Anisotropic plane oscillator in magnetic field

Regulated hidden symmetry

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Problem about charged symmetric plane oscillator in the magnetic field was solved by V.A. Fock (Zs. f. Phys. **47** (1928) 446-448). Two years later for the particular case without oscillator field this result was published by L.D. Landau. The corresponding energy levels are known as Landau levels. Anisotropic case was considered by Margulis et al. (1996,2004) and T.K. Rebane, ZheTF, 2012. Simple expressions for wave functions were obtained only recently.

Pedagogical. Solution of QM problem

At $B=0$ the Hamiltonian reads $\hat{H}=\dfrac{\hat{p}_x^2+\hat{p}_y^2}{2m}+\dfrac{m\omega_1^2x^2}{2}+\dfrac{m\omega_2^2y^2}{2}$.
Let magnetic field is directed along z-axis. Simplest description is in gauge $\boldsymbol{A}=(0,\,Bx,\,0)$. Now ($\omega_B=eB/mc$)

$$\hat{H}=\frac{\hat{p}_z^2}{2m}+\hat{H}_\perp, \quad \hat{H}_\perp=\frac{\hat{p}_x^2}{2m}+\frac{\hat{p}_y^2}{2m}-\omega_B x\hat{p}_y+\frac{m(\omega_1^2+\omega_B^2)x^2}{2}+\frac{m\omega_2^2y^2}{2}.$$

Problem – mixed term $x\hat{p}_y$.

Solution – 1) canonical transformation $\widehat{Y} = -\widehat{p}_y/(m\omega_2)$, $\widehat{p}_Y = ym\omega_2$.

$$\Rightarrow \quad \widehat{H}_\perp = \frac{\widehat{p}_x^2}{2m} + \frac{\widehat{p}_Y^2}{2m} + m\omega_B\omega_2 x\widehat{Y} + \frac{m(\omega_1^2+\omega_B^2)x^2}{2} + \frac{m\omega_2^2\widehat{Y}^2}{2}\,.$$

2) Subsequent diagonalization is simple rotation: $\begin{pmatrix} x \\ Y \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

($c = \cos\theta$, $s = \sin\theta$) and the same transformation for $\widehat{p}_x$ and $\widehat{p}_Y$.

$$\widehat{H} = \frac{\widehat{p}_1^2}{2m} + \frac{m\Omega_1^2\widehat{x}_1^2}{2} + \frac{\widehat{p}_2^2}{2m} + \frac{m\Omega_2^2\widehat{x}_2^2}{2} \equiv \hbar\Omega_1\left(\hat{a}_1^+\hat{a}_1 + \frac{1}{2}\right) + \hbar\Omega_2\left(\hat{a}_2^+\hat{a}_2 + \frac{1}{2}\right)$$

with

$$\Omega_{1,2} = \frac{1}{2}\left[\sqrt{(\omega_1+\omega_2)^2+\omega_B^2} \pm \sqrt{(\omega_1-\omega_2)^2+\omega_B^2}\right],$$

$$E_{\perp(n_1 n_2)} = \hbar\Omega_1(n_1+1/2) + \hbar\Omega_2(n_2+1/2)\,.$$

The annihilation operators in the basic coordinates are

$$\hat{a}_1 = \frac{1}{2}\sqrt{\frac{m\Omega_1}{\hbar}}\left(cx - is\frac{\hbar}{m\omega_2}\frac{d}{dy} + c\frac{\hbar}{m\Omega_1}\frac{d}{dx} + isy\frac{\omega_2}{\Omega_1}\right),$$

$$\hat{a}_2 = \frac{1}{2}\sqrt{\frac{m\Omega_2}{\hbar}}\left(-sx - ic\frac{\hbar}{m\omega_2}\frac{d}{dy} - s\frac{\hbar}{m\Omega_2}\frac{d}{dx} + icy\frac{\omega_2}{\Omega_2}\right).$$

The wave function of ground state is found from equations

$$\hat{a}_1\psi_{00}(x,y) = \hat{a}_2\psi_{00}(x,y) = 0 \;\Rightarrow$$

$$\psi_{00}(x,y) = \left(\frac{4c^2\omega_1\omega_2}{\pi^2}\right)^{1/4} \exp\left(-c(\omega_1 x_1^2 + \omega_2 x_2^2) + ic_{12}xy\right)$$

where $c = \dfrac{m\sqrt{\omega_B^2 + (\omega_1+\omega_2)^2}}{2\hbar(\omega_1+\omega_2)}$, $\quad c_{12} = \dfrac{m\omega_B\omega_2}{\hbar(\omega_1+\omega_2)} \equiv \dfrac{eB\omega_2}{\hbar c(\omega_1+\omega_2)}$.

For other wave function we have

$$\psi_{n_1 n_2} = (\hat{a}_1^{+})^{n_1} (\hat{a}_2^{+})^{n_2} \psi_{00} / \sqrt{n_1! n_2!}\,.$$

Note that in the gauge (T.K. Rebane)

$$\boldsymbol{A}_R = B(-\frac{\omega_2}{\omega_1 + \omega_2} y,\, \frac{\omega_1}{\omega_1 + \omega_2} x,\, 0)$$

term c_{12} is absent.

Hidden symmetries

With variation of a magnetic field our system passes through states with hidden symmetries.

If magnetic field is such that $\Omega_1 = r\Omega_2$ with $r = m/n$, i.e. at

$$B = B_r = \frac{mc}{e}\sqrt{\omega_1\omega_2(r-1)^2/r - (\omega_1 - \omega_2)^2}\,,$$

the states become degenerated and additional conserved operators appear

$$\widehat{C}_2^{(1/r)} = \left(\widehat{a}_{1z}^{+}\right)^n (\widehat{a}_{2z})^m\,, \quad \widehat{C}_1^{(2/r)} = \left(\widehat{a}_{2z}^{+}\right)^m (\widehat{a}_{1z})^n\,.$$

In particular, to get $r = 2$ we should have the magnetic field

$$B = B_2 = \frac{mc}{e}\sqrt{\omega_1\omega_2/2 - (\omega_1 - \omega_2)^2}$$

and the additional conserved operators read

$$\widehat{C}_2^{(1/2)} = \hat{a}_{1z}^{+}\,(\hat{a}_{2z})^2\,, \quad \widehat{C}_1^{(2/2)} = \left(\hat{a}_{2z}^{+}\right)^2\hat{a}_{1z}.$$

In this case $E_N = \hbar\Omega_2(N + 3/2)$ (with $N = 2n_1 + n_2$). Degree of degeneracy is $n + 1$ for both $N = 2n$ and $N = 2n + 1$.

For most of results the basic anisotropy is inessential, so that many problems one can treat at $\omega_1 = \omega_2$. In this case $\Omega_1 - \Omega_2 = \omega_B$. $\Rightarrow$ Ω_1 describes rotation in the same direction as given by field, Ω_2 – in the opposite direction.

In the real system, the basic oscillator potential is an approximation valid for distances smaller than some R. The levels $E_n = \hbar\omega(n + 1/2)$ can be described by a harmonic approximation at such n that the size of wave function $x_0\sqrt{n} < R$ (here $x_0 = \sqrt{\hbar/(m\omega)}$).

If $r = 1 + 1/p$, first really degenerated state corresponds to $n \approx p$. At large p (r close to 1), it can be beyond harmonic region for potential.

Small unharmonic terms in the potential like $\varepsilon x^2 y$ result in redistribution of energy among modes like at Fermi resonance when magnetic field pass through the mentioned critical value.

How to observe?

We invite proposals.

1. The IR wave after passing through a plane with such oscillator acquires a particular circular polarization according to the state (n_1, n_2) of electron in a magnetic field. At the variation of magnetic field beyond resonance, this polarization varies weakly. After passing through resonance value, this polarization state can changes strongly.

Antioscillator (potential hill) Questions

Let us consider initially unstable system

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} - \frac{m\omega_1^2 x^2}{2} - \frac{m\omega_2^2 y^2}{2}.$$

Whether magnetic field can stabilize this system (due to final value of Larmor radius)? To simplify equations, we will write all only for basically symmetric case.

The same construction as above results in Hamiltonian at large enough magnetic field

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{m\Omega_1^2 \hat{x}_1^2}{2m} - \left(\frac{\hat{p}_2^2}{2m} + \frac{m\Omega_2^2 \hat{x}_2^2}{2m} \right)$$

with

$$E_{n_1 n_2} = \hbar\Omega_1(n_1 + 1/2) - \hbar\Omega_2(n_2 + 1/2)\,,$$

$$\Omega_{1,2} = \frac{1}{2}\left[\omega_B \pm \sqrt{\omega_B^2 - 4\omega^2}\right]$$

1. Levels are discrete, but there is no ground state
2. Let $\Omega_1/\Omega_2 = r$.

If $r = p/q$ is rational number, states are infinitely degenerated.
If r is irrational number, degeneracy is absent.

Where such system may be realized?

Facts: All troubles are related to large distances from the top of hill where harmonic approximation is broken. For the states localized around this top our approximation may be good enough.

The possibilities

Let our potential is transformed to the approximately constant at $R > A$.

One can consider the degenerated electron gas with Fermi level slightly below 0. In this case all negative levels are occupied and some new electron excitations correspond $E \sim 0$. They may be described by our hamiltonian.

B. One can try to invent new set of problems. For systems, having no regular ground state (antioscillator, attraction like g/r^4, etc. one can consider class of physical phenomena, for which these singular effects are irrelevant.