Can effects of quantum gravity be observed in the cosmic microwave background?

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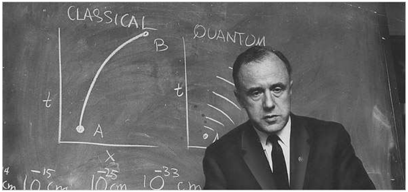
Conservative route to quantum gravity

‘Quantization’:

- The mechanical Hamilton–Jacobi equation leads to the Schrödinger equation;
- the gravitational Hamilton–Jacobi equation (Peres 1962) leads to the Wheeler–DeWitt equation.

Quantum geometrodynamics



- Question: what is the quantum wave equation that immediately gives Einstein's equations in the semiclassical limit?
- Answer: the Wheeler–DeWitt equation

$$\boxed{\hat{H}\Psi = 0}$$

Constraints of this type also occur in loop quantum gravity

Semiclassical (Born–Oppenheimer) approximation

Ansatz:

$$|\Psi[h_{ab}]\rangle = C[h_{ab}]\mathrm{e}^{\mathrm{i}m_{\mathrm{P}}^2 S[h_{ab}]}|\psi[h_{ab}]\rangle$$

and expansion with respect to the Planck-mass squared.

Highest order: One evaluates $|\psi[h_{ab}]\rangle$ along a solution of the classical Einstein equations, $h_{ab}(\mathbf{x}, t)$, corresponding to a solution, $S[h_{ab}]$, of the Hamilton–Jacobi equations;

$$\dot{h}_{ab} = N G_{abcd} \frac{\delta S}{\delta h_{cd}} + 2D_{(a} N_{b)}$$

$$\frac{\partial}{\partial t}\,|\psi(t)\rangle := \int \mathrm{d}^3x\,\dot{h}_{ab}(\mathbf{x},t)\,\frac{\delta}{\delta h_{ab}(\mathbf{x})}|\psi[h_{ab}]\rangle$$

This leads to a functional Schrödinger equation for quantized matter fields in the chosen external classical gravitational field:

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle &= \hat{H}^{\mathrm{m}}|\psi(t)\rangle \\ \hat{H}^{\mathrm{m}} &:= \int \mathrm{d}^3x\left\{N(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_{\perp}(\mathbf{x}) + N^a(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_a(\mathbf{x})\right\} \end{aligned}$$

$\hat{H}^{\mathrm{m}}$: matter-field Hamiltonian in the Schrödinger picture, parametrically depending on (generally non-static) metric coefficients of the curved space–time background.

WKB time t controls the dynamics in this approximation

Quantum gravitational corrections

The next order in the Born–Oppenheimer approximation gives

$$\hat{H}^{\mathrm{m}} \to \hat{H}^{\mathrm{m}} + \frac{1}{m_{\mathrm{P}}^2} \times (\text{various terms})$$

(C. K. and T. P. Singh (1991); A. O. Barvinsky and C. K. (1998))

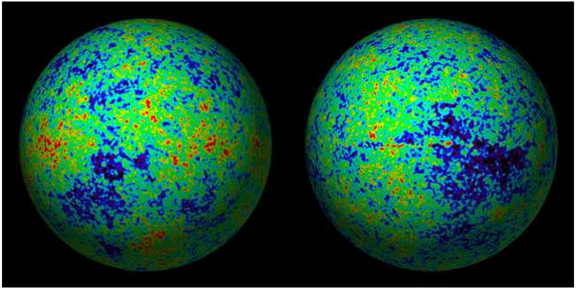
Example: Quantum gravitational correction to the trace anomaly in de Sitter space:

$$\delta\epsilon \approx -\frac{2G\hbar^2 H_{\mathrm{dS}}^6}{3(1440)^2\pi^3 c^8}$$

(C.K. 1996)

Observations

Does the anisotropy spectrum of the cosmic background radiation contain information about quantum gravity?



C.K. and M. Krämer, *Phys. Rev. Lett.*, **108**, 021301 (2012); see also our first-prize winning essay for the Gravity Research Foundation.

Minisuperspace background

Wheeler–DeWitt equation for small fluctuations in a flat Friedmann–Lemaître universe with scale factor $a \equiv \exp(\alpha)$ and inflaton field ϕ

Choose the simplest potential:

$$\mathcal{V}(\phi) = \frac{1}{2} m^2 \phi^2 ;$$

any other potential obeying at the classical level the slow-roll condition $\dot{\phi}^2 \ll |\mathcal{V}(\phi)|$ should lead to similar results.

Minisuperspace Wheeler–DeWitt equation

$$\mathcal{H}_0\Psi_0(\alpha,\phi) \equiv \frac{\mathrm{e}^{-3\alpha}}{2}\left[\frac{1}{m_{\mathrm{P}}^2}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + \mathrm{e}^{6\alpha}m^2\phi^2\right]\Psi_0(\alpha,\phi) = 0$$

- $\hbar = c = 1$
- $m_{\mathrm{P}} = \sqrt{3\pi/2G} \approx 2.65 \times 10^{19}$ GeV
- $\phi \to \phi/\sqrt{2}\pi$
- assume in the following $\partial^2\Psi_0/\partial\phi^2 \ll \mathrm{e}^{6\alpha}m^2\phi^2\Psi_0$ and substitute $m\phi$ by $m_{\mathrm{P}}H$, where H is the quasistatic Hubble parameter of inflation (Born–Oppenheimer approximation)

Introduction of inhomogeneities

$$\phi \to \phi(t) + \delta\phi(\mathbf{x}, t)$$

Perform a decomposition into Fourier modes with wave vector $\mathbf{k}$, $k \equiv |\mathbf{k}|$,

$$\delta\phi(\mathbf{x}, t) = \sum_k f_k(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,.$$

The Wheeler-DeWitt equation including the fluctuation modes then reads (Halliwell and Hawking 1985)

$$\left[\mathcal{H}_0 + \sum_{k=1}^{\infty} \mathcal{H}_k\right] \Psi\big(\alpha, \phi, \{f_k\}_{k=1}^{\infty}\big) = 0$$

with

$$\mathcal{H}_k = \frac{1}{2}\, \mathrm{e}^{-3\alpha} \left[-\frac{\partial^2}{\partial f_k^2} + \left(k^2\, \mathrm{e}^{4\alpha} + m^2\, \mathrm{e}^{6\alpha}\right) f_k^2\right]$$

$$\mathcal{H}_k = \frac{1}{2}\, \mathrm{e}^{-3\alpha} \left[- \frac{\partial^2}{\partial f_k^2} + \left(k^2\, \mathrm{e}^{4\alpha} + m^2\, \mathrm{e}^{6\alpha} \right) f_k^2 \right]$$

Ansatz:

$$\Psi\left(\alpha, \phi, \{f_k\}_{k=1}^{\infty}\right) = \Psi_0(\alpha, \phi) \prod_{k=1}^{\infty} \widetilde{\Psi}_k(\alpha, \phi, f_k)\,.$$

The components $\Psi_k(\alpha, \phi, f_k) := \Psi_0(\alpha, \phi)\widetilde{\Psi}_k(\alpha, \phi, f_k)$ obey

$$\frac{1}{2}\, \mathrm{e}^{-3\alpha} \left[\frac{1}{m_{\mathrm{P}}^2}\, \frac{\partial^2}{\partial \alpha^2} + \mathrm{e}^{6\alpha}\, m_{\mathrm{P}}^2\, H^2 - \frac{\partial^2}{\partial f_k^2} + W_k(\alpha) f_k^2 \right] \Psi_k(\alpha, \phi, f_k) = 0$$

with

$$W_k(\alpha) := k^2\, \mathrm{e}^{4\alpha} + m^2\, \mathrm{e}^{6\alpha}\,,$$

Born–Oppenheimer approximation

Following the general scheme, we make the ansatz

$$\Psi_k(\alpha, f_k) = \mathrm{e}^{\mathrm{i}\, S(\alpha, f_k)}$$

and expand $S(\alpha, f_k)$ in terms of powers of m_P^2,

$$S(\alpha, f_k) = m_\mathrm{P}^2\, S_0 + m_\mathrm{P}^0\, S_1 + m_\mathrm{P}^{-2}\, S_2 + \ldots$$

We insert this ansatz into the full Wheeler–DeWitt equation and compare consecutive orders of m_P^2.

- $\mathcal{O}(m_{\mathrm{P}}^4)$: S_0 is independent of f_k
- $\mathcal{O}(m_{\mathrm{P}}^2)$: S_0 obeys the Hamilton–Jacobi equation

$$\left[\frac{\partial S_0}{\partial \alpha}\right]^2 - V(\alpha) = 0\,, \quad V(\alpha) := \mathrm{e}^{6\alpha} H^2$$

  solved by $S_0(\alpha) = \pm \mathrm{e}^{3\alpha} H/3$

- $\mathcal{O}(m_{\mathrm{P}}^0)$: Write $\psi_k^{(0)}(\alpha, f_k) \equiv \gamma(\alpha)\, \mathrm{e}^{\mathrm{i}\, S_1(\alpha, f_k)}$ and impose a condition on $\gamma(\alpha)$ that makes it equal to the standard WKB prefactor. After introducing the 'WKB time' according to

$$\frac{\partial}{\partial t} := -\, \mathrm{e}^{-3\alpha} \frac{\partial S_0}{\partial \alpha}\, \frac{\partial}{\partial \alpha}\,,$$

  one finds that each $\psi_k^{(0)}$ obeys a Schrödinger equation,

$$\mathrm{i}\, \frac{\partial}{\partial t}\, \psi_k^{(0)} = \mathcal{H}_k \psi_k^{(0)}.$$

Quantum gravitational corrections

$\mathcal{O}(m_{\mathrm{P}}^{-2})$: decompose $S_2(\alpha, f_k)$ as

$$S_2(\alpha, f_k) \equiv \varsigma(\alpha) + \eta(\alpha, f_k)$$

and demand that $\varsigma(\alpha)$ be the standard second-order WKB correction. The wave functions

$$\psi_k^{(1)}(\alpha, f_k) := \psi_k^{(0)}(\alpha, f_k)\, \mathrm{e}^{\mathsf{i}\, m_{\mathrm{P}}^{-2}\, \eta(\alpha, f_k)}$$

then obey the quantum gravitationally corrected Schrödinger equation

$$\mathsf{i}\,\frac{\partial}{\partial t}\,\psi_k^{(1)} = \mathcal{H}_k \psi_k^{(1)} - \frac{\mathrm{e}^{3\alpha}}{2 m_{\mathrm{P}}^2 \psi_k^{(0)}}\left[\frac{\left(\mathcal{H}_k\right)^2}{V}\psi_k^{(0)} + \mathsf{i}\frac{\partial}{\partial t}\left(\frac{\mathcal{H}_k}{V}\right)\psi_k^{(0)}\right]\psi_k^{(1)}$$

Solution of the uncorrected Schrödinger equation

Ansatz:

$$\psi_k^{(0)}(t, f_k) = \mathcal{N}_k^{(0)}(t)\,\mathrm{e}^{-\frac{1}{2}\,\Omega_k^{(0)}(t)\,f_k^2}$$

This leads to

$$\begin{aligned}
\dot{\mathcal{N}}_k^{(0)}(t) &= -\frac{\mathrm{i}}{2}\,\mathrm{e}^{-3\alpha}\,\mathcal{N}_k^{(0)}(t)\,\Omega_k^{(0)}(t), \\
\dot{\Omega}_k^{(0)}(t) &= \mathrm{i}\,\mathrm{e}^{-3\alpha}\Big[-\big(\Omega_k^{(0)}(t)\big)^2 + W_k(t)\Big].
\end{aligned}$$

Solution:

$$\Omega_k^{(0)}(\xi) = \frac{k^3}{H^2\xi}\,\frac{1}{\xi - \mathrm{i}} + \mathcal{O}\left(\frac{m^2}{H^2}\right)$$

$$\xi(t) := k/(Ha(t))$$

Unperturbed power spectrum

In the slow-roll regime, the density contrast is given by

$$\delta_k(t) \approx \frac{\delta\rho_k(t)}{\mathcal{V}_0} = \frac{\dot{\phi}(t)\,\dot{\sigma}_k(t)}{\mathcal{V}_0}\,,$$

with

$$\sigma_k^2(t) := \left\langle \psi_k | f_k^2 | \psi_k \right\rangle = \sqrt{\frac{\Re\mathfrak{e}\,\Omega_k}{\pi}} \int_{-\infty}^{\infty} f_k^2\, \mathrm{e}^{-\frac{1}{2}[\Omega_k^*(t)+\Omega_k(t)] f_k^2}\, \mathsf{d} f_k = \frac{1}{2\,\Re\mathfrak{e}\,\Omega_k(t)}$$

$$\delta_k(t_{\text{enter}}) = \frac{4}{3}\,\frac{\mathcal{V}_0}{\dot{\phi}^2}\,\delta_k(t_{\text{exit}}) = \frac{4}{3}\,\frac{\dot{\sigma}_k(t)}{\dot{\phi}(t)}\bigg|_{t\,=\,t_{\text{exit}}}$$

$$\Delta^2_{(0)}(k) := 4\pi k^3 \left|\delta_k(t_{\text{enter}})\right|^2 \propto \frac{H^4}{\left|\dot{\phi}(t)\right|^2_{t_{\text{exit}}}}$$

(approximately scale-invariant power spectrum)

Solution of the corrected Schrödinger equation

Ansatz:

$$\psi_k^{(1)}(t, f_k) = \left(\mathcal{N}_k^{(0)}(t) + \frac{1}{m_{\rm P}^2}\,\mathcal{N}_k^{(1)}(t)\right) \times \exp\left[-\frac{1}{2}\left(\Omega_k^{(0)}(t) + \frac{1}{m_{\rm P}^2}\,\Omega_k^{(1)}(t)\right) f_k^2\right]$$

Inserting this into the corrected Schrödinger equation leads to

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\,\Omega_k^{(1)}(\xi) = \frac{2\,\mathrm{i}\,\xi}{\xi - \mathrm{i}}\,\Omega_k^{(1)}(\xi) + \frac{3\,\xi^3}{2}\,\frac{2\xi - \mathrm{i}}{(\xi - \mathrm{i})^3}\,,$$

which can be solved analytically up to a numerical integration.

Modification of the power spectrum

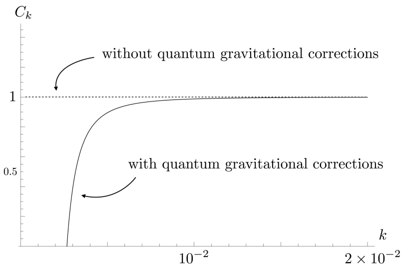


Figure: Size of the corrections for $H = 10^{14}$ GeV.

$$\Delta^2_{(1)}(k) = \Delta^2_{(0)}(k)\, C_k^2$$

$$C_k := \left(1 - \frac{43.56}{k^3}\,\frac{H^2}{m_{\mathsf{P}}^2}\right)^{-\frac{3}{2}}\left(1 - \frac{189.18}{k^3}\,\frac{H^2}{m_{\mathsf{P}}^2}\right)$$

introduces scale dependence

Discussion

- Effect is most pronounced for large scales
- Accuracy is fundamentally limit by cosmic variance
- Suppression of power for large scales
- From the non-observation of this effect, one finds the bound
$$H \lesssim 1.4 \times 10^{-2}\, m_{\mathsf{P}} \sim 4 \times 10^{17}\,\text{GeV}$$
- But there already exists a stronger constraint on this scale from the bound on the tensor-to-scalar ratio r :
$$H \lesssim 10^{-5} m_{\mathsf{P}} \sim 10^{14}\,\text{GeV}$$

Comparison with loop quantum cosmology

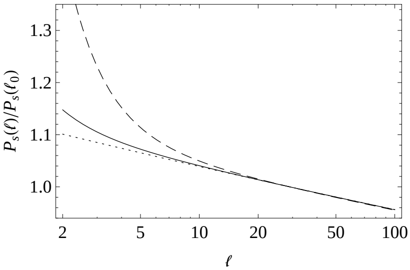


Figure: Primordial power spectrum for a certain model of loop quantum cosmology (upper curve). The dotted line is the classical case, and the solid line is the experimental upper bound. From: M. Bojowald, G. Calcagni, and S. Tsujikawa, *Phys. Rev. Lett.*, **107**, 211302 (2011).

Loop quantum cosmology predicts an *enhancement* of power at large scales.

Summary

- Concrete prediction from a conservative approach to quantum gravity
- It is consistent with existing observational limits
- No additional trans-Planckian effects are needed to understand these predictions
- In the present case, the effect is too small to be observable, but maybe one can find testable predictions along these lines
- Comparison with other approaches: loop quantum cosmology predicts an enhancement at large scales, while other approaches (non-commutative geometry, string-inspired cosmology) seem to predict a suppression