Critical Nucleus Charge in a Superstrong Magnetic Field: Effect of Screening

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(Phys. Rev. D 85, 044058 (2012), http://arxiv.org/abs/1112.1891)

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System of units

$$\hbar = c = 1$$

Gauss units:

$$e^2 = \alpha = 1/137.03599...$$

Magnetic fields:

$$B_a = m_e^2 e^3 = 2.3 \cdot 10^9 G$$

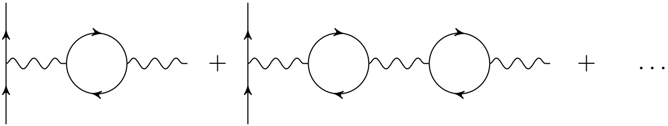
$$B_0 = m_e^2 / e = 4.4 \cdot 10^{13} G$$

Potential of the pointlike charge in a superstrong magnetic field

One loop contribution

(A.E.Shabad, V.V.Usov, Phys. Rev. D. 77 (2008), 025001)

(B.Machet, M.I.Vysotsky, Phys. Rev. D 83 (2011), 025022)



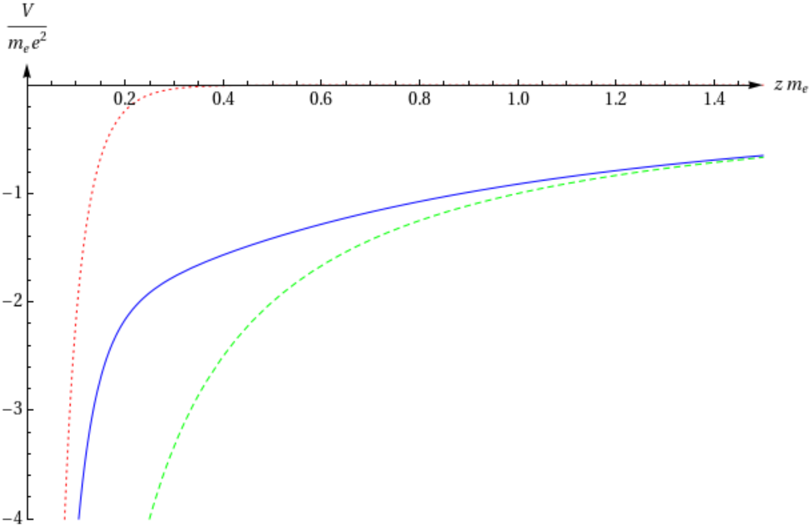
$$\Phi(z,\rho) = 4\pi e \int \frac{e^{i\bar{k}_\perp \bar{\rho} + ik_\parallel z} dk_\parallel d^2 k_\perp/(2\pi)^3}{k_\parallel^2 + k_\perp^2 + \frac{2e^3 B}{\pi} \exp(-\frac{k_\perp^2}{2eB}) \frac{k_\parallel^2}{6m_e^2 + k_\parallel^2}}$$

Potential along magnetic field

$$\mathbf{\Phi}(z,0) = \frac{e}{|z|}\left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3B+6m_e^2}|z|}\right] \quad .$$

$$\mathbf{\Phi}(z,0) = \begin{cases} \frac{e}{|z|}e^{(-\sqrt{(2/\pi)e^3B}|z|)} & , \quad |z| < \frac{1}{\sqrt{(2/\pi)e^3B}}\ln\left(\sqrt{\frac{e^3B}{3\pi m_e^2}}\right) \\ \frac{e}{|z|}(1 - e^{(-\sqrt{6m_e^2}|z|)} & , \quad \frac{1}{\sqrt{(2/\pi)e^3B}}\ln\left(\sqrt{\frac{e^3B}{3\pi m_e^2}}\right) < |z| < \frac{1}{m_e} \\ \frac{e}{|z|} & , \quad \frac{1}{m_e} < |z| \end{cases}$$

Modified potential along z axis ($B = 5 \cdot 10^4 B_0$)



Potential in the transverse plane

(Numerically obtained in the paper A.E. Shabad, V.V. Usov, Phys. Rev.D, 77, 025001)

$$\Phi(z,\rho) = 4\pi e \int \frac{e^{i\bar{k}_\perp \bar{\rho} + i k_\parallel z} dk_\parallel d^2 k_\perp / (2\pi)^3}{k_\parallel^2 + k_\perp^2 + \frac{2e^3 B}{\pi} \exp(-\frac{k_\perp^2}{2eB}) \frac{k_\parallel^2}{6m_e^2 + k_\parallel^2}} \ .$$

$$\Downarrow \text{ at } B \gg m^2/e^3$$

$$\Phi(0,\rho) = \begin{cases} \frac{e}{\rho} \exp(-\sqrt{(2/\pi) e^3 B}\rho) & , \quad \rho < \frac{1}{\sqrt{(2/\pi e^3 B)}} \ln \sqrt{\frac{e^3 B}{3\pi m_e^2}} \\ \sqrt{\frac{3\pi m_e^2}{e^3 B}} \frac{e}{\rho} & , \quad \frac{1}{\sqrt{(2/\pi e^3 B)}} \ln \sqrt{\frac{e^3 B}{3\pi m_e^2}} < \rho \end{cases} ,$$

Potential at large distances

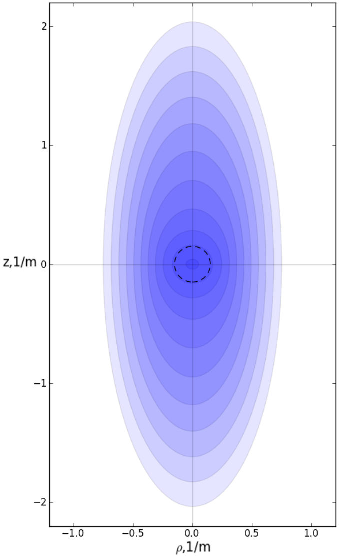
At large distances $z \gg 1/m_e$ $(k_\parallel \ll m_e)$

$$\Phi(\vec{\rho}, z) = \int\limits_{-\infty}^{+\infty} \frac{dk_\parallel}{2\pi} e^{ik_\parallel z} \int \frac{d^2 k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \vec{\rho}} \frac{4\pi e}{\left(1 + \frac{e^3 B}{3\pi m_e^2}\right) k_\parallel^2 + k_\perp^2} =$$

$$= \frac{e}{\sqrt{z^2 + \left(1 + \frac{e^3 B}{3\pi m_e^2}\right) \rho^2}}.$$

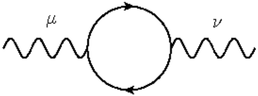
So equipotential lines are ellipses.

Equipotential lines



Vacuum polarization

$$G(k) = e^{-k_\perp^2/eB}(1 - i\gamma_1\gamma_2)\frac{\hat{k}_{0,3} + m_e}{k_{0,3}^2 - m_e^2}$$



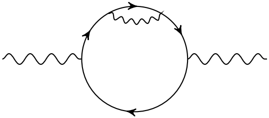
Integral over transverse momentum:

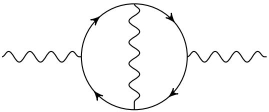
$$\int d^2k_\perp e^{-k_\perp^2/eB} \sim eB$$

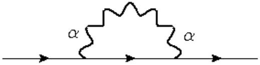
As a result we have:

$$\Pi_{\mu\nu} \simeq e^3 B \exp\left(-\frac{q_\perp^2}{eB}\right)\Pi_{\mu\nu}^{(2)}(q_\parallel)$$

Higher loops







Corresponding integral:

$$\int d^2k_\perp d^2k_\parallel e^{-k_\perp^2/eB}\gamma_\alpha(1-i\gamma_1\gamma_2)\frac{\hat{k}_{0,3}+m_e}{k_{0,3}^2-m_e^2}\gamma_\alpha\frac{1}{(k-q)_\parallel^2-(k-q)_\perp^2-m_e^2}$$

Matrix expression:

$$\gamma_\mu(1-i\gamma_1\gamma_2)\hat{k}_{0,3}\gamma_\mu = -2[\hat{k}_{0,3}-i\hat{k}_{0,3}\gamma_2\gamma_1] = -2\hat{k}_{0,3}(1+i\gamma_1\gamma_2) \ ,$$

But $(1+i\gamma_1\gamma_2)(1-i\gamma_1\gamma_2)=0!$

So the initial integral is proportional to m_e and not $\sqrt{eB}$.

Energy levels in the modified potential

Energy levels of hydrogen atom with the account of screening (Schrödinger)

(B.M. Karnakov, V.S. Popov, JETP, 97, 890)
Karnakov–Popov equation:

$$\ln\left(\frac{B}{B_a}\right) = \lambda + 2\ln\lambda + 2\psi\left(1-\frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1+|m|)$$

where λ defines corresponding energy $E_m = -(m_e e^4/2)\lambda^2$

(B.Machet, M.I.Vysotsky, Phys. Rev. D 83 (2011), 025022)
With the account of screening for ground energy level we get:

$$\ln\left(\frac{B/B_a}{1+\frac{e^6}{3\pi}\frac{B}{B_a}}\right) = \lambda + 2\ln\lambda + 2\psi(1-\frac{1}{\lambda}) + \ln 2 + 3\gamma$$

For $B \to \infty$ we get $\lambda_\infty \approx 11.21$ or $E_0^\infty = -1.7keV$.

Why do we need to solve Dirac equation?

- For superstrong magnetic fields $a_H = \frac{1}{\sqrt{eB}} \ll \frac{1}{m_e} \Rightarrow$ relativistic effects can be important
- For large Z binding energy becomes comparable to the electron mass

Used method

For the fields $B \gg m^2 e^3 Z^2$ $(a_H = \frac{1}{\sqrt{eB}} \ll \frac{1}{m_e Z e^2} = a_B)$

$$(\alpha(\mathbf{p} + \mathbf{eA}) + V + \beta)\psi = \varepsilon\psi$$

$$\Downarrow$$

$$\frac{d^2\chi}{dz^2} + 2m_e(E - U)\chi = 0 \ ,$$

$$\begin{aligned} E &= \frac{\varepsilon^2 - m_e^2}{2m_e}, \\ U &= \frac{\varepsilon}{m_e}\bar{V} - \frac{1}{2m_e}\bar{V}^2 + \frac{\bar{V}''}{4m_e(\varepsilon + m_e - V)} + \frac{3/8(\bar{V}')^2}{m_e(\varepsilon + m_e - V)^2} \ , \\ \bar{V} &= \frac{1}{a_H^2}\int\limits_0^\infty V(\sqrt{\rho^2 + z^2}) \exp\left(-\frac{\rho^2}{2a_H^2}\right)\rho d\rho \end{aligned}$$

Results for $Z = 1$ without taking screening into account

$$\bar{V}(z) = -\frac{Ze^2}{a_H}\sqrt{\frac{\pi}{2}}\exp\left(\frac{z^2}{2a_H^2}\right)\operatorname{erfc}\left(\frac{|z|}{\sqrt{2}a_H}\right)$$

$$E = \frac{\varepsilon^2 - m_e^2}{2m_e} \equiv -\frac{m_e e^4}{2}\lambda^2$$

For λ we get:

$\frac{B}{B_0}$	Karnakov–Popov equation (Schrödinger)	Numerical results (Schrödinger)	Numerical results (Dirac)
10^0	5.736	5.736	5.734
10^1	7.373	7.374	7.371
10^2	9.141	9.142	9.136
10^3	11.00	11.00	10.99
10^4	12.93	12.93	12.91
10^5	14.91	14.91	14.88
10^6	16.93	16.93	16.89
10^7	18.98	18.98	18.92
10^8	21.06	21.06	20.98
10^9	23.16	23.15	23.05
10^{10}	25.27	25.27	25.14

Results for $Z = 1$ with the account of screening

$$\bar{V}(z) = -\frac{Ze^2}{a_H}\left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3B+6m_e^2}|z|}\right]\sqrt{\frac{\pi}{2}}\exp\left(\frac{z^2}{2a_H^2}\right)\mathrm{erfc}\left(\frac{|z|}{\sqrt{2}a_H}\right)$$

For λ we get:

$\frac{B}{B_0}$	Machet–Vysotsky formula (Schrödinger)	Numerical results (Schrödinger)	Numerical results (Dirac)
10^0	5.736	5.735	5.734
10^1	7.367	7.371	7.368
10^2	9.082	9.095	9.090
10^3	10.53	10.584	10.575
10^4	11.11	11.231	11.219
10^5	11.20	11.357	11.346
10^6	11.21	11.381	11.369
10^7	11.21	11.386	11.375
10^8	11.21	11.387	11.376

Pointlike Coulomb center with charge $Z > 137$ can not exist.

1945-1970

But if the finite size of a nucleus is taken into account then Dirac equation can be solved for $Z > 137$. At $Z = Z_{cr} \approx 170$ ground energy level reaches lower continuum ($\varepsilon = -m_e$) and spontaneous electron-positron pair production occurs. Electrons occupy ground level and one can observe two free positrons.

Critical charge in a magnetic field

V.N. Oraevskii, A.I. Rez, V.B. Semikoz, JETP, Vol. 45, No 3, p. 428 (March 1977)

Solution found for $B \gg m^2 e^3 Z^2$; $B \gg \frac{m^2}{e(Ze^2)^2}$

$$Ze^2 \ln\left(2\frac{\sqrt{m_e^2-\varepsilon^2}}{\sqrt{eB}}\right) + \arctan\left(\sqrt{\frac{m_e+\varepsilon}{m_e-\varepsilon}}\right) + \arg\Gamma\left(-\frac{Ze^2\varepsilon}{\sqrt{m_e^2-\varepsilon^2}} + iZe^2\right) - \arg\Gamma(1+2iZe^2) - \frac{Ze^2}{2}(\ln 2+\gamma) = \frac{\pi}{2} + n\pi \quad ,$$

for ground energy level $n = 0$ for $\varepsilon > 0$ and $n = 1$ for $\varepsilon < 0$.

Critical charge:

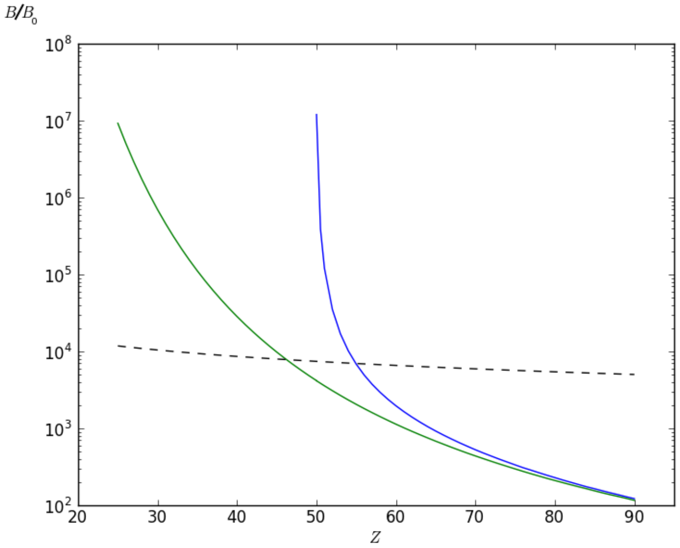
$$\frac{B}{B_0} = 2(Z_{cr}e^2)^2 \exp\left(-\gamma + \frac{\pi - 2\arg\Gamma(1+2iZ_{cr}e^2)}{Z_{cr}e^2}\right).$$

B/B_0	10^2	10^3	$2\cdot 10^4$
Z_{cr}	96	61	41

Results for $Z = 40$

$\frac{B}{B_0}$	ε/m_e (ORS equation)	ε/m_e (Numerical results)	ε/m_e (Numerical results with the account of screening)
10^0	0.819	0.850	0.850
10^1	0.653	0.667	0.667
10^2	0.336	0.339	0.346
10^3	-0.158	-0.159	-0.0765
10^4	-0.758	-0.759	-0.376
$2 \cdot 10^4$	-0.926	-0.927	-0.423
...	at $B/B_0 \approx 2.85 \cdot 10^4, \; \varepsilon = -m_e$		...
10^5	—	—	-0.4887
10^6	—	—	-0.5241
10^7	—	—	-0.5351
10^8	—	—	-0.5386
10^9	—	—	-0.5397

Critical nucleus charge with the account of screening



Conclusions

- Analytical formula for the modified potential in the transverse plane was derived;
- There is no large contributions to the potential from higher loops;
- For $Z = 1$ relativistic effects are not important;
- For large Z the effect of screening significantly changes the energy of the ground level;
- Only ions with $Z \gtrsim 52$ can reach criticality.

Thanks for attention!