The gauge structure of generalised geometry

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Background

Duality symmetries in string theory/M-theory mix gravitational and non-gravitational fields. Manifestation of such symmetries calls for a generalisation of the concept of geometry.

It has been proposed that the compactifying space (torus) is enlarged to accomodate momenta in representations of a duality group.

This leads to *doubled geometry*
in the context of T-duality [Hull et al.; Hitchin;...]

and *generalised/exceptional geometry*
in the context of U-duality. [Hull; Berman et al.; Coimbra et al;...]

Compactify from 11 to $11-n$ dimensions on T^n. As is well known, all fields and charges fall into representations of $E_n(n)$.

n	$E_{n(n)}$	
4	$SL(5)$	
5	$Spin(5,5)$	
6	$E_{6(6)}$	
7	$E_{7(7)}$	
8	$E_{8(8)}$	

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4	$SL(5)$	**10**
5	$Spin(5,5)$	**16**
6	$E_{6(6)}$	**27**
7	$E_{7(7)}$	**56**
8	$E_{8(8)}$	**248**

I will focus on diffeomorphisms, and how they generalise. The ordinary diffeomorphisms go together with gauge transformations for the 3-form and (dual) 6-form fields (and for high enough n also gauge transformations for dual gravity) in an $E_n(n)$ representation $\overline{\mathbf{R}}$. This is the "coordinate representation". The derivative transforms in $\mathbf{R}$.

One has to decide how tensors transform.

The generic recipe is to mimic the Lie derivative for ordinary diffeomorphisms:

$$L_U V^m = U^n \partial_n V^m - \partial_n U^m V^n$$

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In the case of U-duality, the role of GL is assumed by $E_{n(n)} \times \mathbb{R}$, and

$$\mathscr{L}_U V^M = U^N \partial_N V^M + \Pi^M{}_Q{},^N{}_P \partial_N U^P V^Q$$

where $\Pi^M{}_Q{},^N{}_P = -\alpha_n P^M_{\text{adj}\,Q}{},^N{}_P + \beta_n \delta^M_Q \delta^N_P$.

n	repr. of U^M	α_n	β_n
4	$\overline{\mathbf{10}}$	3	$\frac{1}{5}$
5	$\overline{\mathbf{16}}$	4	$\frac{1}{4}$
6	$\overline{\mathbf{27}}$	6	$\frac{1}{3}$
7	$\mathbf{56}$	12	$\frac{1}{2}$

For these values of the coefficients, the transformations form an algebra for $n \leq 7$:

$$[\mathscr{L}_U, \mathscr{L}_V] W^M = \mathscr{L}_{[\![U,V]\!]} W^M$$

where the "Courant bracket" is $[\![U, V]\!]^M = \frac{1}{2}(\mathscr{L}_U V^M - \mathscr{L}_V U^M)$, provided that the derivatives fulfill a "*section condition*".

The *section condition* ensures that fields locally depend only on an n-dimensional subspace of the coordinates, on which a $GL(n)$ subgroup acts. It reads

$$(\partial \otimes \partial)|_{\mathbf{R_2}} = 0$$

n	$\mathbf{R_1}$	$\mathbf{R_2}$
3	$(\mathbf{3}, \mathbf{2})$	$(\overline{\mathbf{3}}, \mathbf{1})$
4	$\mathbf{10}$	$\overline{\mathbf{5}}$
5	$\mathbf{16}$	$\mathbf{10}$
6	$\mathbf{27}$	$\overline{\mathbf{27}}$
7	$\mathbf{56}$	$\mathbf{133}$
8	$\mathbf{248}$	$\mathbf{1} \oplus \mathbf{3875}$

I will digress a little on this condition.

The interpretation of the section condition is that the momenta locally are chosen so that they may span a linear subspace of cotangent space with maximal dimension, such that any pair of covectors p, p' in the subspace fulfill $(p \otimes p')|_{\mathbf{R_2}} = 0$.

The corresponding statement in T-duality is $\eta^{MN}\partial_M \otimes \partial_N = 0$, where η is the $O(d,d)$-invariant metric. The maximal linear subspace is a d-dimensional isotropic subspace, and it is determined by a pure spinor Λ. Once a Λ is chosen, the section condition can be written $\Gamma^M \Lambda \partial_M = 0$.

(In double field theory, the condition may be weakened, so that only $\eta^{MN}\partial_M\partial_N = 0$. This seems difficult here.)

What are the corresponding U-duality covariant statements, *i.e.*, how does the concept of a pure spinor generalise, and what is the linear condition that picks out allowed momenta?

These questions have to be adressed case by case. For all $n \leq 7$, such objects exist, and are given by the following table ($n = 8$ not worked out). Take an object Λ in $\mathbf{R_3}$ with a purity constraint $\Lambda^2|_{\mathbf{P}} = 0$, and let $(\Lambda\partial)|_{\mathbf{R_4}} = 0$. This gives the maximal solution to the section condition, and selects an n-dimensional subspace.

n	$\mathbf{R_1}$	$\mathbf{R_2}$	$\mathbf{R_3}$	$\mathbf{R_4}$	$\mathbf{P}$
3	$(\mathbf{3},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	—
4	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\mathbf{5}$	$\overline{\mathbf{10}}$	—
5	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$	$\mathbf{10}$
6	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{78}$	$\mathbf{351}$	$\mathbf{650}$
7	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{1539}$	$\mathbf{1463}$

The representations $\mathbf{R_{p+1}}$ (almost) coincide with the p-brane charges in the uncompactified directions, and form part of a tensor hierarchy.

The generalised diffeomorphisms do not satisfy a Jacobi identity. On general grounds, it can be shown that the "Jacobiator" is proportional to $(\!([\![U,V]\!], W)\!)$ + cycl, where $(\!(U,V)\!) = \frac{1}{2}(\mathscr{L}_U V + \mathscr{L}_V U)$.

It is important to show that the Jacobiator in some sense is trivial. It turns out that $\mathscr{L}_{(\!(U,V)\!)} W = 0$ (for $n \leq 7$), and the interpretation is that it is a gauge transformation with a parameter representing reducibility.

In doubled geometry, this reducibility is just the scalar reducibility of a gauge transformation: $\delta B_2 = d\lambda_1$, with the reducibility $\delta\lambda_1 = d\lambda'_0$.

In generalised geometry, the reducibility turns out to be more complicated.

The tensor gauge transformations are reducible. A 2-form transformation has a 1-form reducibility and a 0-form second order reducibility, so that the effective number of gauge parameters in n dimensions is $\binom{n}{2} - n + 1 = \binom{n-1}{2}$, and analogously for a a 5-form parameter $\binom{n-1}{5}$.

Including diffeomorphisms, the effective number of generalised diffeomorphisms should be $n + \binom{n-1}{2} + \binom{n-1}{5}$, as long as dual gravity does not enter.

A parameter constructed as $U^M[\xi] = \partial_N \xi^{MN}$, where ξ is in the representation conjugate to the section condition, will generate a zero transformation through $\mathscr{L}_{U[\xi]}$. This is the first order reducibility.

The relation for $U[\xi]$ will in turn be reducible, in the sense that for an $\eta[\xi] \sim \partial\xi$ in a certain representation, $U[\xi[\eta]] = 0$, and so on. In all cases, the reducibility is infinite (if $E_{n(n)}$ covariance is demanded).

The (ghost) structure of this reducibility will be identical to the one for the (weak) section condition, seen as an algebraic condition on an object X.

Write a partition function for the constrained object by counting the homogeneous functions of degree k of the constrained object X:

$$Z(t) = \sum_{k=0}^{\infty} \dim(r_k) t^k$$

$$Z_3(t) = (1-t)^{-4}(1+2t) \ ,$$

$$Z_4(t) = (1-t)^{-7}(1+3t+t^2) \ ,$$

$$Z_5(t) = (1-t)^{-11}(1+t)(1+4t+t^2) \ ,$$

$$Z_6(t) = (1-t)^{-17}(1+t)(1+9t+19t^2+9t^3+t^4) \ ,$$

$$\begin{aligned} Z_7(t) = (1-t)^{-28}(&1+28t+273t^2+1248t^3+3003t^4+4004t^5+3003t^6+1248t^7 \\ &+273t^8+28t^9+t^{10}) \ , \end{aligned}$$

$$\begin{aligned} Z_8(t) = (1-t)^{-58}(1+t)(&1+189t+14080t^2+562133t^3+13722599t^4 \\ &+220731150t^5+2454952400t^6+19517762786t^7+113608689871t^8 \\ &+492718282457t^9+1612836871168t^{10}+4022154098447t^{11} \\ &+7692605013883t^{12}+11332578013712t^{13}+12891341012848t^{14} \\ &+11332578013712t^{15}+7692605013883t^{16}+4022154098447t^{17} \\ &+1612836871168t^{18}+492718282457t^{19}+113608689871t^{20} \\ &+19517762786t^{21}+2454952400t^{22}+220731150t^{23}+13722599t^{24} \\ &+562133t^{25}+14080t^{26}+189t^{27}+t^{28}) \ . \end{aligned}$$

The effective number of independent gauge parameters is read off as the negative power of the first factor.

For $n \leq 7$, they match the number of diffeomorphisms, 2-form and 5-form (for $n \geq 6$) transformations calculated above. For $n = 8$, the number also matches the number obtained by including $n\binom{n-1}{7}$ for a vector-valued 7-form.

n	diffeo	2-form	5-form	dual diffeo	total
3	3	1			4
4	4	3			7
5	5	6	0		11
6	6	10	1		17
7	7	15	6	0	28
8	8	21	21	8	58

Comments and questions:

What can be done for $n \geq 8$? The counting of parameters seems meaningful for $n = 8$, although there is yet no construction of the gauge algebra.

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Solutions to the (weak) section condition provides interesting generalisations of pure spinor cônes. Can one go beyond supergravity in some meaningful way, *e.g.* by relaxing the strong section condition?

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