Unstable oscillator and the tachyon field

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Motivation

- Instability of quantum systems with conservative Hamiltonians.

  Oscillator and string (field)

- Tachyon

  Faster – than – light particles $\Rightarrow$ neutrino (?)
  Physics of black hole

- Quantization $\Rightarrow$ S – matrix

  (1) Asymptotic states for $t \to \pm\infty$.

  (2) Causal Green function (propagator).

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Classical dynamics

Stable	Unstable
$L = \frac{1}{2}(p^2 - \omega^2 q^2)$	$L = \frac{1}{2}(p^2 + \Omega^2 q^2)$
$\ddot{q}(t) + \omega^2 q(t) = 0$	$\ddot{q}(t) - \Omega^2 q(t) = 0$
$q(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$	$q(t) = C_1 e^{-\Omega t} + C_2 e^{\Omega t}$
$H = \frac{1}{2}(p^2 + \omega^2 q^2)$	$H = \frac{1}{2}(p^2 - \Omega^2 q^2)$

Quantization ?

Hamiltonian and instability

$$\mathbf{H} = \frac{1}{2}\int dk\left[\pi^2(k) + (k^2+\mu^2)\phi^2(k)\right] \xrightarrow{\mu^2 \to -m^2} H_{st} + H_{un}$$

$$H_{st} = \frac{1}{2}\int\limits_{k^2>m^2} dk\left[\pi^2(k) + \omega^2(k)\phi^2(k)\right], \qquad \omega^2(k) = (k^2 - m^2) > 0$$

$$H_{un} = \frac{1}{2}\int\limits_{k^2<m^2} dk\left[\pi^2(k) - \Omega^2(k)\phi^2(k)\right], \qquad \Omega^2(k) = (k^2 - m^2) > 0$$

$$[\phi(k), \pi(k')] = i\delta(k - k')$$

$$\Downarrow$$

$$H = \frac{p^2}{2} - \frac{\Omega^2}{2}q^2 \longrightarrow \frac{\Omega}{2}(p^2 - q^2) \quad \Rightarrow \quad \text{instability}$$

Stable	Unstable
$H=\frac{\omega}{2}(p^2+q^2)=\omega\left(a^+a+\frac{1}{2}\right)$	$H=\frac{\Omega}{2}(p^2-q^2)=\omega\left(-BA-\frac{i}{2}\right)$
$a^{\pm}=\frac{q\mp ip}{\sqrt{2}},\quad [a,a^+]=1$	$A=\frac{q-p}{\sqrt{2}},\quad B=\frac{q+p}{\sqrt{2}},\quad [A,B]=i$
$Ha^{\pm}-a^{\pm}H=\pm\omega a^{\pm}$	$[H,A]=i\Omega A,\ [H,B]=-i\Omega B$
$Ha^n\Psi_E=(E-n\omega)a^n\Psi_E$	$HA^n\Phi_E=(E+i\,n\Omega)A^n\Phi_E$
$(E-n\omega)\to-\infty$	$e^{-iHt}A^n\Phi_E\to e^{-iEt+\Omega t}A^n\Phi_E\to\infty$
$a\Psi_0=0\ \Rightarrow\ \Psi_0(q)=e^{-\frac{\omega}{2}q^2}\in L^2$	$A\Phi_0=0\ \Rightarrow\ \Phi_0(q)=e^{i\frac{\Omega}{2}q^2}\notin L^2$
$\Psi_n=\frac{(a^+)^n}{n!}\Psi_0$	$\Phi_n=\frac{B^n}{n!}\Phi_0$
$H\Psi_n=\omega\left(n+\frac{1}{2}\right)\Psi_n$	$H\Phi_n=-i\Omega\left(n+\frac{1}{2}\right)\Phi_n$
$\Psi_n(t)=e^{-iHt}\Psi_n=e^{-i\omega\left(n+\frac{1}{2}\right)t}\Psi_n$	$\Phi_n(t)=e^{-iHt}\Phi_n=e^{-\Omega\left(n+\frac{1}{2}\right)t}\Phi_n$

$E_n>0$ $\qquad$ $\boxed{\omega\ \Rightarrow\ -i\Omega}$ $\qquad$ $\Phi_n(t)=O(e^{-\Omega t})$

Unstable states

$$\Psi(q) = \int ds\ F(s) e^{isB}\Phi_0(q) = \int ds\ F(s) e^{i\frac{s^2}{2}} e^{isq\sqrt{2\Omega}} e^{i\frac{\Omega}{2}q^2} \in L^2.$$

$$(\Psi^+, \Psi) = \int ds\ F^2(s) < \infty, \qquad F(s) \in L^2.$$

The time dependence

$$\Psi_t(q) = e^{-itH}\Psi(q) = e^{-\frac{\Omega}{2}t} \int ds\ F(s) e^{i\frac{s^2}{2}e^{-2\Omega t}} e^{isqe^{-\Omega t}\sqrt{2\Omega}} e^{i\frac{\Omega}{2}q^2}$$

$$|\Psi_t(q)|^2 \to \mathrm{const}\ e^{-\Omega t}, \qquad t \to \infty.$$

Average values of the coordinate and its square

$$\langle q \rangle_t = -\frac{e^{-\Omega t}}{\sqrt{2\Omega}} \int ds\ sF^2(s),$$

$$\langle q^2 \rangle_t = \frac{1}{2\Omega} \int ds\ \left[e^{2\Omega t}(F'(s))^2 + e^{-2\Omega t} s^2 F^2(s) \right] = O(e^{2\Omega t}).$$

Uncertainty of the location of the particle grows exponentially.

Matrix elements

$$(\Psi_0, F(a^+)\Psi_0) = \int dq\; e^{-\frac{\omega}{2}q^2} \cdot F\left(q - \frac{1}{\omega}\frac{d}{dq}\right) e^{-\frac{\omega}{2}q^2}$$
$$= \int dq\; F\left(q + \frac{1}{\omega}\frac{d}{dq}\right) e^{-\frac{\omega}{2}q^2} \cdot e^{-\frac{\omega}{2}q^2}$$
$$= (F(a)\Psi_0, \Psi_0) = F(0)(\Psi_0, \Psi_0).$$

$$\boxed{\omega \;\Rightarrow\; -i\Omega}$$

$$(\Psi_0, F(a^+)\Psi_0) \Rightarrow \int dq\; e^{i\frac{\Omega}{2}q^2} \cdot F\left(q - \frac{i}{\Omega}\frac{d}{dq}\right) e^{i\frac{\Omega}{2}q^2} = (\Phi_0, F(B)\Phi_0)$$
$$= \int dq\; F\left(q + \frac{i}{\Omega}\frac{d}{dq}\right) e^{i\frac{\Omega}{2}q^2} \cdot e^{i\frac{\Omega}{2}q^2} = (F(A)\Phi_0, \Phi_0) = F(0)(\Phi_0, \Phi_0).$$

$$(\Phi_0, F(B)\Phi_0) = (F(A)\Phi_0, \Phi_0) = F(0)(\Phi_0, \Phi_0)$$
$$(F(B)\Phi_0, \Phi_0) = (\Phi_0, F(A)\Phi_0) = F(0)(\Phi_0, \Phi_0)$$

Coherence states

The macroscopic stable oscillation :

$$\Psi_f = e^{\frac{f}{\sqrt{2}}a^+}\Phi_0, \qquad \Psi_f(t) = e^{-iHt}\Psi_f = e^{\frac{f}{\sqrt{2}}a^+e^{-i\omega t}}\Psi_0$$

$$\langle q(t)\rangle_f = \frac{(\Psi_f^+(t), q\Psi_f(t))}{(\Psi_f^+(t), \Psi_f(t))} = \frac{f}{\sqrt{\omega}}\cos(\omega t)$$

The macroscopic unstable motion :

$$\Phi_f = e^{-i\frac{f}{\sqrt{2}}B}\Phi_0, \qquad \Phi_f(t) = e^{-iHt}\Phi_f = e^{-\frac{\Omega}{2}t}e^{-i\frac{f}{\sqrt{2}}Be^{-\Omega t}}\Phi_0$$

$$\langle q(t)\rangle_f = \frac{(\Phi_f(t), q\Phi_f(t))}{(\Phi_f(t), \Phi_f(t))} = \frac{f}{\sqrt{\Omega}}e^{-\Omega t}$$

$$q(t) = e^{iHt} q e^{-iHt} = \frac{1}{\sqrt{2\Omega}} \left(Ae^{-\Omega t} + Be^{\Omega t} \right)$$

.

$$D_c(t-t') = \frac{\left(\Phi_0, T(q(t)q(t'))\Phi_0\right)}{(\Phi_0,\Phi_0)} = \frac{i}{2\Omega} e^{-\Omega|t-t'|}$$

i - "norm" of the unstable state

$$\frac{\left(\Phi_0, T\left\{ e^{\int_{t_0}^{t_1} dt\ q(t)J(t)} \right\} \Phi_0\right)}{(\Phi_0,\Phi_0)} = e^{\frac{1}{2} \iint_{t_0}^{t_1} dt dt'\ J(t) D_c(t-t') J(t')}$$

Stable and unstable oscillators

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{2}(P^2 - \Omega^2 Q^2) + hqQ - qJ.$$

$$\ddot{q}(t) + \omega^2 q(t) + hQ = J$$
$$\ddot{Q}(t) - \Omega^2 Q(t) + hq(t) = 0$$

$$q(t) = \int dt' G(t-t') J(t),$$

$$G(t-t') = \int \frac{dE}{2\pi} \cdot \frac{e^{-iE(t-t')}}{E^2 - \omega^2 + i0 + \frac{h^2}{E^2+\Omega^2}}$$

$$\approx \frac{1}{2\omega} e^{-i\omega|t-t'|} - \frac{h^2}{\omega^4} \cdot \frac{i}{2\Omega} e^{-\Omega|t-t'|}, \qquad h \ll \Omega\omega \ll \omega^2$$

Tachyon field in QFT

$$L(t) = \frac{1}{2} \int dx \left[(\dot{\phi}(t,x))^2 - (\nabla \phi(t,x))^2 + m^2 \phi^2(t,x) \right].$$

$$\phi(x) = \int \frac{dk}{(2\pi)^{\frac{3}{2}}} \phi(k) e^{i(kx)}, \qquad \pi(x) = \int \frac{dk}{(2\pi)^{\frac{3}{2}}} \pi(k) e^{-i(kx)}$$

$$[\phi(k), \pi(k')] = i\delta(k - k').$$

$$H = \frac{1}{2} \int dk \left[\pi^2(k) + (k^2 - m^2)\phi^2(k) \right] = H_{st} + H_{un}$$

$$H_{st} = \frac{1}{2} \int\limits_{k^2 > m^2} dk \left[\pi^2(k) + (k^2 - m^2)\phi^2(k) \right],$$

$$H_{un} = \frac{1}{2} \int\limits_{k^2 < m^2} dk \left[\pi^2(k) - (m^2 - k^2)\phi^2(k) \right].$$

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$$[\phi(k), \pi(k')] = i\delta(k-k').$$

$$H = \frac{1}{2}\int dk \left[\pi^2(k) + (k^2 - m^2)\phi^2(k)\right] = H_{st} + H_{un}$$

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$$H_{un} = \frac{1}{2}\int\limits_{k^2<m^2} dk \left[\pi^2(k) - (m^2 - k^2)\phi^2(k)\right].$$

Stable region $k^2 > m^2$

$$\phi_{st}(t,x) = \int\limits_{k^2>m^2} \frac{dk}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}} \left(a_k\ e^{-i\omega_k t+ikx} + a_k^+\ e^{i\omega_k t-ikx}\right)$$

$$[a_k, a_{k'}^+] = \delta(k-k'), \qquad \omega_k = \sqrt{k^2-m^2}$$

Unstable region $k^2 < m^2$

$$\phi_{un}(t,x) = \int\limits_{k^2<m^2} \frac{dk}{(2\pi)^{\frac{3}{2}}\sqrt{2\Omega_k}} \left(A_k\ e^{-\Omega_k t+ikx} + B_k\ e^{\Omega_k t-ikx}\right)$$

$$[A_k, B_{k'}] = i\delta(k-k'), \qquad \Omega_k = \sqrt{m^2-k^2}$$

The tachyon field

$$\phi(t,x) = \phi_{st}(t,x) + \phi_{un}(t,x), \qquad |0\rangle = \Psi_0 \Phi_0.$$

$$\langle 0|\phi(t,x) a_k^+|0\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}\, e^{-i\omega_k t + ikx},$$

$$\langle 0|\phi(t,x) B_k|0\rangle = \frac{i}{(2\pi)^{\frac{3}{2}}\sqrt{2\Omega_k}}\, e^{-\Omega_k t + ikx} \Longrightarrow 0$$

Commutator

$$[\phi(t,x),\phi(t',x')] = \epsilon(t-t')D(t-t',x-x'),$$

$$D(t,x) = -i\int \frac{d^3k}{(2\pi)^3}\left[\theta_{k^2-m^2}\frac{\sin(\omega_k t)}{\omega_k} + \theta_{m^2-k^2}\frac{\sinh(\Omega_k t)}{\Omega_k}\right]e^{ikx}$$

$$= -\frac{1}{2|x|}\int\limits_{-\infty}^{\infty} ds\frac{s\sin(\sqrt{s^2-m^2}\;t)}{\sqrt{s^2-m^2}}e^{is|x|} = 0 \quad \text{for} \quad |x|>|t|$$

$$D_\mu(t,x) = \frac{1}{2\pi}\left[\delta(t^2-x^2) - \mu^2\theta(t^2-x^2)\frac{J_1(\mu\sqrt{t^2-x^2})}{2\mu\sqrt{t^2-x^2}}\right]$$

$$\xrightarrow{\mu\to -im} \frac{1}{2\pi}\left[\delta(t^2-x^2) + m^2\theta(t^2-x^2)\frac{I_1(m\sqrt{t^2-x^2})}{2m\sqrt{t^2-x^2}}\right].$$

Propagator of the tachyon

$$\mathbf{T}(\phi(t,x)\phi(t',x')) = D_c(t-t',x-x') + \vdots\phi(t,x)\phi(t',x')\vdots,$$

$$D_c(t,x) = \int \frac{d^3k}{(2\pi)^3}\left[\theta_{k^2-m^2}\frac{e^{-i\omega_k|t|}}{2\omega_k} + i\theta_{m^2-k^2}\frac{e^{-\Omega_k|t|}}{2\Omega_k}\right]e^{ikx}$$

$$= \int\limits_{C_c} \frac{d^4k}{(2\pi)^4 i}\cdot\frac{e^{-ikx}}{(-k^2-m^2-i0)}, \quad k^2 = k_0^2 - k^2.$$

$$C_c = \{k_0 : -\infty < k_0 < \infty\}$$

Causality

I. Lorenz transformation

$$\begin{cases} \mathbf{x} = \frac{\mathbf{x}' - \mathbf{v}\mathbf{t}'}{\sqrt{1-\beta^2}} \\ \mathbf{t} = \frac{\mathbf{t}' - \beta \frac{\mathbf{x}'}{\mathbf{c}}}{\sqrt{1-\beta^2}} \end{cases} \Rightarrow \quad \mathbf{V} = \frac{\mathbf{V}' + \mathbf{v}}{1 + \frac{\mathbf{v}\mathbf{V}'}{\mathbf{c}^2}}, \begin{cases} \mathbf{V} < \mathbf{c} & \Longrightarrow & \mathbf{V}' < \mathbf{c}; \\ \mathbf{V} > \mathbf{c} & \Longrightarrow & \mathbf{V}' > \mathbf{c}. \end{cases}$$

II. The Cauchy problem

$$\left[\left(\frac{\partial}{\partial \mathbf{t}}\right)^2 - \left(\frac{\partial}{\partial \mathbf{x}}\right)^2 \pm \mathbf{m}^2\right] \mathbf{u}(\mathbf{t},\mathbf{x}) = \mathbf{0}.$$

The character of the equation is defined by the differential operator and does not depend on the sign of the constant term $\pm\mathbf{m}^2$.

The Cauchy problem is correctly formulated for any sign of $\pm\mathbf{m}^2$.

$$\mathbf{u}(\mathbf{t},\mathbf{x}) = \theta(\mathbf{c}^2\mathbf{t}^2 - \mathbf{x}^2)\begin{cases} \sim \mathbf{e}^{\pm \mathbf{imt}}, & +\mathbf{m}^2, \quad \text{stable oscillations} \\ \sim \mathbf{e}^{+\mathbf{mt}}, & -\mathbf{m}^2, \quad \text{unstable movement} \end{cases}$$

Bogoliubov causality

$$\frac{\delta}{\delta g(x)}\left(\frac{\delta}{\delta g(y)} S[g]\cdot S^{+}[g]\right) = 0 \quad \text{for} \quad \begin{cases} x_0 < y_0, & (x-y)^2 > 0; \\ x \sim y, & (x-y)^2 < 0. \end{cases}$$

This requirement is equivalent to

$$D_{ret}(x-y) = 0 \quad \text{for} \quad \begin{cases} x_0 < y_0, & (x-y)^2 > 0; \\ x \sim y, & (x-y)^2 < 0. \end{cases}$$

$$D_{ret}(x) = \frac{1}{2\pi}\theta(t)\left[\delta(x^2) - \theta(x^2)\mu^2 \frac{J_1(\mu\sqrt{x^2})}{2\mu\sqrt{x^2}}\right]. \qquad x^2 = t^2 - r^2.$$

$$\mu \;\longrightarrow\; -im$$

$$D_{ret}(x) = \frac{1}{2\pi}\theta(t)\left[\delta(x^2) + \theta(x^2)m^2 \frac{I_1(m\sqrt{x^2})}{2m\sqrt{x^2}}\right].$$

The tachyon field does not break the causality.

Conclusion

I. Asymptotic tachyon $\Longrightarrow$ only stable components $\phi_{as}(t, x) = \phi_{st}(t, x)$

$$\langle 0|\phi(t,x)a_k^+|0\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}\, e^{-i\omega_k t + ikx}, \qquad \omega_k^2 = k^2 - m^2 > 0.$$

The unstable tachyon components contribute to propagator

$$D_c(t,x) = \int \frac{d^4q}{(2\pi)^4 i} \cdot \frac{e^{-iqx}}{q^2 + m^2 + i0}, \quad q^2 = q_0^2 - q^2$$

The S-matrix $\Longrightarrow$ sum of Feynman diagrams.

II. Quantization: electro-magnetic field [physical (transverse) and two nonphysical (longitudinal and time) components].
The nonphysical components $\Longrightarrow$ causal propagator.

III. Instability of the world with tachyon

$$\Phi \implies 2T_{tachyon}$$

Conservation of energy in the rest system of the particle Φ with mass M:

$$M = 2\sqrt{k^2 - m^2} \implies k = \sqrt{\frac{M^2}{4} + m^2} > m$$

Any massive particles with mass M are unstable
or
the tachyon should have special quantum numbers.