, has the following form (in the transverse gauge $D_{\mu\varrho}(k) = (\delta_{\mu\varrho} - k_\mu k_\varrho k^{-2})/(k^2 - i\varepsilon)$):

$$(4.69)\qquad G(p) = \frac{-i\gamma_\mu p_\mu + m}{p^2+m^2}\,\frac{1}{[1+p^2/m^2]^{(3e^2/8\pi^2)}} f(p),$$

where

$$f(p) = 1 + \frac{1}{2}\left(\frac{p^2}{m^2}+1\right) + O(p^2+m^2), \qquad p^2+m^2 \to 0.$$

The exact expression obtained from (4.67) for $G(p)$ is

$$(4.70)\qquad G(p) = i\int\limits_0^\infty \mathrm{d}\nu \Bigg[\{-i\gamma_\mu p_\mu + m\}\langle \exp\Big[2ie^2\int\limits_0^\nu \mathrm{d}\nu'\,\mathrm{d}\nu''\, P_\mu(\nu')\Big]\cdot$$
$$\cdot D_{\mu\varrho}(x(\nu') - x(\nu''))\, P_\varrho(\nu')\rangle + 2ie^2\Big\langle\int\limits_0^\nu \gamma_\mu D_{\mu\varrho}(x(\nu') - x)\, P_\varrho(\nu')\,\mathrm{d}\nu'\cdot$$
$$\cdot\exp\Big[2ie^2\int\limits_0^\nu \mathrm{d}\nu'\,\mathrm{d}\nu''\, P_\mu(\nu')\, D_{\mu\varrho}(x(\nu') - x(\nu''))\, P_\varrho(\nu'')\Big]\Big\rangle\Bigg]\exp[-i(p^2+m^2)\nu],$$

where the operators $x_\mu(\nu')$, $P_\mu(\nu')$ and the sign $\langle\,\rangle$ are determined by (4.58) and (4.60).

From (4.70) we obtain the following expression for $G(p)$ in the « modified » perturbation theory:

$$(4.71)\qquad G(p) = i\int\limits_0^\infty \mathrm{d}\nu \exp[-i(p^2+m^2-i\varepsilon)\nu]\cdot$$
$$\cdot\left[(-i\gamma_\mu p_\mu + m)\exp\left[\sum b_n\right] + a_0'\exp\left[\sum b_n'\right]\right].$$

Expressing the coefficients b_j as a function of a_n,

$$(4.72)\qquad b_j = \sum_n (-1)^{(\Sigma n_i - 1)}\left(\sum n_i - 1\right)!\prod\left[\frac{1}{n_i!}\left(\frac{a_i}{i!}\right)^{n_i}\right],$$

we obtain the following for a_n and a'_n:

$$(4.73)\qquad \begin{cases} a_n = \left\langle \left(2ie^2\int\limits_0^\nu d\nu'\,d\nu'' P_\mu(\nu) D_{\mu\varrho}(x(\nu') - x(\nu''))\,P_\varrho(\nu^n)\right)^n \right\rangle, \\ a'_n = \dfrac{1}{a'_0}\left\langle 2ie^2\int\limits_0^\nu \gamma_\mu D_{\mu\varrho}(x(\nu') - x)\,P_\varrho(\nu')\,d\nu' \cdot \right. \\ \qquad\qquad \left. \cdot\left(2ie^2\int\limits_0^\nu d\nu'\int\limits_0^\nu d\nu'' P_\mu(\nu') D_{\mu\varrho}(x(\nu') - x(\nu''))\,P_\varrho(\nu'')\right)\right\rangle, \end{cases}$$

in particular,

$$(4.74)\qquad a'_0 = 2e^2\int d^4k\,\gamma_\mu D_{\mu\varrho}(k)\left(p_\varrho - \frac{1}{2}\gamma_\varrho\gamma_{\varrho_1}k_{\varrho_1}\right)\frac{1-\exp[-i(k^2-2pk)\,\nu]}{k^2-2pk}.$$

From (4.73) it is possible to calculate all the coefficients in the general explicit form. We will not give the expression of these coefficients but we will note that it is possible to develop a very simple diagram technique to determine these quantities. The expression

$$G_0(x, \nu - \nu') = \theta(\nu - \nu')\int \frac{d^4k}{\exp[-i(k^2-2pk)(\nu-\nu') + ikx]}$$

represents the fermion propagator; the photon propagator is represented in the usual way and the vertex function is the corresponding operator of quantum electrodynamics.

5. – Application of the method of stationary phase to the operator solution of Green's functions [23].

Introduction. – It is well known that all mathematical problems of quantum mechanics and quantum field theories lead to the calculation of certain integrals of a definite kind. The construction of a method of calculation for integrals of the following is therefore important:

$$(5.1)\qquad \int (D_q)\exp\left[i\,\frac{S[q]}{\hbar}\right],$$

where $S[q]$ is the «action» and (D_q) the «measure» in the functional space:

$$(5.2) \qquad (D_q) = \frac{\prod_{\zeta=0}^{T} \mathrm{d}q(\zeta)}{\int \exp\left[(i/2\hbar)(q', \Omega_0 q')\right] \prod_{\xi=0}^{T} \mathrm{d}q'(\xi)} .$$

Here, Ω_0 is the constant integral operator and $(q, \Omega_0 q)$ is the scalar product

$$(5.3) \qquad S^{(0)}[q] = (q, \tfrac{1}{2}\,\Omega_0 q) = \tfrac{1}{2}\int_0^T \mathrm{d}\xi\,\mathrm{d}\xi'\, q(\xi)\,\Omega_0(\xi\xi')\,q(\xi') ,$$

where, with $S^0[q]$, the unperturbed part of the action is indicated:

$$S[q] = S^{(0)}[q] + \bar{S}[q] .$$

Let us examine a method to obtain the asymptotic expansion for $h \to 0$, of integrals of the kind (5.1).

We shall study the asymptotic behaviour of the solution of Schrödinger equation

$$(5.4) \qquad i\hbar \frac{\partial \Psi}{\partial t} = H\Psi ,$$

or of the equation for Green's functions:

$$(5.5) \qquad \left(-i\hbar \frac{\partial}{\partial t} + H\right) G(\boldsymbol{x}, t; \boldsymbol{x}'t') = \delta(t-t')\,\delta(\boldsymbol{x}-\boldsymbol{x}') .$$

After all, it is an integral formulation of the analogous method of stationary phase which, as is well known, gives us the asymptotic behaviour for $\lambda \to 0$ of oscillating integrals to infinity:

$$(5.6) \qquad \mathscr{I}_n = \int_{E_n} \exp\left[i\,\frac{f(x)}{\lambda}\right] \prod_{i=1}^{N} \mathrm{d}x_i .$$

Here $f(x)$ is a sufficiently «smooth» real function of the variables $x = (x_1 \dots x_N)$, so that the integral (5.6) exists. The stationary points satisfy the relationship

$$(5.7) \qquad \frac{\partial f(x)}{\partial x_i} = 0 , \qquad i = 1, 2, \dots, N,\ \mathrm{d}^2 f \neq 0 .$$

Let us summarize the formal aspect of the method. Let us expand $f(x)$ in Taylor's series near a stationary point $x^k = (x_1^{(k)} \dots x_N^{(k)})$:

$$f(x) = f(x^{(k)}) + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 f(x)}{\partial x_i \partial x_j}\bigg|_{x=x^k}(x-x^k)_i(x-x^k)_j + f_1(\Delta x^k, x^k)\,, \tag{5.8}$$

where with $f_1(\Delta x^k, x^k)$ the following terms of expansion are indicated. Let us substitute expansion (5.8) in (5.6) after expanding the term $\exp[(i/\lambda)f]$. For each point x^k we shall have a contribution

$$\int_{-\infty}^{\infty}\exp\left[\frac{i}{\lambda}f(x^k)\right] + \frac{i}{2\lambda}\sum_{i,j=1}^{N}\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\bigg|_{x=x^{(k)}}(x-x^k)_i(x-x^k)_j \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\lambda}f_1(\Delta x^k, x^k)\right)^n. \tag{5.9}$$

Putting $(x-x^{(k)})_i/\sqrt{\lambda} = Z_i$ and keeping the contributions of all stationary points, we have

$$\begin{aligned}\mathscr{I}_n \simeq \lambda^{N/2}\sum_k \exp\left[\frac{i}{\lambda}f(x^{(k)})\right]\int_{-\infty}^{+\infty}\prod_{i=1}^{N}\mathrm{d}Z_i \exp\left[\frac{i}{2}\right]\sum_{i,j=1}^{N}\Omega_{ij}^{(k)}Z_iZ_j\,\cdot \\ \cdot\sum_{n=0}^{\infty}\left(\frac{i}{\lambda}f_1(\sqrt{\lambda}Z, x^{(k)})\right)^n\cdot\frac{1}{n!} = (2\pi\lambda)^{N/2}\sum_k \exp\left[\frac{i}{\lambda}f(x^{(k)})\right]\cdot \\ \cdot\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\lambda}f_1\left(-i\sqrt{\lambda}\frac{\partial}{\partial y}, x^{(k)}\right)\right)^n \exp\left[i\frac{\pi}{4}\delta^{(k)}\right]\left|\det{}_{(N)}[\Omega^{(k)}]^{-\frac{1}{2}}\cdot\right. \\ \cdot\exp\left[-\frac{i}{2}\sum_{j,m=1}^{N}y_i(\Omega^{-1})_{j,m}Y_m\right]\bigg|_{Y=0},\end{aligned} \tag{5.10}$$

where

$$\Omega_{ij}^{(k)} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}\bigg|_{x=x^{(k)}}. \tag{5.11}$$

$\delta^{(k)}$ is the sign of the quadratic form $Q^{(k)} = \sum_{i,j=1}^{N}\Omega_{ij}^{(k)}Z_iZ_j$ and Ω_{ij}^{-1} is the element i, j of the inverse matrix of Ω. The quantity $\delta^{(k)}$ is written in the form

$$\delta^{(k)} = \gamma_+^{(k)} - \gamma_-^{(k)}\,, \tag{5.12}$$

where $\gamma_+^{(k)}$ and $\gamma_-^{(k)}$ respectively are the number of positive and negative eigenvalues of the form $Q^{(k)}$.

Since $Q^{(k)}$ has no zero eigenvalues

$$\gamma_+^{(k)} = N - \gamma_-^{(k)}\,, \tag{5.13}$$

so that

$$\delta^{(k)} = N - 2\gamma_-^{(k)} \tag{5.14}$$

and so

$$\exp\left[i\frac{\pi}{4}\delta^{(k)}\right] = \exp\left[i\frac{\pi}{4}N\right]\exp\left[-i\frac{\pi}{2}\gamma_-^{(k)}\right]. \tag{5.15}$$

5'1. *Schrödinger nonrelativistic equation.* – Let us now consider Schrödinger's nonrelativistic equation for Green's function:

$$\left\{-i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} + U(\boldsymbol{x}, t)\right\} G(\boldsymbol{x}, t; \boldsymbol{x}', t') = \delta(\boldsymbol{x} - \boldsymbol{x}')\,\delta(t - t'), \tag{5.16}$$

where $\hbar = 1$.

We shall find the asymptotic behaviour of the solution by means of the stationary phase method.

Let us look for a solution of (5.16) in the form [17-24]

$$G(\boldsymbol{x}, t; \boldsymbol{x}', t') = \int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\exp\left[i[\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{x}') - \omega(t - t')]\right] G(\boldsymbol{x}, \boldsymbol{p}, \omega, t). \tag{5.17}$$

By substituting (5.17) in (5.16) we obtain the following equation for $G(\boldsymbol{x}, \boldsymbol{p}, \omega, t)$:

$$\left\{-i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} - i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m} + \frac{\boldsymbol{p}^2}{m} - \omega + U(\boldsymbol{x}, t)\right\} G(\boldsymbol{x}, \boldsymbol{p}, \omega, t) = 1, \tag{5.18}$$

which gives the solution

$$\begin{aligned} G(\boldsymbol{x}, \boldsymbol{p}, \omega, t) &= \frac{1}{-i(\partial/\partial t) - \boldsymbol{\nabla}^2/2m - i(\boldsymbol{p}\cdot\boldsymbol{\nabla})/m + \boldsymbol{p}^2/2m - \omega + U(\boldsymbol{x}, t) - i\varepsilon} = \\ &= i\int_0^{\infty} d\nu\, T_\nu \exp\left[\int_0^{\nu}\left\{-i\partial_\mu(\nu') - \frac{\boldsymbol{\nabla}^2(\nu')}{2m} + \frac{\boldsymbol{p}^2}{2m} - i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}(\nu')}{m} - \omega + U\big(\boldsymbol{x}(\nu'), t(\nu')\big) - i\varepsilon\right\}\cdot\right. \\ &\qquad \left.\cdot(-i)\,d\nu'\right] = i\int_0^{\infty} d\nu \exp\left[-i\nu\left(\frac{\boldsymbol{p}^2}{2m} - \omega\right)\exp[-\varepsilon\nu]\phi(\nu, \boldsymbol{p}, \omega, \boldsymbol{x}, t)\right], \end{aligned} \tag{5.19}$$

where $\phi(\nu, \boldsymbol{p}, \omega, \boldsymbol{x}, t)$ satisfies the equation

$$i\frac{\partial\phi(\nu)}{\partial\nu} = \left\{-i\frac{\partial}{\partial t} - \frac{\boldsymbol{\nabla}^2}{2m} - i\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m} + U(\boldsymbol{x}, t)\right\}\phi(\nu) \tag{5.20}$$

with the condition $\phi(0) = 1$.

Let us now introduce an operator $\boldsymbol{\tau}(\nu)$ representing an additional interaction; eq. (20) becomes

$$(5.21)\qquad i\,\frac{\partial\phi(\nu,\boldsymbol{\tau})}{\partial\nu}=\left\{-\frac{\boldsymbol{\nabla}^2}{2m}-i\,\frac{\partial}{\partial t}-i\,\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}-i\boldsymbol{\tau}(\nu)\cdot\boldsymbol{\nabla}+U(\boldsymbol{x},t)\right\}\phi(\nu,\tau)\,,$$

where

$$(5.22)\qquad \phi(0,\boldsymbol{\tau})=1 \text{ and } \phi(\nu,0)=\phi(\nu).$$

Equation (5.21) is equivalent to the functional equation

$$(5.23)\qquad i\,\frac{\partial\phi(\nu,\boldsymbol{\tau})}{\partial\nu}=\left\{-\frac{1}{2m}\,\frac{\delta^2}{\delta\boldsymbol{\tau}^2(\nu)}-i\,\frac{\partial}{\partial t}-i\,\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}-i\boldsymbol{\tau}(\nu)\boldsymbol{\nabla}+U(\boldsymbol{x},t)\right\}\phi(\nu,\boldsymbol{\tau})$$

with $\phi(0,\boldsymbol{\tau})=1$. Equation (5.23) admits the solution

$$(5.24)\qquad \phi(\nu,\boldsymbol{\tau})=\exp\left[\frac{i}{2m}\int_0^\nu\frac{\delta^2}{\delta\boldsymbol{\tau}^2(\xi)}\,\mathrm{d}\xi\right]Y(\nu,\boldsymbol{\tau})\,,$$

where $Y(\nu,\boldsymbol{\tau})$ satisfies the equation

$$(5.25)\qquad i\,\frac{\partial Y(\nu,\boldsymbol{\tau})}{\partial\nu}=\left\{-i\,\frac{\partial}{\partial t}-i\,\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}-i\boldsymbol{\tau}(\nu)\boldsymbol{\nabla}+U(\boldsymbol{x},t)\right\}Y(\nu,\boldsymbol{\tau})$$

with $Y(0,\boldsymbol{\tau})=1$.

Equation (5.25) is a first-order differential equation and we seek the solution in the form

$$(5.26)\qquad Y(\nu,\tau)=\exp\left[-\int_0^\nu\mathrm{d}\xi\left[\frac{\partial}{\partial t}+\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}+\boldsymbol{\tau}(\xi)\,\boldsymbol{\nabla}\right]\right]Y_1(\nu,\boldsymbol{\tau})\,,$$

where $Y_1(\nu,\boldsymbol{\tau})$ satisfies the equation

$$(5.27)\qquad i\,\frac{\partial Y_1(\nu,\boldsymbol{\tau})}{\partial\nu}=R^{-1}(\nu)\,U(\boldsymbol{x},t)R(\nu)\,Y_1(\nu,\boldsymbol{\tau})=$$

$$=U\left(t+\nu,\,\boldsymbol{x}+\frac{\boldsymbol{p}\nu}{m}+\int_0^\nu\boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\right)Y_1(\nu,\,\boldsymbol{\tau})$$

with the condition $Y_1(0,\boldsymbol{\tau})=1$.

Therefore,

$$(5.28)\qquad Y_1(\nu, \boldsymbol{\tau}) = \exp\left[-i\int_0^{\nu} U\left(t+\nu', \boldsymbol{x}+\frac{\boldsymbol{p}}{m}\nu'+\int_0^{\nu'}\boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\nu'\right].$$

By substituting (5.28) in (5.26) we obtain

$$(5.29)\qquad Y(\nu, \boldsymbol{\tau}) = \exp\left[-i\int_0^{\nu} U\left(t-(\nu-\nu'), \boldsymbol{x}-\frac{\boldsymbol{p}}{m}(\nu-\nu')-\int_{\nu'}^{\nu}\boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\nu'\right]$$

and therefore

$$(5.30)\qquad \phi(\nu, \boldsymbol{\tau}) = \exp\left[\frac{i}{2m}\int_0^{\nu}\frac{\delta^2}{\delta\boldsymbol{\tau}^2(\xi)}\,\mathrm{d}\xi\right]\cdot$$
$$\cdot\exp\left[-i\int_0^{\nu} U\left(t-(\nu-\nu'), \boldsymbol{x}-\frac{\boldsymbol{p}}{m}(\nu-\nu')-\int_{\nu'}^{\nu}\boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\nu'\right].$$

In (5.17) it is possible to integrate on ω. Thus we obtain

$$(5.31)\qquad G(\boldsymbol{x}, t, \boldsymbol{x}', t') = \int_{-\infty}^{\infty}\frac{\mathrm{d}^3p}{(2\pi)^3}\,G(\boldsymbol{p}, \boldsymbol{x}, t, t')\exp\left[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')\right],$$

where

$$(5.32)\qquad G(\boldsymbol{p}, \boldsymbol{x}, t, t') = \exp\left[-i\,\frac{\boldsymbol{p}^2}{2m}(t-t')\right] i\theta(t-t')\phi(\nu=t-t', 0)\,.$$

Equation (5.30) for $t-t'=T$ becomes

$$(5.33)\qquad \phi(\nu=t-t', 0) =$$
$$=\int(D^3\tau)\exp\left[i\int_0^{T}\left[\frac{m\boldsymbol{\tau}^2(\nu)}{2}-U\left(t'+\nu, \boldsymbol{x}-\frac{\boldsymbol{p}}{m}(\tau-\nu)-\int_{\nu}^{T}\boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\right)\right]\mathrm{d}\nu\right],$$

where

$$(5.34)\qquad (D^{(3)}\tau) = \frac{\prod\limits_{\xi=0}^{T}\mathrm{d}^3\tau(\xi)}{\int\prod\limits_{\xi=0}^{T}\mathrm{d}^3\tau'(\xi)\exp\left[i\int_0^{T}(m\boldsymbol{\tau}'^2(\xi)/2)\,\mathrm{d}\xi\right]}\,.$$

By integration on $\mathrm{d}^3 p$ (5.31) becomes

$$G(\boldsymbol{x}, t; \boldsymbol{x}', t') = i\theta(T)\int (D^{(3)}\boldsymbol{\tau})\,\delta\Big(\boldsymbol{x} - \boldsymbol{x}' - \int_0^{\nu} \boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\Big)\cdot \tag{5.35}$$

$$\cdot \exp\left[i\int_0^{T} \mathrm{d}\nu \left[\frac{m\boldsymbol{\tau}^2(\nu)}{2} - U\Big(t' + \nu, \boldsymbol{x} - \int_{\nu}^{T} \boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\Big)\right]\right] =$$

$$= i\theta(T)\int (D^{(3)}\boldsymbol{\tau})\,\delta\Big(\boldsymbol{x} - \boldsymbol{x}' - \int_0^{T} \boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\Big)\cdot$$

$$\cdot \exp\left[i\int_0^{T}\left[\frac{m\boldsymbol{\tau}^2}{2} - U\Big(t' + \nu, \boldsymbol{x}' + \int_0^{\nu} \boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\Big)\right]\mathrm{d}\nu\right].$$

Equation (5.35) provides the functional solution. In analogy with the stationary phase method of the previous Section, we determine the extremum of the functional

$$S[\boldsymbol{\tau}] = \int_0^{T}\left[\frac{m\boldsymbol{\tau}^2(\nu)}{2} - U\Big(t' + \nu, \boldsymbol{x} - \int_{\nu}^{T} \boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\Big)\right]^2 \mathrm{d}\nu \tag{5.36}$$

with the condition

$$\boldsymbol{x} - \boldsymbol{x}' = \int_0^{T} \boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\,. \tag{5.37}$$

Let us consider the functional

$$S[\boldsymbol{\tau}, \boldsymbol{\lambda}] = S[\boldsymbol{\tau}] + \boldsymbol{\lambda}\int_0^{T} \boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\,. \tag{5.38}$$

The functional derivative calculated in the extremal point gives the condition

$$\frac{\delta S[\boldsymbol{\tau}, \boldsymbol{\lambda}]}{\delta \tau_\alpha(\eta)} = m\,\tau_\alpha(\eta) + \int_0^{T} \frac{\partial U\big(t' + \nu, \boldsymbol{x}(\nu)\big)}{\partial x_\alpha}\,\theta(\eta - \nu)\,\mathrm{d}\nu + \lambda_\alpha = 0\,, \tag{5.39}$$

where $\alpha = 1, 2, 3$,

$$\boldsymbol{x}(\nu) = \boldsymbol{x} - \int_{\nu}^{T} \boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\,. \tag{5.40}$$

The differentiation of (5.39) and (5.40) gives

$$\begin{cases} m\,\dfrac{\mathrm{d}^2 x_\alpha(\eta)}{\mathrm{d}\eta^2} = -\dfrac{\partial U\big(t' + \eta, \boldsymbol{x}(\eta')\big)}{\partial x_\alpha}\,; \\ x_\alpha(T) = x_\alpha, \qquad x_\alpha(0) = x'_\alpha, \end{cases} \qquad \alpha = 1, 2, 3. \tag{5.41}$$

The solution of system (5.41) is not unique. For definite conditions a solution of (5.41) as a function of $\boldsymbol{x}(0)$ with $\dot{\boldsymbol{x}}(0)$ can be given:

$$x_\alpha(\eta) = x_\alpha(\eta, \boldsymbol{x}(0), \dot{\boldsymbol{x}}(0)) . \tag{5.42}$$

Then, the number of extrema coincides with the number of real solutions of the system with respect to $\dot{\boldsymbol{x}}(0)$:

$$x_\alpha(T, \boldsymbol{x}', \dot{\boldsymbol{x}}(0)) = x_\alpha . \tag{5.43}$$

Taking $\dot{\boldsymbol{x}}^{(k)}(0)$ from (5.43) we obtain k extremum

$$\boldsymbol{x}^{(k)}(\eta) = \boldsymbol{x}(\eta, \boldsymbol{x}'; \dot{\boldsymbol{x}}^{(k)}(0)) , \tag{5.44}$$

so that

$$\boldsymbol{\tau}^{(k)}(\eta) = \frac{\mathrm{d}x^{(k)}(\eta)}{\mathrm{d}\eta} . \tag{5.45}$$

Let us develop the functional $S[\boldsymbol{\tau}]$ in a Taylor's functional series near the k-th extremal:

$$S[\boldsymbol{\tau}] = S[\boldsymbol{\tau}^{(k)}] + \frac{1}{2}\int_0^T \mathrm{d}\eta'\mathrm{d}\eta'' \frac{\delta^2 S[\boldsymbol{\tau}]}{\delta\tau_\alpha(\eta')\,\delta\tau_\beta(\eta'')}\bigg|_{\boldsymbol{\tau}=\boldsymbol{\tau}^{(k)}} \Delta\tau_\alpha(\eta')\Delta\tau_\beta(\eta'') + S_1[\Delta\boldsymbol{\tau}^{(k)}, \boldsymbol{\tau}^{(k)}], \tag{5.46}$$

where

$$\Delta\tau_\alpha(\eta) = \tau_\alpha(\eta) - \tau_\alpha^{(k)}(\eta) .$$

Here, S_1 indicates the higher-order terms. Substituting expansion (5.46) into (5.35) we obtain its asymptotic behaviour:

$$\begin{aligned}
&\sum_k \exp\left[iS[\boldsymbol{\tau}^{(k)}]\right]\int (D^3\tau)\,\delta\left(\int_0^T \boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\right)\cdot \\
&\cdot\exp\left[\frac{i}{2}\int_0^T \mathrm{d}\eta' \int_0^T \mathrm{d}\eta''\Omega_{\alpha\beta}^{(k)}(\eta', \eta'')\,\tau_\alpha(\eta')\,\tau_\beta(\eta'')\right]\sum_{n=0}^{\infty}\frac{1}{n!}\left(iS_1[\boldsymbol{\tau}; \boldsymbol{\tau}^{(k)}]\right)^n = \\
&= \sum_k \exp\left[iS[\boldsymbol{\tau}^{(k)}]\right]\sum_{n=0}^{\infty}\frac{1}{n!}\left(iS_1\left[-i\frac{\delta}{\delta \boldsymbol{J}}, \boldsymbol{\tau}^{(k)}\right]\right)^n \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\int (D^{(3)}\tau)\cdot \\
&\cdot\exp\left[\frac{i}{2}\int_0^T \mathrm{d}\eta'\,\mathrm{d}\eta''\Omega_{\alpha\beta}^{(k)}(\eta', \eta'')\,\tau_\alpha(\eta')\,\tau_\beta(\eta'') + i\int_0^T \mathrm{d}\eta\,\tau_\alpha(\eta)\left(p_\alpha + J_\alpha(\eta)\right)\right]\bigg|_{J=0},
\end{aligned} \tag{5.47}$$

where

$$\Omega^{(k)}_{\alpha\beta}(\eta', \eta'') = \frac{\delta^2 S[\tau]}{\delta\tau_\alpha(\eta')\,\delta\tau_\beta(\eta'')}\bigg|_{\tau=\tau^{(k)}}.$$

The Gaussian integral in (5.47) gives

$$(5.48)\quad I(p) = \int (D^{(3)}\tau)\exp\left[\frac{i}{2}\int_0^T d\eta'\,d\eta''\,\Omega^{(k)}_{\alpha\beta}(\eta', \eta'')\,\tau_\alpha(\eta')\,\tau_\beta(\eta'') + i\int_0^T d\eta\big(p_\alpha + J_\alpha(\eta)\big)\,\tau_\alpha(\eta)\right] = \left|\operatorname{Det}\left[\frac{1}{m}\,\Omega^{(k)}\right]\right|^{-\frac{1}{2}}\exp\left[-i\,\frac{\pi}{2}\,\gamma^{(k)}\right]\cdot \exp\left[-\frac{i}{2}\int_0^T d\eta'\,d\eta''\big(p_\alpha + J_\alpha(\eta')\big)\,G_{\alpha\beta}(\eta', \eta'')\big(p_\beta + J_\beta(\eta'')\big)\right],$$

where $\operatorname{Det}[(1/m)\,\Omega^{(k)}]$ is defined by Fredholm's tensorial operator integral of nucleus $(1/m)\,\Omega^{(k)}_{\alpha\beta}(\eta', \eta'')$; $G^{(k)}_{\alpha\beta}(\eta', \eta'')$ is its Green's function which satisfies the system of integral equations

$$(5.49)\qquad \int_0^T \Omega^{(k)}_{\alpha\gamma}(\eta, \xi)\,G^{(k)}_{\alpha\beta}(\xi, \eta') = \delta_{\alpha\beta}\,\delta(\eta - \eta').$$

If, with $\gamma^{(k)}$, we indicate the number of eigenvalues of the form

$$\int_0^T d\eta'\,d\eta''\,\Omega^{(k)}_{\alpha\beta}(\eta', \eta'')\,\tau_\alpha(\eta')\,\tau_\beta(\eta''),$$

by integration on d^3p in (5.47) we obtain

$$(5.50)\quad \begin{cases} \displaystyle\int \frac{d^3p}{(2\pi)^3}\,I(p) = \left|\operatorname{Det}\left[\frac{1}{m}\,\Omega^{(k)}\right]\right|^{-\frac{1}{2}}\exp\left[-i\,\frac{\pi}{2}\,\gamma^{(k)}\right]\cdot \\ \qquad\cdot\exp\left[-\frac{i}{2}\displaystyle\int_0^T d\eta'\,d\eta''\,\tau_\alpha(\eta')\,G^{(k)}_{\alpha\beta}(\eta', \eta'')\,J_\beta(\eta'')\right]\cdot \\ \qquad\cdot\displaystyle\int \frac{d^3p}{(2\pi)^3}\exp\left[-\frac{i}{2}\,p_\alpha p_\beta \int_0^T d\eta'\,d\eta''\,G^{(k)}_{\alpha\beta}(\eta', \eta'') - ip_\alpha \int_0^T d\eta'\,d\eta''\,G^{(k)}_{\alpha\beta}(\eta', \eta''),\right. \\ J_\beta(\eta'') = \left|\operatorname{Det}\left[\frac{1}{m}\,\Omega^{(k)}\right]\det_{(3)}[\Lambda]^{(k)}\right|^{-\frac{1}{2}}\exp\left[-i\,\frac{\pi}{2}\,\gamma^{(k)} - i\,\frac{\pi}{4}\,\delta^{(k)}_1\right](2\pi)^{-\frac{3}{2}}\cdot \\ \qquad\cdot\exp\left[-\frac{i}{2}\displaystyle\int_0^T d\eta'\,d\eta''\,J_\alpha(\eta')\,G^{(k)}_{\alpha\beta}(\eta', \eta'')\,J_\beta(\eta'') + \frac{i}{2}\,Q^{(k)}_\alpha\,(\Lambda^{(k)-1})_{\alpha\beta}\,Q^{(k)}_\beta\right], \end{cases}$$

where

$$(5.51)\qquad \begin{cases} \Lambda^{(k)}_{\alpha\beta} = \int\limits_0^T \mathrm{d}\eta'\,\mathrm{d}\eta''\, G^{(k)}_{\alpha\beta}(\eta', \eta'')\,, \\ Q^{(k)}_{\alpha} = \int\limits_0^T \mathrm{d}\eta'\,\mathrm{d}\eta''\, G^{(k)}_{\alpha\beta}(\eta', \eta'')\, J_\beta(\eta'')\,, \end{cases}$$

and $\delta_1^{(k)}$ indicates the sign of the form $\left(\sum\limits_{\alpha,\beta=1}^{3} P_\alpha P_\beta \Lambda^k_{\alpha\beta}\right)$. It is therefore necessary to determine the Green's function and Fredholm determinant of the integral operator in order to obtain the asymptotic part of the generating functional.

The generating functional π of the action has the form

$$(5.52)\qquad \frac{\delta^2 S}{\delta\tau_\alpha(\eta)\,\delta\tau_\beta(\eta')} = m\delta_{\alpha\beta}\delta(\eta-\eta') - \int\limits_0^T \mathrm{d}\nu\, \frac{\partial^2 U\big(t'+\nu, \boldsymbol{x}(\nu)\big)}{\partial x_\alpha\,\partial x_\beta}\,\theta(\eta-\nu)\theta(\eta'-\nu)\mathrm{d}\nu =$$

$$= m\delta_{\alpha\beta}\delta(\eta-\eta') - \theta(\eta-\eta')\int\limits_0^{\eta'} \mathrm{d}\nu\, \frac{\partial^2 U\big(t'+\nu, \boldsymbol{x}(\nu)\big)}{\partial x_\alpha\,\partial x_\beta} - \theta(\eta'-\eta)\int\limits_0^{\eta} \mathrm{d}\nu\, \frac{\partial^2 U\big(t'+\nu, \boldsymbol{x}(\nu)\big)}{\partial x_\alpha\,\partial x_\beta} =$$

$$= m\delta_{\alpha\beta}\delta(\eta-\eta') - \int\limits_0^{\min(\eta,\eta')} \frac{\partial^2 U\big(t'+\nu, \boldsymbol{x}(\nu)\big)}{\partial x_\alpha\,\partial x_\beta}\,\mathrm{d}\nu\,.$$

Calculating (5.52) in the k-th extremum, we obtain the following expression for (5.51):

$$(5.53)\qquad \begin{cases} mG_{\alpha\varrho}(\eta,\eta') - R_{\alpha\beta}(\eta)\int\limits_\eta^T G_{\beta\varrho}(\xi,\eta')\,\mathrm{d}\xi - \int\limits_0^\eta R_{\alpha\beta}(\xi)\,G_{\beta\varrho}(\xi,\eta')\,\mathrm{d}\xi = \delta_{\alpha\beta}\delta(\eta-\eta')\,, \\ R_{\alpha\beta}(\xi) = \int\limits_0^\xi \frac{\partial^2 U\big((t'+\nu), \boldsymbol{x}(\nu)\big)}{\partial x_\alpha\,\partial \beta_\beta}\,\mathrm{d}\nu\,. \end{cases}$$

Introducing the function

$$(5.54)\qquad \int\limits_\eta^T G_{\gamma\beta}(\xi,\eta')\,\mathrm{d}\xi = \Gamma_{\gamma\beta}(\eta,\eta')\,, \qquad G_{\gamma\beta}(\eta,\eta') = -\,\frac{\mathrm{d}\Gamma_{\gamma\beta}(\eta,\eta')}{\mathrm{d}\eta}\,,$$

for $\Gamma_{\alpha\beta}$ we easily obtain

$$(5.55)\qquad \begin{cases} m\,\dfrac{\mathrm{d}^2\Gamma_{\alpha\varrho}(\eta,\eta')}{\mathrm{d}\eta^2}+\dfrac{\partial^2 U(t'+\nu,\,\boldsymbol{x}(\eta))}{\partial x_\alpha\,\partial x_\beta}\,\Gamma_{\beta\varrho}(\eta,\eta')=-\,\delta_{\alpha\varrho}\,\delta'(\eta-\eta')\,,\\[2mm] \Gamma_{\alpha\varrho}(T,\eta')=0\,,\qquad \left.\dfrac{\mathrm{d}\Gamma_{\alpha\varrho}(\eta,\eta')}{\mathrm{d}\eta}\right|_{\eta=0}=-\dfrac{1}{m}\,\delta(\eta')\,.\end{cases}$$

The solution of (5.55) can be written

$$(5.56)\qquad \Gamma_{\alpha\varrho}(\eta,\eta')=\frac{1}{2m}\,[\Gamma^0_{\alpha\varrho}(\eta,\eta')+\operatorname{sign}(\eta'-\eta)\gamma_{\alpha\varrho}(\eta,\eta')]\,,$$

where Γ^0 and γ satisfy the relationships

$$(5.57)\qquad \begin{cases} \Gamma^0_{\alpha\beta}(\eta,\eta')|_{\eta=T}=\gamma_{\alpha\beta}(T,\eta')\,,\qquad \left.\dfrac{\mathrm{d}\Gamma^0_{\alpha\beta}}{\mathrm{d}\eta}\right|_{\eta=0}=-\left.\dfrac{\mathrm{d}\gamma_{\alpha\beta}(\eta,\eta')}{\mathrm{d}\eta}\right|_{\eta=0},\\[2mm] \gamma_{\alpha\beta}(\eta,\eta')|_{\eta=\eta'}=\delta_{\alpha\beta}\,,\qquad \left.\dfrac{\mathrm{d}\gamma_{\alpha\beta}(\eta,\eta')}{\mathrm{d}\eta}\right|_{\eta=\eta'}=0\,.\end{cases}$$

Equation (5.55) without the right-hand side becomes

$$(5.58)\qquad m\,\frac{\mathrm{d}^2x_\alpha(\eta)}{\mathrm{d}\eta^2}=-\,\frac{\partial U(t'+\eta,\,\boldsymbol{x}(\eta))}{\partial x_\alpha}\,,\qquad x_\alpha(0)=x^0_\alpha\,,\qquad \dot{x}_\alpha(0)=\dot{x}^{(0)}_\alpha\,,$$

so that the system of solutions is

$$\frac{\partial x_\alpha(\eta)}{\partial x_\beta(0)}\,,\qquad \frac{\partial x_\alpha(\eta)}{\partial \dot{x}_\beta(0)}\,.$$

In this way

$$(5.59)\qquad \gamma_{\alpha\beta}(\eta,\eta')\;=\frac{\partial x_\alpha(\eta)}{\partial x_\varrho}\,C^{(1)}_{\varrho\beta}(\eta')+\frac{\partial x_\alpha(\eta)}{\partial \dot{x}_\varrho(0)}\,C^{(2)}_{\varrho\beta}(\eta')\,,$$

$$(5.60)\qquad \Gamma^{(0)}_{\alpha\beta}(\eta,\eta')=\frac{\partial x_\alpha(\eta)}{\partial x_\varrho(0)}\,\overline{C}^{(1)}_{\varrho\beta}(\eta')+\frac{\partial x_\alpha(\eta)}{\partial \dot{x}_\varrho(0)}\,\overline{C}^{(2)}_{\varrho,\beta}(\eta')\,,$$

where $C^{(1,2)}_{\alpha,\beta}(\eta')$ and $\overline{C}^{(1,2)}_{\alpha,\beta}(\eta')$ satisfy the system

$$(5.61)\qquad \frac{\partial x_\alpha(\eta')}{\partial x_\varrho(0)}\,C^{(1)}_{\varrho,\beta}(\eta')+\frac{\partial x_\alpha(\eta')}{\partial \dot{x}_\varrho(0)}\,C^{(2)}_{\varrho\beta}(\eta')=\delta_{\alpha\beta}\,,$$

$$(5.62)\qquad \frac{\partial \dot{x}_\alpha(\eta')}{\partial x_\varrho(0)}\,C^{(1)}_{\varrho\beta}(\eta')+\frac{\partial \dot{x}_\alpha(\eta')}{\partial \dot{x}_\varrho(0)}\,C^{(2)}_{\varrho\beta}(\eta')=0\,,\qquad\qquad \overline{C}^{(2)}_{\alpha,\beta}=-\,C^{(2)}_{\alpha\beta},$$

$$(5.63)\qquad \frac{\partial x_\alpha(T)}{\partial x_\varrho(0)}\,\overline{C}^{(1)}_{\varrho\beta}(\eta')=\gamma_{\alpha\beta}(T,\eta')+\frac{\partial x_\alpha(T)}{\partial \dot{x}_\varrho(0)}\,C^{(2)}_{\varrho\beta}(\eta')\,.$$

Therefore

$$(5.64)\qquad G_{\alpha\beta}(\eta,\eta') = -\frac{1}{2m}\left(\frac{\partial\dot{x}_\alpha(\eta)}{\partial x_\varrho(0)}\,\bar{C}^{(1)}_{\varrho.\beta}(\eta') - \frac{\partial\dot{x}_\alpha(\eta)}{\partial_\varrho\dot{x}(0)}\,C^{(2)}_{\varrho\beta}(\eta)\right) + \\ + \frac{1}{m}\,\delta_{\alpha\beta}\,\delta(\eta-\eta') - \frac{1}{2m}\,\mathrm{sign}\,(\eta'-\eta)\left(\frac{\partial\dot{x}_\alpha(\eta)}{\partial x_\varrho(0)}\,C^{(1)}_{\varrho\beta}(\eta') + \frac{\partial\dot{x}_\alpha(\eta)}{\partial\dot{x}_\varrho(0)}\,C^{2}_{\varrho\beta}(\eta')\right).$$

Here we note that $\partial\dot{x}_\alpha(\eta)/\partial\dot{x}_\beta(0) \equiv \partial x_\alpha(\eta)/\partial x_\beta(0)$.

From (5.61)-(5.63) it follows that

$$(5.65)\qquad G(\eta,\eta') = \frac{1}{m}\left[\frac{\partial x(T)/\partial\dot{x}(0)}{\partial x(T)/\partial x(0)}\,\frac{\partial\dot{x}(\eta)}{\partial x(0)}\,\frac{\partial\dot{x}(\eta')}{\partial x(0)} - \theta(\eta-\eta')\,\frac{\partial\dot{x}(\eta')}{\partial x(0)}\,\frac{\partial x(\eta)}{\partial x(0)} - \right. \\ \left. - \theta(\eta'-\eta)\,\frac{\partial\dot{x}(\eta)}{\partial x(0)}\,\frac{\partial x(\eta')}{\partial x(0)} + \delta(\eta-\eta')\right].$$

From (5.62) it is possible to determine Fredholm's determinant of the generating functional:

$$(5.66)\qquad \mathrm{Det}\,(1+\lambda k) = \exp\,[\mathrm{Sp}\ln(1+\lambda k)] = \exp\left[\mathrm{Sp}\int\limits_0^\lambda \frac{\mathrm{d}\lambda' k}{1+\lambda' k}\right],$$

which gives the expression

$$(5.67)\quad \mathrm{Det}\left[\frac{1}{m}\,\Omega\right] = \exp\left[-\int\limits_0^1 \mathrm{d}\lambda \int\limits_0^T \mathrm{d}\eta' \int\limits_0^T \mathrm{d}\eta''\, G_{\alpha\beta}(\eta',\eta'';\lambda)\, m \int\limits_0^{\min(\eta',\eta'')} \frac{\partial^2 U\big(t'+\nu,\,\boldsymbol{x}(\nu)\big)}{\partial x_\alpha\,\partial x_\beta}\,\mathrm{d}\nu\right],$$

where $G_{\alpha\beta}(\eta,\eta';\lambda)$ is the solution of eq. (5.53) with the substitution $U\to\lambda U$. Equation (5.58) is written

$$(5.68)\qquad \frac{1}{\lambda}\,m\,\frac{\mathrm{d}^2 x_\alpha(\eta)}{\mathrm{d}\eta^2} = -\frac{\partial U\big(t'+\eta;\,\boldsymbol{x}(\eta)\big)}{\partial x_\alpha}$$

with $x_\alpha(\eta) = x_\alpha\big(\lambda;\eta;\boldsymbol{x}(0);\boldsymbol{x}(0)\big)$, and by integrating (5.67) we obtain, as a final result,

$$(5.69)\qquad \mathrm{Det}\left[\frac{1}{m}\,\Omega\right] = \det_{(3)}\left\{\frac{\partial x(T)}{\partial x(0)}\right\} \equiv \frac{D\big(\boldsymbol{x}(T)\big)}{D\big(\boldsymbol{x}(0)\big)}\,.$$

In other cases too, we can demonstrate that the integral operator with the nucleus [25]

$$(5.70)\qquad \delta_{ij}\,\delta(\eta-\eta') + \int\limits_0^{\min(\eta,\eta')} Q_{ij}(\xi)\,\mathrm{d}\xi\,,\qquad i,j=1,2,\ldots,n\,,\ 0\leqslant\eta,\eta'\leqslant T\,,\ Q_{ij}=Q_{ji}\,,$$

has Fredholm's determinant

$$D = \det_{(n)}[\varphi(T)]\,, \tag{5.71}$$

where $\varphi_{ij}(\eta)$ is the solution of the system

$$\frac{d^2}{d\eta^2}\varphi_{ij}(\eta) - Q_{ik}(\eta)\varphi_{kj}(\eta) = 0 \tag{5.72}$$

with the initial conditions

$$\varphi_{ij}(0) = \delta_{ij}\,, \qquad \varphi'_{ij}(0) = 0\,. \tag{5.73}$$

For $N = 3$ we obtain once more (5.69).

Let us calculate

$$\Lambda_{\alpha\beta} = \int_0^T d\eta' \int d\eta''\, G_{\alpha\beta}(\eta', \eta'')\,. \tag{5.74}$$

It is possible to calculate $\Lambda_{\alpha\beta}$ using the explicit expression of $G_{\alpha\beta}(\eta', \eta'')$, but it is easier to obtain it directly from the equation. From (5.53) we obtain, by integrating,

$$m\int_0^T d\eta'\, G_{\alpha\varrho}(\eta, \eta') - R_{\alpha\beta}(\eta)\int_\eta^T d\xi \int_0^T d\eta'\, G_{\beta\varrho}(\xi, \eta') - \int_0^\eta d\xi\, R_{\alpha\beta}(\xi)\int_0^T d\eta'\, G_{\beta\varrho}(\xi, \eta') = \delta_{\alpha\varrho}\,.$$

So, if $\Gamma_{\alpha\varrho}(\eta) = \int_\eta^T d\xi \int_0^T d\eta'\, G_{\alpha\varrho}(\xi, \eta')$,

$$m\frac{d^2\Gamma_{\alpha\varrho}(\eta)}{d\eta^2} + \frac{\partial^2 U(t' + \eta, \boldsymbol{x}(\eta))}{\partial x_\alpha \partial x_\beta}\Gamma_{\beta\varrho}(\eta) = 0\,, \qquad \Gamma_{\alpha\varrho}(T) = 0\,, \tag{5.75}$$

$$\left.\frac{d\Gamma_{\alpha\varrho}(\eta)}{d\eta}\right|_{\eta=0} = -\frac{1}{m}\,\delta_{\alpha\varrho}\,, \tag{5.76}$$

the solution has the form

$$\Gamma_{\alpha\beta}(\eta) = \frac{\partial x_\alpha(\eta)}{\partial x_\varrho(0)}\, C^{(1)}_{\varrho\beta} + \frac{\partial x_\alpha(\eta)}{\partial \dot{x}_\varrho(0)}\, C^{(2)}_{\varrho\beta}\,. \tag{5.77}$$

As

$$C^{(2)}_{\alpha\beta} = -\frac{1}{m}\,\delta_{\alpha\beta}\,, \tag{5.78}$$

we have

$$\frac{\partial x_\alpha(T)}{\partial x_\varrho(0)}\, \tilde{C}^{(1)}_{\varrho\beta} = \frac{1}{m}\,\frac{\partial x_\alpha(T)}{\partial \dot{x}_\beta(0)}\,. \tag{5.79}$$

In this way,

$$\Gamma_{\alpha\beta}(\eta) = \frac{\partial x_\alpha(\eta)}{\partial x_\varrho(0)}\, \tilde{C}^{(1)}_{\varrho\beta} - \frac{1}{m}\,\frac{\partial x_\alpha(\eta)}{\partial \dot{x}_\beta(0)}\,, \tag{5.80}$$

where

$$\int_0^T \mathrm{d}\eta\,\mathrm{d}\eta'\, G_{\alpha\beta}(\eta,\eta') = \Gamma_{\alpha\beta}(0) = \tilde{C}^{(1)}_{\alpha\beta}\,. \tag{5.81}$$

Then, for (5.79), we obtain

$$\frac{\partial x_\alpha(T)}{\partial x_\varrho(0)}\, \Lambda_{\varrho\beta} = \frac{1}{m}\,\frac{\partial x_\alpha(T)}{\partial \dot{x}_\beta(0)} \tag{5.82}$$

and again we obtain

$$\det{}_{(3)}\Lambda = \frac{1}{m^3}\,\frac{\det_{(3)}[\partial x(T)/\partial \dot{x}(0)]}{\det_{(3)}[\partial x(T)/\partial x(0)]}\,. \tag{5.83}$$

From (5.83) and (5.69) we have

$$\operatorname{Det}\left[\frac{1}{m}\,\Omega\right]\det{}_{(3)}\Lambda = \frac{1}{m^3}\det{}_{(3)}\left[\frac{\partial x(T)}{\partial \dot{x}(0)}\right]. \tag{5.84}$$

Substituting (5.84) and (5.65) in the expression for $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ we obtain

$$\begin{aligned} G(\boldsymbol{x}, t; \boldsymbol{x}', t') \simeq\; & i\theta(t-t')\left(\frac{2\pi\hbar}{m}\right)^{-\frac{3}{2}} \sum_k \exp\left[(i/\hbar)\, S[\boldsymbol{\tau}^{(k)}]\right] \left|\det{}_{(3)}\left[\frac{\partial x^{(k)}(T)}{\partial \dot{x}^{(k)}(0)}\right]\right|^{-\frac{1}{2}} \cdot \\ & \cdot \exp\left[-\frac{3}{4}\pi i - \frac{i\pi}{2}\,(\gamma^{(k)} - \gamma_1^{(k)})\right] \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{\hbar}\, S_1\left[-i\hbar^{\frac{1}{2}}\,\frac{\delta}{\delta \boldsymbol{J}},\, \boldsymbol{\tau}^{(k)}\right]\right)^n \cdot \\ & \cdot \exp\left[-\frac{i}{2}\int_0^T \mathrm{d}\eta'\,\mathrm{d}\eta''\, J_\alpha(\eta')\, G^{(k)}_{\alpha\beta}(\eta',\eta'')\, J_\beta(\eta'') + \frac{i}{2}\, Q^{(k)}_\alpha (\Lambda^{(k)-1})\, Q^{(k)}_\beta\right]\Bigg|_{J=0}, \end{aligned} \tag{5.85}$$

where $\gamma^{(k)}$ is the number of zeros of $\det_{(3)}[\partial x(\eta)/\partial x(0)]$ in the k-th extremum for $0 \leqslant \eta \leqslant T$; $\gamma_1^{(k)}$ is the number of negative eigenvalues of the matrix $\Lambda^{(k)}$.

We can demonstrate that $\gamma^{(k)}-\gamma_1^{(k)}=\gamma_2^{(k)}$, where $\gamma_2^{(k)}$ is the number of zeros of $\det_{(3)}[\partial x(\eta)/\partial x(0)]$ in the k-th extremum for $0\leqslant\eta\leqslant T$;

$$(5.86)\qquad S_1\left[-i\hbar^{\frac{1}{2}}\frac{\delta}{\delta\boldsymbol{J}},\boldsymbol{\tau}^{(k)}\right]=\sum_{n=3}^{\infty}\frac{1}{n!}\int_0^T\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n\frac{\delta^n S[\boldsymbol{\tau}]}{\delta\tau_{\alpha_1}(\eta_1)\ldots\delta\tau_{\alpha_n}(\eta_n)}\bigg|_{\boldsymbol{\tau}=\boldsymbol{\tau}^{(k)}}\cdot(-i\hbar)^{n/2}\frac{\delta}{\delta J_{\alpha_1}(\eta_1)}\cdots\frac{\delta}{\delta J_{\alpha_n}(\eta_n)}.$$

Equation (5.85) allows us to obtain the asymptotic behaviour, the most important term of which, in the classical limit, is

$$(5.87)\qquad G(\boldsymbol{x},t;\boldsymbol{x}',t')\simeq i\theta(t-t')\left(\frac{2\pi\hbar}{m}\right)^{-\frac{3}{2}}\exp\left[-\frac{3}{4}\pi i\right]\sum_k\exp\left[(i/\hbar)S[\boldsymbol{\tau}^{(k)}]\right]\cdot\exp\left[-i\frac{\pi}{2}\gamma_2^{(k)}\right]\left|\det\left[\frac{\partial x^{(k)}(T)}{\partial\dot{x}^{(k)}(0)}\right]\right|^{-\frac{1}{2}}.$$

This becomes

$$(5.88)\qquad G(\boldsymbol{x},t;\boldsymbol{x}',t')\quad i\theta(t-t')\left(\frac{2\pi\hbar}{m}\right)^{-\frac{3}{2}}\exp\left[-\frac{3\pi i}{4}\right]\sum_k\exp\left[\frac{i}{\hbar}S[\boldsymbol{\tau}^{(k)}]\right]\cdot$$
$$\cdot\exp\left[-i\frac{\pi}{2}\gamma^{(k)}\right]\left|\det{}_{(3)}\left[\frac{\partial x^{(k)}(T)}{\partial\dot{x}^{(k)}(0)}\right]\right|^{-\frac{1}{2}}\left[1+\frac{i\hbar}{4}\int_0^T\left(\prod_{i=1}^4\mathrm{d}\eta_i\frac{\delta^i}{\delta\tau_{\alpha_i}(\eta_i)}\right)S[\boldsymbol{\tau}]\bigg|_{\boldsymbol{\tau}=\boldsymbol{\tau}^{(k)}}\prod_{k=1}^4\frac{\delta^{k'}}{\delta J_{\alpha_{k'}}(\eta_{k'})}\right]\cdot$$
$$\cdot\exp\left[-\frac{i}{2}\int_0^T\mathrm{d}\eta'\,\mathrm{d}\eta''J_\alpha(\eta')\,G^{(k)}_{\alpha\beta}(\eta',\eta'')\,J_\beta(\eta'')\right]+\frac{i}{2}\,Q^{(k)}_\alpha(\Lambda^{(k)})^{-1}_{\alpha\beta}Q^{(k)}_\beta\bigg|_{J=0},$$

whilst

$$(5.89)\qquad\left(\prod_{i=1}^4\frac{\delta^i}{\delta\tau_{\alpha_i}(\eta_i)}\right)S[\boldsymbol{\tau}]=-\int_0^T\left(\prod_{i=1}^4\theta(\eta_i-\nu)\frac{\partial}{\partial x_{\alpha_i}}\right)U(t'+\nu,\boldsymbol{x}(\nu))\,\mathrm{d}\nu.$$

So far, we have obtained results regarding the asymptotic behaviour of the Green's function of Schrödinger's equation in $\boldsymbol{x}$. Nevertheless, these results can easily be generalized in other situations, too. In particular, we shall show the asymptotic behaviour in the $(\boldsymbol{p},\boldsymbol{x})$ representation. In this case, the Green's function is obtained as a Fourier transform in the difference $(\boldsymbol{x}-\boldsymbol{x}')$ of the function $G(\boldsymbol{p},\boldsymbol{x},t,t')$ in $(\boldsymbol{p},\boldsymbol{x})$:

$$(5.90)\qquad G(\boldsymbol{x},t;\boldsymbol{x}',t')=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\exp\left[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')\right]G(\boldsymbol{p},\boldsymbol{x},t,t'),$$

where $G(\boldsymbol{p}, \boldsymbol{x}, t, t')$ satisfies the equation

$$\text{(5.91)}\qquad \left\{\begin{aligned} &LG(\boldsymbol{p}, \boldsymbol{x}, t, t') = \delta(t, t')\,,\\ &L = -i\frac{\partial}{\partial t} - \frac{\boldsymbol{\nabla}^2}{2m} - i\frac{\boldsymbol{p}\boldsymbol{\nabla}}{m} + \frac{\boldsymbol{p}^2}{2m} + U(\boldsymbol{x}, t)\,.\end{aligned}\right.$$

The solution of this equation is obtained from the formulae (5.31) and (5.32):

$$\text{(5.92)}\qquad \left\{\begin{aligned} &G(\boldsymbol{p}, \boldsymbol{x}, t, t') = i\theta(t-t')\exp\left[-i\frac{\boldsymbol{p}^2}{2m}(t-t')\right]\cdot\\ &\qquad\qquad\cdot\int(D^3\boldsymbol{\tau})\exp\left[i\int_0^T\left[\frac{m\boldsymbol{\tau}^2(\nu)}{2} - U\big(t'+\nu, \boldsymbol{x}(\nu)\big)\right]\mathrm{d}\nu\right],\\ &\boldsymbol{x}(\nu) = \boldsymbol{x} - \frac{\boldsymbol{p}}{m}(T-\nu) - \int_\nu^T\boldsymbol{\tau}(\xi)\,\mathrm{d}\xi\,.\end{aligned}\right.$$

The calculation of the second-member integral of (5.90) is done in a way similar to the proceding one; the only differences being those regarding the approach to the problem of the extremal trajectory and the fact that in (5.92) there is not the analogue of the integral (5.47). The extremal problem consists in finding the extremum of the action in the exponent of the integral (5.92):

$$\text{(5.93)}\qquad m\frac{\mathrm{d}^2x_\alpha(\eta)}{\mathrm{d}\eta^2} = -\frac{\partial U\big(t'+\eta, \boldsymbol{x}(\eta)\big)}{\partial x_\alpha}\,,$$

$$\text{(5.94)}\qquad x_\alpha(T) = x_\alpha\,,$$

$$\text{(5.95)}\qquad \dot{x}_\alpha(0) = \frac{p_\alpha}{m}\,,$$

$$\text{(5.96)}\qquad \tau_\alpha(\eta) = \dot{x}_\alpha(\eta) - \frac{p_\alpha}{m}\,.$$

From the extremals $\tau_\alpha^{(k)}(\eta)$ given by (5.93)-(5.96) the Green's function in the representation (p, x) can be written thus:

$$\text{(5.97)}\qquad G(\boldsymbol{p}, \boldsymbol{x}, t, t') \simeq i\theta(t-t')\exp\left[-i\frac{\boldsymbol{p}^2}{2m}(t-t')\right]\cdot$$
$$\cdot\sum_k\exp\left[\frac{i}{\hbar}S[\boldsymbol{\tau}^{(k)}] - i\frac{\pi}{2}\gamma^{(k)}\right]\left|\det_{(3)}\left[\frac{\partial x^{(k)}(T)}{\partial x^{(k)}(0)}\right]\right|^{-\frac{1}{2}}\cdot$$
$$\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}S_1\left(-i\hbar^{\frac{1}{2}}\frac{\delta}{\delta\boldsymbol{J}}, \boldsymbol{\tau}^{(k)}\right)\exp\left[-\frac{i}{2}\int_0^T\mathrm{d}\eta'\,\mathrm{d}\eta''J_\alpha(\eta')\,G_{\alpha\beta}^{(k)}(\eta', \eta'')\,J_\beta(\eta'')\right]\right)\Bigg|_{J=0},$$

where S is the action in the exponent of the integral (5.92), S_1 is expressed through S by means of the formula (5.86), $G^{(k)}_{\alpha\beta}(\eta', \eta'')$ is defined by means of the formulae (5.62)-(5.64) but in the extremals (5.93)-(5.95); $\gamma^{(k)}$ is the number of zeros of $\det_{(3)}[\partial x(\eta)/\partial x(0)]$ for $0 \leqslant \eta \leqslant T$ in the k-th extremal (5.93)-(5.96). The problem (5.93)-(5.96) is solved in the following way.

Let us find the solution of eq. (5.93) as a function of $\boldsymbol{x}(0)$ and $\dot{\boldsymbol{x}}(0)$:

$$x_\alpha(\eta) = x_\alpha(\eta, \boldsymbol{x}(0), \dot{\boldsymbol{x}}(0)) . \tag{5.98}$$

Then, we find $\boldsymbol{x}(0)$ as the real solution of the system:

$$x_\alpha\left(T, \boldsymbol{x}(0), \frac{\boldsymbol{p}}{m}\right) = x_\alpha . \tag{5.99}$$

The asymptotic solution of Schrödinger's equation

$$i\hbar \frac{\partial \Psi(\boldsymbol{x}, t)}{\partial t} = \left\{-\frac{\hbar^2}{2m}\nabla^2 + U(\boldsymbol{x}, t)\right\} \Psi(\boldsymbol{x}, t) \tag{5.100}$$

with the initial conditions

$$\Psi(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}) \exp\left[\frac{i}{\hbar} S_0(\boldsymbol{x})\right] \tag{5.101}$$

is written

$$\Psi(\boldsymbol{x}, t) \simeq \sum_k \exp\left[\frac{i}{\hbar} S[\boldsymbol{x}^{(k)}] - i\frac{\pi}{2}\gamma^{(k)}\right] \varphi(x^{(k)}(0)) \left|\det_{(3)}\left[\frac{\partial x^{(k)}(T)}{\partial x^{(k)}(0)}\right]\right|^{-\frac{1}{2}}, \tag{5.102}$$

where $S[\boldsymbol{x}^{(k)}]$ is the value of the functional

$$S[\boldsymbol{x}] = S_0(\boldsymbol{x}(0)) + \int_0^t d\eta \left[\frac{m\dot{x}^2(\eta)}{2} - U(\boldsymbol{x}(\eta), \eta)\right] \tag{5.103}$$

in the k-th extremum, determined by the equations

$$\begin{cases} m\ddot{x}_\alpha(\eta) = -\dfrac{\partial U(\boldsymbol{x}(\eta), \eta)}{\partial x_\alpha}, \\ m\dot{x}_\alpha(0) = -\dfrac{\partial S_0(\boldsymbol{x}(0))}{\partial x_\alpha(0)}, \end{cases} \qquad x_\alpha(t) = x_\alpha . \tag{5.104}$$

5·2. *Scalar electrodynamics.* – Let us consider now the relativistic equation of scalar-particle electrodynamics

$$\{-(\partial_\mu - ieA_\mu(x))^2 + m^2\} G(x, y) = \delta(x - y) , \tag{5.105}$$

where $\hbar = c = 1$.

Let us write the solution of (5.105) in the form

$$G(x, y) = i\int_0^\infty d\nu \exp[-\nu(m^2 - i\varepsilon)]\int (D^4 t)\,\delta\Big(x - y - 2\int_0^\nu t(\xi)\,d\xi\Big)\exp[iS[t]], \tag{5.106}$$

where

$$S[t] = \int_0^\nu d\nu'\Big[t_\mu^2(\nu') + 2et_\mu(\nu')A_\mu\Big(x - 2\int_{\nu'}^\nu t(\xi)\,d\xi\Big)\Big], \tag{5.107}$$

$$(D^4 t) = \frac{\prod\limits_{\xi=0}^{\nu} d^4 t(\xi)}{\int \prod\limits_{\xi=0}^{\nu} d^4 t'(\xi)\exp\Big[i\int_0^\nu t'^2(\xi)\,d\xi\Big]}. \tag{5.108}$$

The extremals of (5.107) are determined by

$$\frac{d^2 x_\mu(\eta)}{d\eta^2} - 2eF_{\mu\varrho}(x(\eta))\frac{dx_\varrho(\eta)}{d\eta} = 0, \tag{5.109}$$

$$x_\mu(\nu) = x_\mu, \tag{5.110}$$

$$x_\mu(0) = y_\mu, \tag{5.111}$$

$$x_\mu(\eta) = x_\mu - 2\int_\eta^\nu t_\mu(\xi)\,d\xi. \tag{5.112}$$

The functional derivative S is

$$\frac{\delta S}{\delta t_\mu(\xi)} = 2t_\mu(\xi) + 2eA_\mu(\xi) - 4e\int_0^\nu t_\varrho(\nu')\frac{\partial A_\varrho}{\partial x_\mu}(x(\nu'))\,\theta(\xi - \nu')\,d\nu'. \tag{5.113}$$

From (5.113) we obtain the generating functional

$$\begin{aligned}\frac{\delta^2 S}{\delta t_\mu(\eta)\,\delta t_\chi(\zeta)} = {} & 2\delta_{\mu\chi}\delta(\eta - \zeta) - 4e\frac{\partial A_\mu(x(\eta))}{\partial x_\chi}\theta(\zeta - \eta) - \\ & - 4e\frac{\partial A_\chi(x(\zeta))}{\partial x_\mu}\theta(\eta - \zeta) + 4e\theta(\zeta - \eta)\int_0^\eta d\nu'\frac{dx_\varrho(\nu')}{d\nu'}\frac{\partial^2 A_\varrho(x(\nu'))}{\partial x_\mu \partial x_\chi} + \\ & + 4e\theta(\eta - \zeta)\int_0^\zeta d\nu'\frac{dx_\varrho(\nu')}{d\nu'}\frac{\partial^2 A_\varrho(x(\nu'))}{\partial x_\mu \partial x_\chi}.\end{aligned} \tag{5.114}$$

The Green's function of the integral operator with nucleus (5.114) satisfies the system of equations

$$\int_{\eta}^{\nu} G_{\chi\nu}(\xi,\eta')\,\mathrm{d}\zeta = \Gamma_{\chi\nu}(\eta,\eta')\,, \tag{5.115}$$

$$\frac{\mathrm{d}^2\Gamma_{\mu\nu}(\eta,\eta')}{\mathrm{d}\eta^2} - 2eF_{\mu\chi}\big(x(\eta)\big)\frac{\mathrm{d}\Gamma_{\chi\nu}(\eta,\eta')}{\mathrm{d}\eta} - 2e\,\frac{\partial F_{\mu\varrho}\big(x(\eta)\big)}{\partial x_\chi}\,\frac{\mathrm{d}x_\varrho(\eta)}{\mathrm{d}\eta}\,\Gamma_{\chi\nu}(\eta,\eta') = = -\frac{1}{2}\,\delta_{\mu\nu}\delta'(\eta-\eta')\,, \tag{5.116}$$

$$\begin{cases} \Gamma_{\mu\varrho}(\nu,\eta') = 0\,, \\ \left(\dfrac{\mathrm{d}\Gamma_{\mu\nu}(\eta,\eta')}{\mathrm{d}\eta} + 2e\,\dfrac{\partial A_\mu\big(x(\eta)\big)}{\partial x_\chi}\,\Gamma_{\chi\nu}(\eta,\eta')\right)\Bigg|_{\eta=0} = -\dfrac{1}{2}\,\delta_{\mu\nu}\,\delta(\eta'). \end{cases} \tag{5.117}$$

The solutions of these equations in the k-th extremal give (the index of the extremum being omitted)

$$G_{\alpha\beta}(\eta,\eta') = \frac{1}{2}\,\delta_{\alpha\beta}\,\delta(\eta-\eta') - \frac{1}{4}\,\mathrm{sign}\,(\eta'-\eta)\cdot \cdot\left[\frac{\partial\dot{x}_\alpha(\eta)}{\partial x_\varrho(0)}\,C^{(1)}_{\varrho\beta}(\eta') + \frac{\partial\dot{x}_\alpha(\eta)}{\partial\dot{x}_\varrho(0)}\,C^{(2)}_{\varrho\beta}(\eta')\right] - \frac{1}{4}\left[\frac{\partial x_\alpha(\eta)}{\partial x_\varrho(0)}\,\bar{C}^{(1)}_{\varrho\beta}(\eta') + \frac{\partial\dot{x}_\alpha(\eta)}{\partial\dot{x}_\varrho(0)}\,C^{(2)}_{\varrho\beta}(\eta')\right], \tag{5.118}$$

$$\frac{\partial x_\alpha(\eta')}{\partial x_\varrho(0)}\,C^{(1)}_{\varrho\beta}(\eta') + \frac{\partial x_\alpha(\eta')}{\partial\dot{x}_\varrho(0)}\,C^{(2)}_{\varrho\beta}(\eta') = \delta_{\alpha\beta}\,, \tag{5.119}$$

$$\frac{\partial\dot{x}_\alpha(\eta')}{\partial x_\varrho(0)}\,C^{(1)}_{\varrho\beta}(\eta') + \frac{\partial\dot{x}_\alpha(\eta')}{\partial\dot{x}_\varrho(0)}\,C^{(2)}_{\varrho\beta}(\eta') = 2eF_{\alpha\beta}\big(x(\eta')\big)\,, \tag{5.120}$$

$$\bar{C}^{(2)}_{\alpha\beta}(\eta') + C^{(2)}_{\alpha\beta}(\eta') + 2e\,\frac{\partial A_\alpha\big(x(0)\big)}{\partial x_\chi(0)}\,\big(\bar{C}^{(1)}_{\chi\beta}(\eta') + C^{(1)}_{\chi\beta}(\eta')\big) = 0\,, \tag{5.121}$$

$$\frac{\partial x_\alpha(\nu)}{\partial x_\varrho(0)}\,\big(\bar{C}^{(1)}_{\varrho\beta}(\eta') - C^{(1)}_{\varrho\beta}(\eta')\big) + \frac{\partial x_\alpha(\nu)}{\partial\dot{x}_\varrho(0)}\,\big(C^{(2)}_{\varrho\beta}(\eta') - C^{(2)}_{\varrho\beta}(\eta')\big) = 0\,. \tag{5.122}$$

Using these results we easily obtain

$$G(x,y) \simeq \int_0^\infty \frac{\mathrm{d}\nu}{(4\pi)^2}\,\exp\left[-i\nu(m^2-i\varepsilon)\right]\sum_k \exp\left[iS[\boldsymbol{\tau}^{(k)}]\right]\cdot \cdot\left(\det\left[\frac{\partial x^{(k)}(\nu)}{\partial\dot{x}^{(k)}(0)}\right]\right)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(iS_1\left[\frac{\delta}{\delta J}(-i)\,t^{(k)}\right]\right)^n\cdot \cdot\exp\left[-\frac{i}{2}\int_0^\nu \mathrm{d}\eta'\,\mathrm{d}\eta''\,J_\alpha(\eta')\,G^{(k)}_{\alpha\beta}(\eta',\eta'')\,J_\beta(\eta'') + \frac{i}{2}\,Q^{(k)}_\alpha(\Lambda^{(k)})^{-1}Q^{(k)}_\beta\right]\Bigg|_{J=0}, \tag{5.123}$$

where $S[t^{(k)}]$ is the action (5.107) in the k-th extremal,

$$Q_\alpha^{(k)} = \int_0^\nu d\eta' \, d\eta'' \, G_{\alpha\beta}^{(k)}(\eta, \eta'') J_\beta(\eta') . \tag{5.124}$$

From (5.107) we can determine S_1 and we can obtain an expression similar to (5.86) where $\alpha_1, \dots, \alpha_n = 1, 2, 3, 4$; the matrix $\Lambda^{(k)}$ satisfies the system (5.125) which is calculated in the k-th extremum

$$\left[\frac{\partial x_\alpha(\nu)}{\partial x_\varrho(0)} - 2e \frac{\partial x_\alpha(\nu)}{\partial \dot{x}_\chi(0)} \frac{\partial A_\chi(x(0))}{\partial x_\varrho(0)}\right] \Lambda_{\varrho\beta}^{(k)} = \frac{1}{2} \frac{\partial x_\alpha(\nu)}{\partial \dot{x}_\beta(0)} . \tag{5.125}$$

6. – Construction of the generating functional for Yang-Mills and gravitational fields [26].

6'1. *Yang-Mills' field.* – Recently many authors have investigated Yang-Mills' field in detail. In ref. [28-30] Feynman's rules for massless Yang-Mills' fields have been suggested and in ref. [31-33] the renormalizability for Yang-Mills' fields with mass is discussed. The main difficulties which are met in field theories with gauge concern the description of fictitious longitudinal fields and the unitarity of the S-matrix in physical subspaces.

We shall write the Lagrangian of Yang-Mill's field, in an arbitrary covariant gauge, in such a way that the fictitious particle is free as a consequence of the dynamic equations, and, therefore, the S-matrix is unitary and gauge independent. Feynman's rules, obtained for massless fields, coincide with those of ref. [28-30], and, for massive fields, correspond to a renormalizable theory. The S-matrix, with Feynman's new rules, coincides with the S-matrix obtained in the usual formulation. It is well known that a term $B\partial_\mu A_\mu$ (where B is a fictitious scalar particle) must be added to the Lagrangian so that an electromagnetic-field theory in a transversal gauge can be formulated. The B particle is free and the S-matrix is unitary. In Yang-Mills' theory the direct use of this method fails. Thus if

$$L = L_0 + \partial_\mu A_\mu^a B^a , \tag{6.1}$$

where

$$L_0 = -\tfrac{1}{4} G_{\mu\nu}^a \cdot G_{\mu\nu}^a , \qquad G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \lambda \cdot f^{abc} \cdot A_\mu^b A_\nu^c , \tag{6.2}$$

the following equations for the field B are obtained:

$$\nabla_\mu^{ab} \partial_\mu B^b = 0 , \qquad \nabla_\mu^{ab} = \delta_{ab} \partial_\mu + \lambda f^{acb} A_\mu^c . \tag{6.3}$$

The B particle is not a free one. For this reason Feynman's ordinary rules give a nonunitarity S-matrix.

The nonunitary S-matrix is produced by the generating functional

$$Z = \int \mathrm{d}A\,\delta(\partial_\mu A_\mu) \exp\left[i\int (L + A_\mu J_\mu)\,\mathrm{d}x\right]. \tag{6.4}$$

In order to solve the problem, let us introduce a Lagrangian multiplier in the following way (arbitrary gauge):

$$L = L_0 + \partial_\mu A_\mu^a \cdot B_1^a + \frac{\alpha}{2} B_1^a \cdot B_1^a , \tag{6.5}$$

where

$$B_1^{(a)}(x) = \int \mathrm{d}y D_1^{ab}(x, y; A) B^b(y) , \tag{6.6}$$

where D_1^{ab} is fixed by the request that B should be a free field. By variation of the Lagrangian as a consequence of dynamic equations, we obtain

$$\partial_\mu A_\mu^a = -\alpha B_1^a , \tag{6.7}$$

$$\nabla_\mu^{ab} \partial_\mu B_1^b = 0 . \tag{6.8}$$

Then D_1 must be taken in the form

$$D_1^{ab} = D_{ab} \Box , \qquad \nabla_\mu^{ab} \partial_\mu D^{bc}(x, y; A) = \delta_{ac}\delta(x-y) . \tag{6.9}$$

The generating functional obtained from eqs. (6.5), (6.6), (6.9) is

$$\begin{aligned} Z_\alpha &= \int \mathrm{d}A\,\mathrm{d}B \exp\left[i\int\left(L_0 + \partial_\mu A_\mu \cdot B_1 + \frac{\alpha}{2} B_1 B_1 + A_\mu J_\mu\right)\mathrm{d}x\right] = \\ &= \int \mathrm{d}A \exp\left[i\int\left(L_0 - \frac{1}{2\alpha}\partial_\mu A_\mu \cdot \partial_\nu A_\nu + A_\mu J_\mu\right)\mathrm{d}x + \mathrm{Sp}\ln \nabla_\mu \partial_\mu\right]. \end{aligned} \tag{6.10}$$

In the case of the transversal gauge ($\alpha = 0$) we have (see also [28])

$$Z_0 = \int \mathrm{d}A\,\delta(\partial_\mu A_\mu) \exp\left[i\int (L_0 + A_\mu J_\mu) + \mathrm{Sp}\ln \nabla_\mu \partial_\mu\right]. \tag{6.11}$$

Feynman's rules for the perturbative calculation of the generating functional (6.11) coincide with the rules suggested in ref. [28-30]. It is possible to show that a gauge transformation exists which changes the Lagrangian (6.4) and the generating functional, on the mass shell (S-matrix), with different gauges, into each other. This proves the independence of the S-matrix on the gauge α.

For an infinitesimal variation $\delta\alpha$ the transformation has the form (see also [29])

$$A_\mu^a \to A_\mu^a - \frac{\delta\alpha}{2\alpha} \nabla_\mu^{ab} D^{bc} \partial_\nu A_\nu . \tag{6.12}$$

The corresponding Lagrangian in the Coulomb gauge becomes

$$L = L_0 + \partial_i A_i^a(x) \int d^3y D_2^{ab}(x, y; A) \Delta B^b(y) , \qquad x_0 = y_0 , \tag{6.4a}$$

where D_2 is determined by the condition

$$\nabla_i^{ab} \partial_i D_2^{bc}(x, y; A) = \delta_{ac} \delta^3(x-y) , \qquad x_0 = y_0 . \tag{6.9a}$$

The generating functional has the form

$$Z = \int dA(\partial_i A_i) \exp\left[i\int (L_0 + A_\mu J_\mu) dx + \mathrm{Sp} \ln \nabla_i \partial_i\right] . \tag{6.11a}$$

It is possible to show [36] that the canonical quantization procedure applied to the Lagrangian (6.4a) leads to the generating functional (and to the S-matrix). Moreover, we can demonstrate that the S-matrices in the Coulomb and transversal gauges are the same. Now let us examine the massive Yang-Mill's field. Let us write the Lagrangian in the form

$$L = L_0 + \frac{m^2}{2} A_\mu^a A_\mu^a . \tag{6.13}$$

The S-matrix of this theory is

$$S = :C\int \prod_\alpha dA_\mu^\alpha(x) \exp\left[i\int d^4x \left\{A_\mu^{in}(-\Box + m_1^2) A_\mu + L_0 + \frac{m^2}{2} A_\mu^\alpha A_\mu^\alpha\right\}\right]. \tag{6.14}$$

Let us multiply (6.14) by $\phi(A)$ [28]

$$\phi(A) = \Delta(A) \int d\mu(s) \delta(\partial_\mu A_\mu^{s(\alpha)}) = 1 \tag{6.15}$$

and perform the gauge transformation $A_\mu^\alpha \to A_\mu^{s_\alpha^{-1}}$, then the expression (6.14) transforms into

$$S = :C\int d\mu(S(x)) \int dA_\mu^\alpha \delta(\partial_\mu A_\mu) \Delta \exp i\int d^4x \cdot \\ \cdot \left\{A_\mu^{in}(-\Box + m^2) A_\mu + L_0 + \frac{m^2}{2} \mathrm{Sp}\left[\left(\hat{A}_\mu - \frac{i}{\lambda} S^{-1}(\lambda) \partial_\mu S(\lambda)^2\right)\right],\right. \tag{6.16}$$

where

$$\Delta(A) = \mathrm{Det}\left[\delta^{ab} + \lambda\varepsilon^{abc}A^c_\mu\partial_\mu\frac{1}{\Box}\right], \qquad \hat{A}^S_\mu = S^{-1}(x)\hat{A}_\mu S - \frac{i}{\lambda}S^{-1}\partial_\mu S,$$

$S(x)$ is a unitary 2×2 matrix with $\det S(\alpha) = 1$, $\mathrm{d}\mu(S)$ is the integration measure over the group SU_2 [25]; $\hat{A} = \tau^\alpha A^\alpha$, $\mathrm{Sp}\,\tau^\alpha\tau^\beta = \delta^{\alpha\beta}$. Taking the parametrization of the group transformation in the form

$$S(u) = \frac{1}{\sqrt{1+u^2}}(1 + i\hat{u}),$$

$$(6.17)\qquad \begin{cases} S = :C\displaystyle\int\prod_x \frac{\mathrm{d}u(x)}{(1+u^2)^2}\int \mathrm{d}A^\alpha_\mu \Delta(A)\,\delta(\partial_\mu A_\mu)\cdot \\ \qquad\cdot\exp\left[i\displaystyle\int \mathrm{d}^4x\left[A^{\mathrm{in}}(-\Box + m^2_\alpha)A_\mu + L_0 + \frac{m^2}{2}A^\alpha_\mu A^\alpha_\mu + L_1\right]\right], \\ L_1 = \dfrac{m^2}{2\lambda^2(1+u^2)}\left[(\partial_\mu u^\alpha)^2 - \dfrac{(u\partial_\mu L_1)^2}{1+u^2}\right] - \dfrac{m^2 A^\alpha_\mu}{1+u^2}(\partial_\mu u^\alpha + [u\times\partial_\mu u]^\alpha). \end{cases}$$

Making the substitution $u \to (\lambda/m)u$ and keeping the terms of the lower order in λ we obtain that the integral of u' is equal to $\Delta^{-\frac{1}{2}}(A)$ and

$$(6.18)\qquad S = :C\int \mathrm{d}A^\alpha_\mu \Delta^{\frac{1}{2}}(A)\,\delta(\partial_\mu A^\alpha_\mu)\cdot \\ \cdot\exp\left[i\int \mathrm{d}^4\alpha\left[A^{\mathrm{in}}(-\Box + m^2_i)A_\mu + L_0 + \frac{m^2}{2}A^\alpha_\mu A^\alpha_\mu\right]\right].$$

The expression (6.18) gives rise the perturbative series suggested in [31, 32]. This expression (6.18) gives the asymptotes of S in the limit $m \to \infty$, but in the limit $m \to 0$ the group integral on u'' in (6.17) is equal to one, and the S-matrix coincides with the S-matrix of the massless theory.

6.2. *Feynman's rules for the gravitational field.* – In Subsect. **6.1** we developed a method for constructing Feynman's rules for fields possessing an invariance group. Now we shall apply this method to the gravitational-field theory. Feynman's rules coincide with those of ref. [28-30]. On the other hand, in contrast to [28-30], it is possible to see the unitary of the S-matrix in the present method of construction of Feynman's rules. Let us consider the gravitational-field Lagrangian:

$$(6.19)\qquad L = L_0 + L_1, \qquad L_0 = -\sqrt{-g}R(g),$$

where $g_{\mu\nu}$ is the metric tensor $g = \det g_{\mu\nu}$ and $R(g)$ is the scalar curvature. The term L_1 is added to the invariant Lagrangian so as to specify the gravitational-field gauge. Let us consider the gravitational-field gauge described in terms of the quantities $\hat{g}_{\mu\nu}$:

$$\partial_\mu \hat{g}^{\mu\nu} = x^\nu_{(1)}, \qquad \hat{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} . \tag{6.20}$$

Let us take L_1 in the following form, as we did previously with eq. (6.5):

$$L_1 = \partial_\mu \hat{g}^{\mu\nu} B^1_\nu + \frac{\alpha}{2} B^1_\mu \delta^{\mu\nu} B^1_\nu , \tag{6.21}$$

where $\delta^{\mu\nu}$ is Minkowski's metric tensor and

$$B^1_\nu(x) = \int \mathrm{d}y D^\lambda_{1\nu}(x, y; \hat{g}) B_\lambda(y) . \tag{6.22}$$

Taking into account the vanishing of the covariant divergence of Einstein tensor one obtains

$$\partial_\mu \hat{g}^{\mu\nu} = -\alpha \delta^{\mu\nu} B^1_\mu , \tag{6.23}$$

$$[\delta^\sigma_\lambda \hat{g}^{\mu\nu} \partial_\mu \partial_\nu + \delta^\sigma_\lambda \partial_\mu \hat{g}^{\mu\nu} \partial_\nu + \partial_\mu \hat{g}^{\mu\sigma} \partial_\lambda] B^1_\sigma = 0 . \tag{6.24}$$

In order that the B-particle may be free, D_1 must be taken in the form

$$D^\nu_{1\mu} = [\delta^\nu_\mu \hat{g}^{\lambda\sigma} \partial_\lambda \partial_\sigma + \delta^\nu_\mu \partial_\lambda \hat{g}^{\lambda\sigma} \partial_\sigma + \partial_\lambda \hat{g}^{\lambda\nu} \partial_\mu]^{-1} \Box . \tag{6.25}$$

Finally one obtains for the generating functional

$$Z^{(1)}_\alpha = \int \mathrm{d}\hat{g}^{\mu\nu} \mathrm{d}B_\lambda \exp\left[i \int \mathrm{d}x (L_0 + L_1 + \hat{g}^{\mu\nu} J_{\mu\nu})\right] = \tag{6.26}$$
$$= \int \mathrm{d}\hat{g}^{\mu\nu} \exp\left[i \int \mathrm{d}x \left(L_0 - \frac{1}{2\alpha} \partial_\lambda \hat{g}^{\lambda\mu} \delta_{\mu\nu} \partial_\sigma \hat{g}^{\sigma\nu} + \hat{g}^{\mu\nu} J_{\mu\nu}\right) + \mathrm{Sp} \log [D^\nu_{1\mu}]^{-1}\right] .$$

In the transversal gauge ($\alpha = 0$) the above expression reduces to the simpler form

$$Z^{(1)}_0 = \int \mathrm{d}\hat{g}^{\mu\nu} \delta(\partial_\mu g^{\mu\nu}) \exp\left[i \int \mathrm{d}x (L_0 + \hat{g}^{\mu\nu} J_{\mu\nu}) + 4\, \mathrm{Sp} \log \hat{g}^{\mu\nu} \partial_\mu \partial_\nu\right] . \tag{6.27}$$

Feynman's rules for the perturbative calculation of the generating functional (6.26) coincide with those of ref. [28].

Now let us consider the class of the gauges of the gravitational field described in terms of the quantity

$$\delta^{\sigma\lambda}(\partial_\sigma g_{\mu\nu} - \tfrac{1}{2}\partial_\mu g_{\sigma\lambda}) = \chi_\mu^{(2)} . \tag{6.28}$$

Repeating the previous process we find the corresponding function D_2:

$$D_{2\nu}^{\mu} = [Q_{\sigma\nu}]^{-1}\Box , \tag{6.29}$$

$$Q_{\sigma\nu} = g_{\sigma\nu}\delta^{\alpha\beta}\partial_\alpha\partial_\beta + (\partial_\alpha g_{\sigma\beta} - \tfrac{1}{2}\partial_\sigma g_{\alpha\beta})(\delta^{\gamma\alpha}\delta_\nu^\beta\partial_\gamma + \delta^{\gamma\beta}\delta_\nu^\alpha\partial_\gamma - \delta^{\alpha\beta}\partial_\nu) . \tag{6.30}$$

The generating functional in the gauge (6.28) is

$$Z_\alpha^{(2)} = \int \mathrm{d}g_{\mu\nu} \exp\left[i\int \mathrm{d}x\left(L_0 - \frac{1}{2\alpha}\chi_\mu^{(2)}\delta^{\mu\nu}\chi_\mu^{(2)} + g_{\mu\nu}J^{\mu\nu}\right) + \mathrm{Sp}\ln Q_{\sigma\nu}\right]. \tag{6.31}$$

For $\alpha = -1$ we obtain the version of the perturbation theory of ref. [29, 30]. In proving the gauge-invariance of the S-matrix, we note that the set of interpolating fields

$$g_{\mu\nu}^{p} = g^{p} g_{\mu\nu} \qquad \text{and} \qquad g_{\mu\nu}^{p} = g^{p} g^{\mu\nu} \tag{6.32}$$

brings us to the same S-matrix, for arbitrary power of p, as described in Borcher's theorem. It follows, therefore, that the S-matrix is independent of the choice of functional variables of integration within class (6.27). A proof of the α independence of the S-matrix can be effected in the same way as in Subsect. **6·1**, that is, by constructing the gauge transformation of the field $\hat{g}_{\mu\nu}$ which changes the generating functional on the mass shell for different α's into each other.

A proof of the independence of the S-matrix of this kind of gauge is effected by means of the method used in ref. [28]. In a recent work by Popov and Faddiev [34] Feynman's rules for the gravitational field are obtained in a noncovariant gauge, as suggested by Dirac [35].

We note that the present method makes it possible for us to prove the equivalence of the S-matrix in both a covariant and a noncovariant gauge.

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