Classical Analog of Quantum Schwarzschild Black Holes and Mystery of $\log 3$

Victor Berezin[1]

[1]Institute for Nuclear Research, RAS, Moscow

Classical Black Holes

- Event Horizon
  [S.W.Hawking R.Penrose]

What is a black hole? = event horizon. Global feature $\rightarrow$ one should know the whole history, both past and future. Classically, nothing can be gone out.

- General Black Hole

Schwarzschild-Kerr-Newman solution.
Depends on only few parameters: mass m, electric charge(s) e and angular momentum J.

- "No-Hair"-Property
  [J.A.Wheeler D.Cristodoulou R.Ruffini + ...]

Resembles thermal equilibrium.
Reversible and irreversible processes while extracting energy out of black holes. $\rightarrow dA \geq 0,\ A$ - horizon area

- Quasi-normal frequencies
  [T.Regge J.A.Wheeler H.-P.Nollert L.Motl A.Neitzke S.Hod R.Schiappa W.H.Press S.Chandrasekhar + ... + ...]

Process of becoming bald - global feature.
Behavior of perturbations - falling into inside and radiating away.
Decaying modes of complex frequencies.
For Schwarzschild black holes

$$G\,m\,w_n = 0.0437123 - \frac{i}{4}\left(n + \frac{1}{2}\right) + O[(n+1)^{-1/2}]. \qquad (1)$$

- decreasing relaxation times and asymptotically constant value for $Re\,w_n$. The very existence of quasi-normal modes resembles pure dying tones of a ringing bell. These resonances are the characteristic sound of the black hole itself.

$$Re\,w = 0.0437123 = \frac{\log 3}{8\pi\,G\,m} \qquad (2)$$

In what follows - units $\hbar = c = \kappa = 1$. Gravitational constant G is the only dimensional quantity. $m_{Pl} = \sqrt{\frac{\hbar c}{G}}, l_{Pl} = \sqrt{\frac{\hbar G}{c^3}}$.

Quasi-classical Picture

- Black hole thermodynamics
  [J.Bekenstein J.M.Bardeen B.Carter S.W.Hawking]

Four laws:

$$dm = \frac{\varkappa_H}{8\pi G} dA + \Phi_H de + \Omega_H dJ \qquad (3)$$

A -horizon area, e - electric charge, J - angular momentum - extensive parameters, and $\varkappa_H$, Φ_H, Ω_H - surface gravity, Coulomb potential and angular velocity - intensive parameters at the event horizon.

$\varkappa_H = const$ - zero law, $dA \geq 0$ -second law

$$\frac{\varkappa}{8\pi G} dA \longrightarrow TdS \qquad (4)$$

T - temperature, S - entropy.

S.Hawking - temperature is real. Schwarzschild:

$$T_H = \frac{1}{8\pi G m} = \frac{\varkappa_H}{2\pi} \quad \rightarrow \quad S = \frac{A}{4G} \qquad (5)$$

Both temperature and entropy are global entities. Black hole has no volume - only horizon area. The origin of entropy - quantum: we need to count the number(!) of possible microstates. Hawking temperature is also of quantum nature: black holes evaporate quantum mechanically, not classically!

- Rindler Space-Time. Unruh Temperature

$2 - dim$ locally flat, metric

$$\begin{aligned} ds_2^2 &= dt^2 - dx^2 = e^{2a\xi}\left(d\eta^2 - d\xi^2\right) = a^2\,\rho^2 d\eta^2 - d\rho^2\,, \\ t &= \frac{1}{a}e^{a\xi}\sinh a\eta\,,\ x = \pm\frac{1}{a}e^{a\xi}\cosh a\eta\,;\quad \rho = \frac{1}{a}e^{a\xi} \end{aligned} \quad (6)$$

Event horizons: $t - x^2 = 0 \ \rightarrow\ \xi = const\,,\ t = \pm\infty$.
Rindler observers - uniform acceleration $\varkappa = a$.
W.Unruh - quantum field theory on Rindler manifold $\rightarrow$ temperature

$$T_U = \frac{a}{2\pi} = \frac{\varkappa}{2\pi} \quad (7)$$

Chain of physical features:
Event horizon $\rightarrow$ Hidden information $\rightarrow$ Entropy $\rightarrow$ Thermodynamics $\rightarrow$ Temperature
Temperature is not an invariant but the temporal component of a heat flow 4-vector. T_U is measured by observer for whom $g_{00} = 1$ (or who is using its own proper time). Local temperature $T_{loc} = T_U / \sqrt{g_{00}}$. Thermal equilibrium condition - $T_{loc} = const$.
Einstein equivalence principle $\rightarrow$:
Rindler observers are uniformly accelerated $\rightarrow$ Schwarzschild observers at constant radius undergo constant acceleration

$$\varkappa = a \ \rightarrow \ \varkappa = \frac{1}{4Gm} \tag{8}$$

Unruh temperature at the horizon = Hawking temperature.

- Topological Temperature

Simple way to calculate Unruh and Hawking temperature
Transformation to the imaginary time (Whick rotation):
$\eta \to i\tau \longrightarrow ds^2 \to dl^2 = d\rho^2 + a^2\rho^2 d\tau^2$. In general, conical singularity.
Minkowski time: $i\,t = \rho \sin a\,\tau \to \tau$ - cyclic coordinate with period $\Theta = 2\pi/a \to$ conical singularity disappears.
Temperature = inverse of imaginary time period.
Black hole of general type: one should consider $2 - dim$ metric in the vicinity of event horizon.

- Entropy Quantization. $\log 2$ *vs* $\log 3$ [J.Bekenstein T.Damour R.Ruffini V.F.Mukhanov G.Gour S.Hod A.Strominger C.Vafa K.Krasnov A.Ashtekar + ...]

Horizon area - classical adiabatic invariant + minimal increase $\Delta A_{min} \approx 4\, L_{Pl}^2$ while capturing neutral or electrically charged particles + Ehrenfest principle $\longrightarrow$ equidistant discrete spectrum for horizon area A and, thus, the entropy

$$S_{BH} = \gamma\, N\,, \quad N = 1, 2, 1... \tag{9}$$

Statistical physics argument $\longrightarrow$

$$\gamma = \log k\,, \quad k = 2, 3, ... \tag{10}$$

Information theory + "It from Bit" claim by J.A.Wheeler $\longrightarrow$

$$\gamma = \log 2 \tag{11}$$

Loop quantum gravity $\rightarrow$ $S_{BH} = N \log(2j_{min} + 1)$, j_{min}-minimal (nonzero) spin value depending on underlying symmetry group. $SU(2) \rightarrow j_{min} = \frac{1}{2} \rightarrow \gamma = \log 2$.
Quasi-normal frequencies + Bohr's correspondence principle

$$\Delta m_{min} = Re\, w_{QN} = T\, \Delta S_{min} = \frac{\gamma}{8\pi G} \tag{12}$$

$$\longrightarrow \quad \gamma = \log 3 \tag{13}$$

Mystery!

- Wheeler-DeWitt Equation
  [V.A.Berezin A.M.Boyarsky A.Yu.Neronov]

Spherically symmetric self-gravitating thin dust shell
- simplest generalization of a point particle
1)has the dynamical degree of freedom - shell radius - only one
,
2) allows full account for back reaction of the matter source on the space-time metric.
Stationary Schroedinger-like equation in finite differences

$$\Psi(m, m_{in}, S+i\zeta)+\Psi(m, m_{in}, S-i\zeta) = \frac{F_{in}+F_{out}-\frac{M^2}{4m^2S}}{\sqrt{F_{in}}\sqrt{F_{out}}}\Psi(m, m_{in}, S), \quad (14)$$

$m = m_{out} = m_{tot}$ - the total mass of the system, m_{in} - the Schwarzschild mass inside, M is the bare mass of the shell, $S = \frac{R^2}{4G^2m^2}$ R - radius, $F = 1 - \frac{2Gm}{R}$, $\zeta = \frac{m_{Pl}^2}{2m^2}$.

$(F)^{1/2}$ -analytical function with branching at the horizons $(F = 0)$, at zero radius singularities and at both infinities. This reflects nontrivial causal structure of complete Schwarzschild space-time.
Wave function Ψ "lives" in both asymptotically flat regions
$\rightarrow$ two boundary conditions at two infinities.

- Discrete Spectrum

Compare asymptotical behavior of solutions at branching points
$\rightarrow$ two quantization conditions:

$$\begin{aligned} \frac{2(\Delta m)^2 - M^2}{\sqrt{M^2 - (\Delta m)^2}} &= \frac{2m_{Pl}^2}{\Delta m + m_{in}}\, n\,, \\ M^2 - (\Delta m)^2 &= 2(1+2p)\, m_{Pl}^2\,, \end{aligned} \tag{15}$$

$\Delta m = m_{out} - m_{in}$, n and $p \geq 0$ are integers.

Two quantum numbers (n, p) - two parameters $(\Delta m, M)$ for fixed m_{in} $\rightarrow$ shell does not collapse (gravitational hydrogen atom).

After switching on radiation - collapse starts with production (necessarily) of new particles (shells) $\rightarrow$ m_{in} increases. Such a process can proceed in many different ways - the origin of entropy.

When could quantum collapse be stopped?

Natural limit - crossing Einstein-Rosen bridge - transition to semi-closed world requires insertion of an infinite volume - zero probability.

This occurs exactly at $n = 0$!

- "No-Memory"

$n = 0$ - special point in the spectrum.
The shell in this state does no "feel" what is going on onside $\rightarrow$ it "feels" only itself!
Like the "no-hair"-property of classical black holes.
Finally, when all shells (both primary one and newly born) are in corresponding states $n_i = 0$, the whole system does not "remember" its own history.
It is this "no-memory" state that can be called "the quantum black hole".
The total Δm_i and bare M_i masses of all the shells obey the relation

$$\Delta m_i = \frac{1}{\sqrt{2}} M_i \qquad (16)$$

Subsequent quantum Hawking radiation can proceed via some collective excitations.

- Einstein Equations

Number of shells $N \gg 1$ $\rightarrow$ quasi-classics $\rightarrow$ "almost" classical description.
Wave function $\Psi(x)$ $\longrightarrow$

$$\Psi^{\star}(x)\Psi(x) \longrightarrow \rho(x) \tag{17}$$

- number density or mass density.
Back reaction on space-time geometry $\rightarrow$ Einstein equations
Spherical symmetry
Quantum stationary states $\rightarrow$ static matter distribution $\rightarrow$ non-zero effective pressure.
Static metric

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{18}$$

Here r is the radius of a sphere with the area $A = 4\pi r^2, \nu = \nu(r), \lambda = \lambda(r)$.

Energy-momentum tensor T_μ^ν: no preferred direction for local observers inside distribution $\to$ isotropy

$$T_\mu^\nu = diag(\varepsilon, -p, -p, -p) \qquad (19)$$

ε - energy density, p - effective pressure.
Einstein equations ("prime" denotes differentiation in r):

$$\begin{aligned} -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} &= 8\pi G\varepsilon\,, \\ -e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) + \frac{1}{r^2} &= -8\pi G p\,, \\ -\frac{1}{2}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) &= -8\pi G p\,. \qquad (20) \end{aligned}$$

Integrating the first equation $\rightarrow$

$$e^{-\lambda} = 1 - \frac{2Gm(r)}{r}, \tag{21}$$

Here

$$m(r) = 4\pi \int_0^r \varepsilon r' 2 dr' \tag{22}$$

is "running" total mass that must be identified with m_{in}.
Bare mass function

$$M(r) = \int \varepsilon dV = 4\pi \int_0^r \varepsilon e^{\frac{\lambda}{2}} r'^2 dr' \tag{23}$$

- "No-Memory" Condition

$$
\begin{aligned}
m(r) &= a r\,, \quad a = const \\
e^{-\lambda} &= 1 - 2Ga = const
\end{aligned}
\tag{24}
$$

None of the local observers has a privilege.
Resembles thermal equilibrium.
Energy density

$$
\varepsilon = \frac{a}{4\pi G r^2} \tag{25}
$$

(Zeldovich machine > 40 years ago)
Bare mass function

$$
M(r) = \frac{a r}{\sqrt{1 - 2Ga}}\,. \tag{26}
$$

Pressure

$$
\begin{aligned}
p(r) &= \frac{b}{4\pi r^2} \\
b &= \frac{1}{G}\left(1 - 3Ga - \sqrt{1-2Ga}\sqrt{1-4Ga}\right) \\
a &\leq \frac{1}{4G} \rightarrow b \leq a \rightarrow v_{sound} \leq c
\end{aligned}
\tag{27}
$$

Finally,

$$
e^{\nu} = C_0 r^{\frac{4b}{a+b}} = C_0 r^{2G\frac{a+b}{1-2Ga}} . \tag{28}
$$

- Boundary Condition

Curvature singularity for $b \leq a$. But, for $a = b = \frac{1}{4G}$ this singularity happily disappears, and we have

$$\begin{aligned} g_{00} &= e^{\nu} = C_0^2 r^2 , \\ g_{11} &= -e^{\lambda} = -\sqrt{2}, \\ \varepsilon &= p = \frac{1}{16\pi G r^2} \quad \longrightarrow \end{aligned} \tag{29}$$

stiffest possible equation of state.
To ensure statics we must include into our model some surface tension Σ ($\rightarrow$ liquid). It plays the role of a potential barrier for tunneling processes.

Constant of integration C_0 - from matching at $r = r_0$.

$$\begin{aligned} e^{-\lambda}(r_0) &= e^{\nu}(r_0) = 1 - \frac{2\,G\,m_0}{r_0} \quad \rightarrow \\ C_0^2 &= \frac{1}{2r_0^2}\,; \quad \Delta p = \frac{2\Sigma}{\sqrt{2}r_0}\,; \\ e^{\nu} &= \frac{1}{2}\left(\frac{r}{r_0}\right)^2; \quad p_0 = \varepsilon_0 = \frac{1}{16\pi G r_0^2}\,; \\ m &= m_0 = \frac{r_0}{4G}\,. \end{aligned} \tag{30}$$

$r_0 = 2\,r_g$ - twice the gravitational radius.
Bare mass $M = \sqrt{2}\,m$ - the same relation as for quantum shells in the "no-memory" state $n = 0$!.

- Horizon. Temperature

Surface $r = 0$?
Not a trivial singularity:

$$ds^2(r = 0) = 0. \tag{31}$$

Looks like an event horizon.
$(t - r)$-part of the metric

$$ds_2^2 = \frac{1}{2}\left(\frac{r}{r_0}\right)^2 dt^2 - 2\,dr^2 \tag{32}$$

- locally flat Rindler space-time with $a = \varkappa = \frac{1}{2\,r_0}$.
Unruh temperature

$$T_U = \frac{1}{4\pi\,r_0} = const \quad \rightarrow \tag{33}$$

thermal equilibrium. This temperature is exactly one half of Hawking temperature

$$T_U = \frac{1}{16\pi\,G\,m} = \frac{1}{2}T_H \tag{34}$$

- Global $\longrightarrow$ Local

By definition, the surface $r = 0$ cannot be crossed. Thus, the event horizon in our model becomes local.
The temperature is also local, $T_{loc} = T_U\, e^{-\frac{\nu}{2}} = \frac{1}{2\sqrt{2\pi}\, r}$, and does not depend on the boundary value r_0. Important feature: if one removes some outer layer, nothing would be changed inside.
The quantum nature of radiation and the fact that the entropy has a discrete equidistant spectrum suggest that our distribution consists, actually, of some number of quasi-particles which can be called "gravitational phonons".
Thus, having at hand local intensive parameters: effective pressure $p(r)$, temperature $T_{loc}(r)$, chemical potential $\mu(r)$, and extensive parameters: Bare mass M, volume V, entropy S and "particle" number N, we are now ready to construct the local thermodynamics.

- Thermodynamics

First law

$$dM = \varepsilon dV = T_{loc} dS - p dV + \mu dN \,. \tag{35}$$

Local form

$$\varepsilon(r) = T_{loc}(r) s(r) - p(r) + \mu(r) n(r) \,, \tag{36}$$

$s(r)$ and $n(r)$ - entropy and particle densities.

$s(r)$–? $S_{tot} = 4\pi G m^2 = \frac{\pi r_0^2}{4G} \quad \rightarrow$

$$\begin{aligned} s(r) &= \frac{1}{8\sqrt{2} G r} \\ T_{loc}(r) s(r) &= \frac{1}{32\pi G r} = \frac{1}{2}\varepsilon \\ \mu(r) n(r) &= \frac{3}{2}\varepsilon \,. \end{aligned} \tag{37}$$

Free energy

$$f(r) = T_{loc}(r)s(r) = \frac{1}{2}\varepsilon(r)\,,$$
$$F(r) = \frac{1}{2}M = 2\,T_{loc}(r_0)\,S_{tot} \qquad (38)$$

as measured by local observer. Distant observe at infinity measures the total mass $m = \frac{1}{\sqrt{2}}M$ and the Hawking temperature $T_H = 2\,T_U = \frac{2}{\sqrt{2}}T_{loc}(r_0)$; $\longrightarrow$

$$F_\infty = m - T_H\,S_{tot} = \frac{1}{2}m\,. \qquad (39)$$

This guarantees the usual Schwarzschild black hole thermodynamic relation $d\,m = T_H\,dS$.

- Entropy Quantization. Partition Function

Since in thermal equilibrium

$$T_{loc}\sqrt{g_{00}} = const\ , \quad \mu\sqrt{g_{00}} = const \tag{40}$$

one automatically gets the equidistant entropy quantization

$$\begin{aligned} \frac{T_{loc}(r)\,s(r)}{\mu(r)\,n(r)} &= \frac{1}{\gamma}\frac{s(r)}{n(r)} \longrightarrow \\ S &= \gamma N\ , \quad N = 1, 2, \ldots \end{aligned} \tag{41}$$

To calculate spacing coefficient γ we should know partition function $Z_{tot} = (Z_1)^N$, where Z_1 is partition function for one quasi-particle ("phonon"). By definition

$$Z_1 = \Sigma_n e^{-\frac{\varepsilon_n}{T}}\ . \tag{42}$$

ε_n are energy levels and, since $\frac{\varepsilon_n}{T}$ is invariant under the change of time variable (clocks), we can use the proper time of local observers, so the temperature is just the Unruh temperature, $T = T_U = const$.

Black holes are characterized by some inherent frequency ω. This follows from the very existence of quasi-normal modes. We assume the simplest ("phonon") equidistant spectrum

$$\varepsilon_n = \omega n\,, \quad n = 1, 2, 3...$$
$$Z_1 = \frac{e^{-\frac{\omega}{T}}}{1 - e^{-\frac{\omega}{T}}}\,. \qquad (43)$$

The total partition function equals $Z_{tot} = (Z_1)^N$. The partition function is an invariant. For any small part of our system one has$f dV = -T_{loc} \log Z_{small} \longrightarrow$

$$\int \frac{f}{T_{loc}} dV = -\Sigma \log Z_{small} = -\log Z_{tot}\,. \qquad (44)$$

L.h.s. equals

$$\int \frac{f}{T_{loc}} dV = \frac{1}{2} \int \frac{\varepsilon}{T_{loc}} dV = 2\sqrt{2}\pi \int_0^{r_0} \frac{\varepsilon}{T_{loc}} r^2 dr = \frac{\pi r_0^2}{4G} = \frac{\pi r_g^2}{G} = S_{tot} \tag{45}$$

From this it follows

$$e^{-S} = Z_{tot} = (Z_1)^N \longrightarrow \frac{e^{-\frac{\omega}{T}}}{1 - e^{-\frac{\omega}{T}}} = e^{-\gamma} \longrightarrow e^{\gamma} = e^{\frac{\omega}{T}} - 1 \,. \tag{46}$$

- Solving the "Mystery of $\log 3$"

Irreversible process of converting the mass (energy) of the system into radiation. It takes place at the boundary $r = r_0$, and the thin shell with zero surface energy density and surface tension Σ serves as such a converter, supplying the radiation with extra energy and extra entropy, i.e., acts like a "brick wall". One can imaging that the near-boundary layer of thickness Δr_0 is converting into radiation, thus decreasing the inner region boundary to $(r_0 - \Delta r_0)$. Its energy equals $\Delta M = \varepsilon \Delta V$. But the radiated quantum receives, in addition, the energy released from the work done by surface tension due to its shift, which equals exactly $\Sigma d(4\pi r_0^2) = p\Delta V = \varepsilon \Delta V = \Delta M$. Therefore, energy of quanta are doubled. Clearly, they have double frequency and exhibit double temperature $\longrightarrow$

$$\frac{Re\, w}{T_H} = \frac{\omega}{T_U} = \log 3 \qquad (47)$$

Substituting into partition function gives

$$e^{\gamma} = e^{\frac{\omega}{T}} - 1 = 3 - 1 = 2 \longrightarrow \tag{48}$$

$$\gamma = \log 2\,. \tag{49}$$

AND THIS IS GOOD

Thank you all very much!

THE END