Chaotic destruction of Anderson localization in nonlinear lattices

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Anderson localization

- electron transport in disordered solids
- wave propagation in a random medium
- quantum chaos
- recent progress: cold bosons in disordered optical traps

Main effect in one dimension

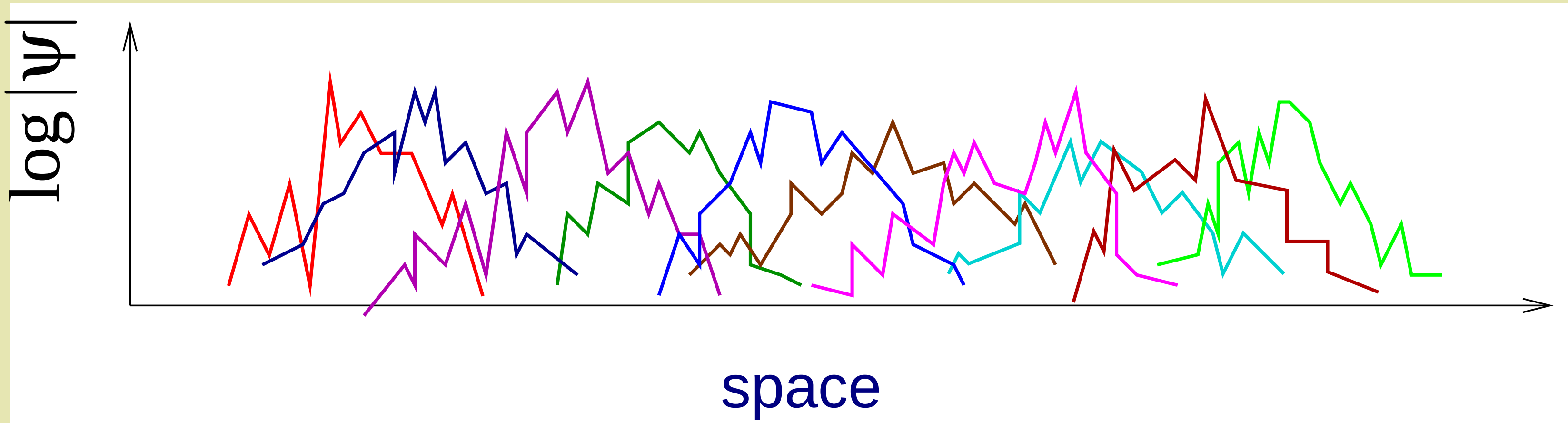
- absence of continuous spectrum in the linear disordered problem
- no propagating waves
- all eigenmides of discrete spectrum are exponentially localized
- localization length depends decreases with disorder and depends on energy (frequency)
- applicable to any wave system (lattice, PDE)

Discrete Anderson model

$$i\frac{d\psi_n}{dt} = E_n\psi_n + \psi_{n+1} + \psi_{n-1}$$

Here E_n is a ranadom on-site potential, we take E_n as independent random variables distributed uniformly in the range $-W/2 < E_n < W/2$ (usually we chose $W = 4$)

All eigenstates are exponentially localized $|\psi_n| \sim \exp(-\frac{|n-n_0|}{\lambda})$

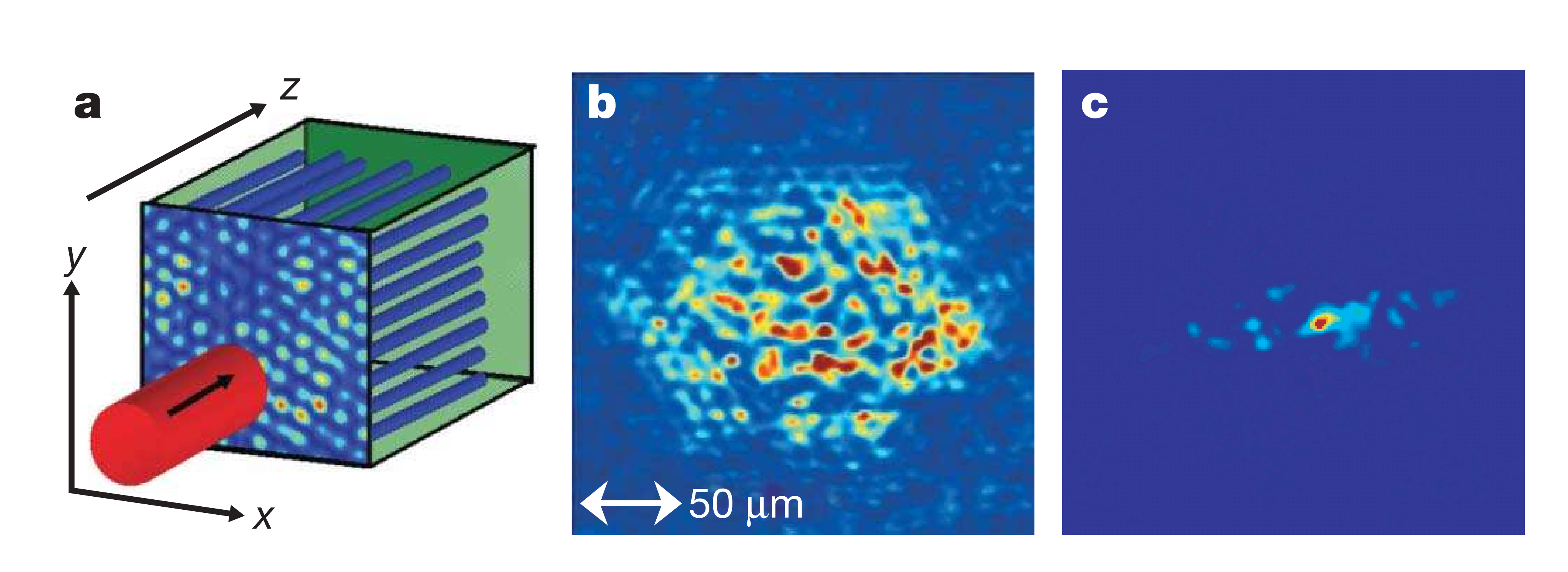


Nonlinear effects

- Bose-Einstein condensate is described by a **nonlinear** Gross-Pitaevskii equation
- wave propagation in a **nonlinear** disordered medium
- disordered chains of **nonlinear** oscillators

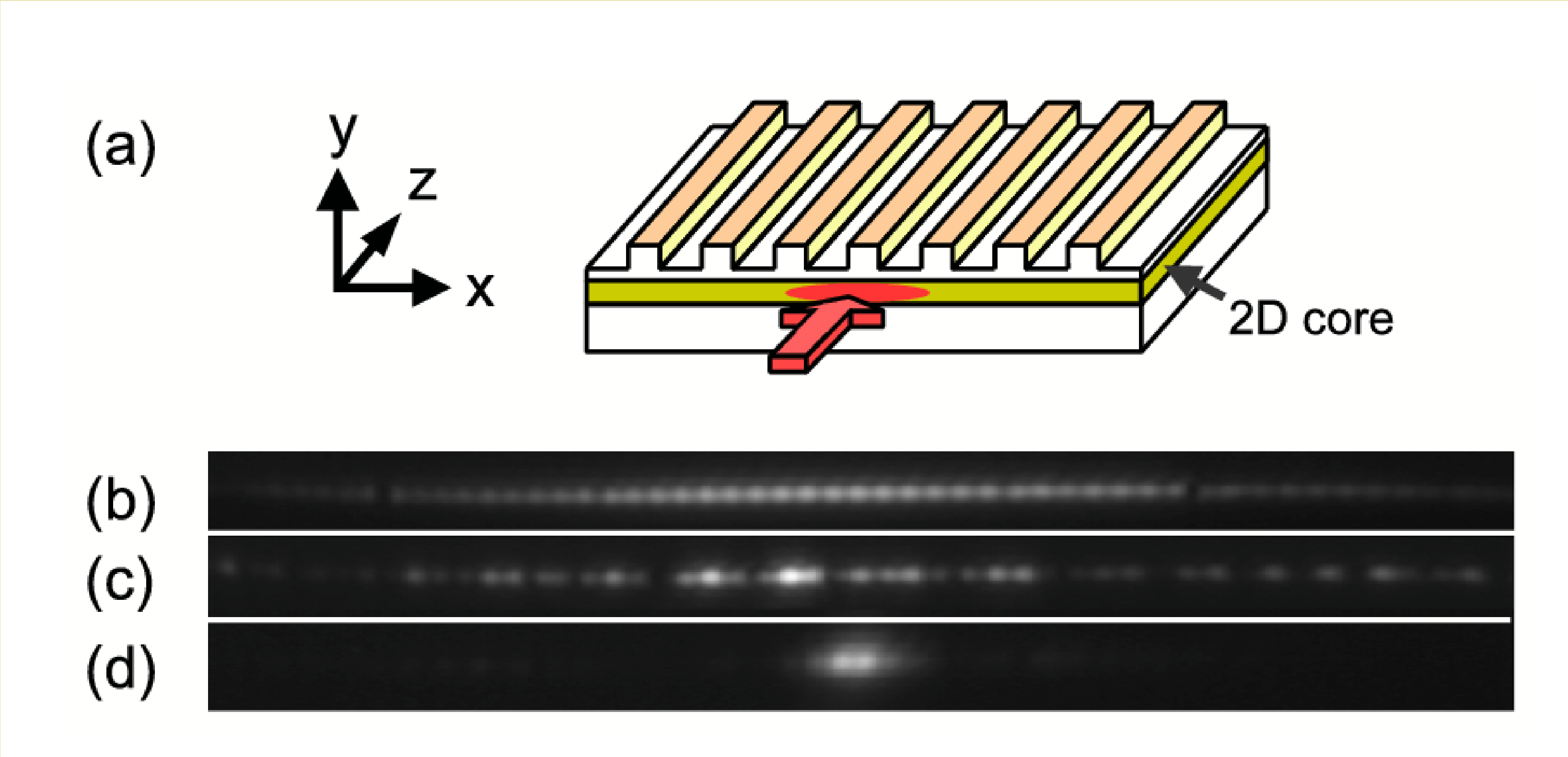
Q: Does nonlinearity enhance or destroy localization?

Optical experiments I



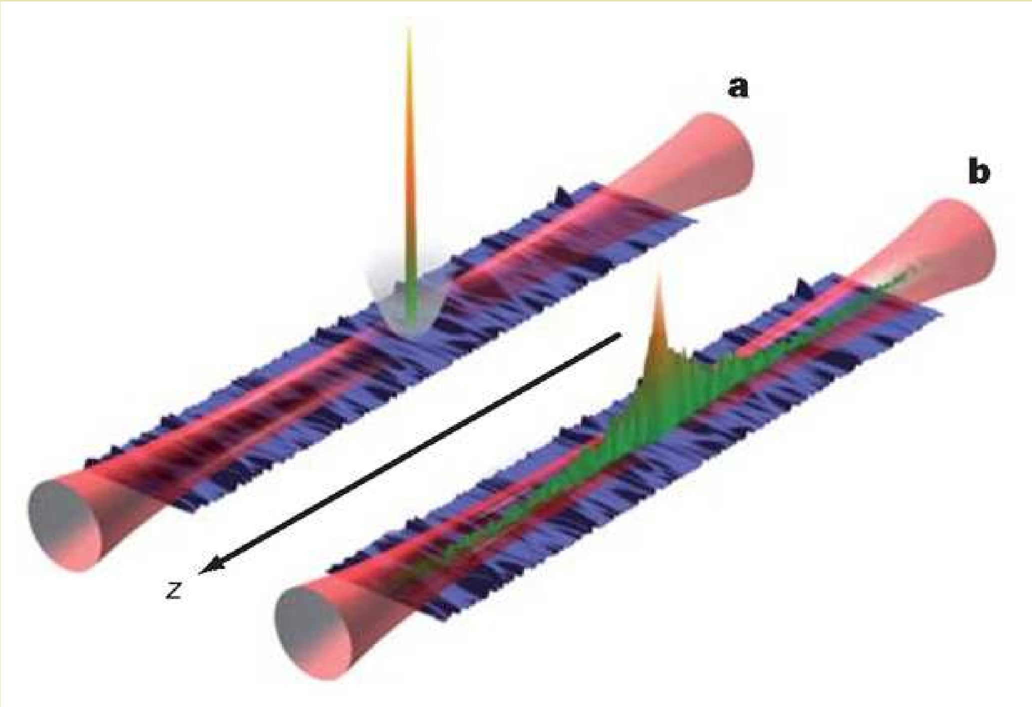
Schwartz, Bartal, Fishman and Segev, Nature 446 (2007)

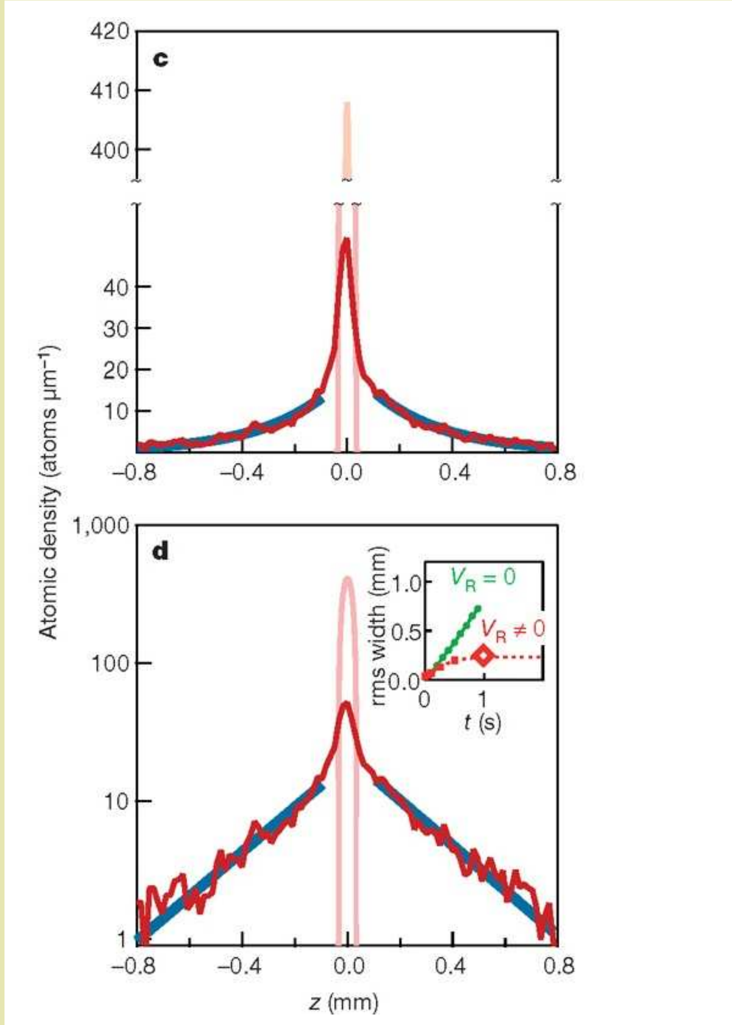
Optical experiments II



Lahini, Avidan, Pozzi, Sorel, Morandotti, Christodoulides and Silberberg, PRL 013906 (2008)

Cold atoms experiments





Billy et al., Nature 453, 891 (2008)

Basic model: DANSE

We study Discrete Anderson Nonlinear Schrödinger Equation

$$i\frac{\partial\psi_n}{\partial t} = E_n\psi_n + \beta|\psi_n|^2\psi_n + \psi_{n+1} + \psi_{n-1}$$

β characterizes nonlinearity

In the context of optical experiments: propagation direction plays a role of time

$$i\frac{\partial\mathcal{E}_n}{\partial z} = E_n\mathcal{E}_n + \beta|\mathcal{E}_n|^2\mathcal{E}_n + \mathcal{E}_{n+1} + \mathcal{E}_{n-1}$$

Different Setups for Nonlinear Disordered Lattices

- **Spreading Problem:** seed an initially localized wavepacket in an infinite lattice and look how it spreads
- **Weak Chaos Problem:** consider a finite lattice and study properties of chaos/regularity at small energy densities (work with S. Fishman, not in this talk)
- **Scattering problem:** consider a finite lattice and study how the field is transmitted/reflected by a random layer

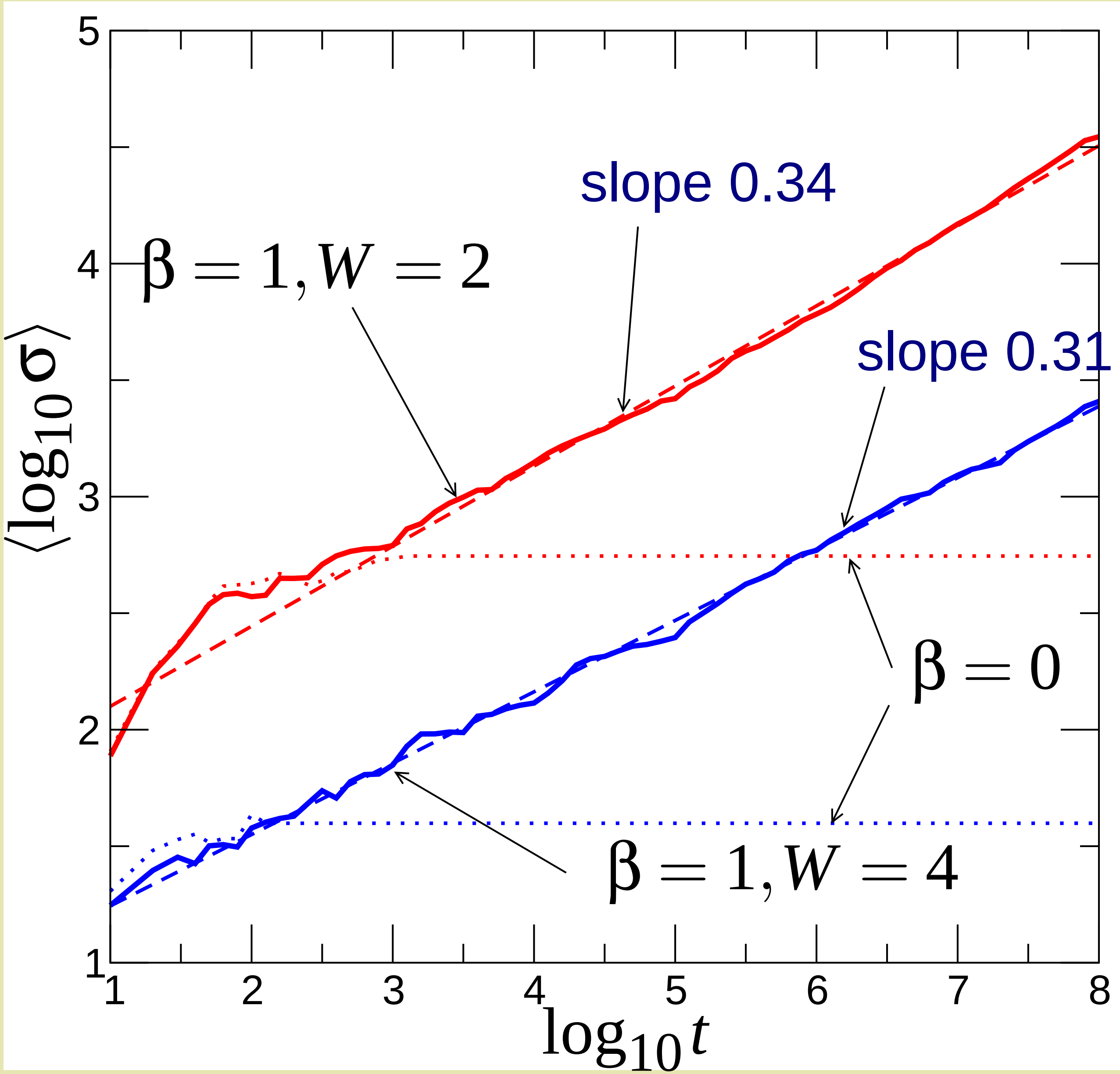
Spreading Problem

How an initially localized field $|\psi_n(0)|^2 = \delta_{n,0}$ is spreading?

Note that $\sum_n |\psi_n(t)|^2 = 1 = const$

We characterize this with the averaged squared width, i.e. the second moment $\langle(\Delta n)^2\rangle = \sigma(t) = \sum_n (n - \langle n \rangle)^2 |\psi_n(t)|^2$. The averaging over disorder realizations was performed for the logarithm of this quantity, i.e. for $\log \sigma$.

Wave packet spreading I



Analytical estimate

In the basis of linear localized modes, the evolution of the amplitudes C_m of these modes is due to their nonlinear coupling,

$$i\dot{C}_m = \omega_m C_m + \beta \sum C_{m_1} C_{m_2} C_{m_3}$$

Assuming randomness of the phases, we can estimate the rate of excitation of a newly involved mode as $\sim |C|^6 \sim 1/(\Delta n)^3$. On the other hand, excitation of a new mode is none other than diffusive spreading of the field, thus $\frac{d}{dt}(\Delta n)^2 \sim 1/(\Delta n)^3$. Solution of this equation yields subdiffusive spreading

$$(\Delta n)^2 \propto t^{2/5}$$

Basic question:

Does spreading persist at very large times?

Numerics suggests:

- Initial wavepacket spreads seemingly unboundedly
- Subdiffusion spreading with exponent ≈ 0.35

But to answer questions

- Does it last forever?
- Does it depend on the nonlinearity constant?

We need to

- Study very large lattices
- At very large times

Arguments against spreading

- In course of spreading nonlinearity becomes weaker
- A Kolmogorov-Arnold-Moser regime is approached?
- But the number of degrees of freedom (effective dimension) grows
- Limit of small density, large number of degrees of freedom

Our approach: to establish **scaling** properties

Scaling approach to spreading

Use Nonlinear Diffusion Equation as heuristic model

$$\frac{\partial \rho}{\partial t} = D\frac{\partial}{\partial x}\left(\rho^a \frac{\partial \rho}{\partial x}\right), \qquad \text{with} \qquad \int \rho \, dx = E$$

Self-similar solution

$$\rho(x,t) = \frac{1}{[D(t-t_0)]^{1/(2+a)}}\left(E - \frac{ax^2}{2(a+2)[D(t-t_0)]^{2/(a+2)}}\right)^{\frac{1}{a}}$$

yields subdiffusion

$$X = \sqrt{2\frac{2+a}{a}}E^{a/(2+a)}[D(t-t_0)]^{1/(2+a)}$$

One parameter scaling

Reformulate

$$X = \sqrt{2\frac{2+a}{a}} E^{a/(2+a)} (D(t-t_0))^{1/(2+a)}$$

as scaling functions:

$$\frac{X}{E} \sim \left(\frac{t-t_0}{E^2}\right)^{1/(2+a)} \qquad \frac{1}{X}\frac{\mathrm{d}t}{\mathrm{d}X} \sim \left(\frac{E}{X}\right)^{-a} \qquad a(w) = -\frac{\mathrm{d}\log\frac{1}{X}\frac{\mathrm{d}t}{\mathrm{d}X}}{\mathrm{d}\log w}$$

where $w = E/X$ is the characteristic density

Spreading in a strongly nonlinear lattice

We modify a weakly nonlinear disordered Klein-Gordon lattice

$$H=\sum_k \frac{p_k^2+\omega_k^2 q_k^2}{2}+\frac{(q_{k+1}-q_k)^2}{2}+\beta\frac{(q_{k+1}-q_k)^4}{4}$$

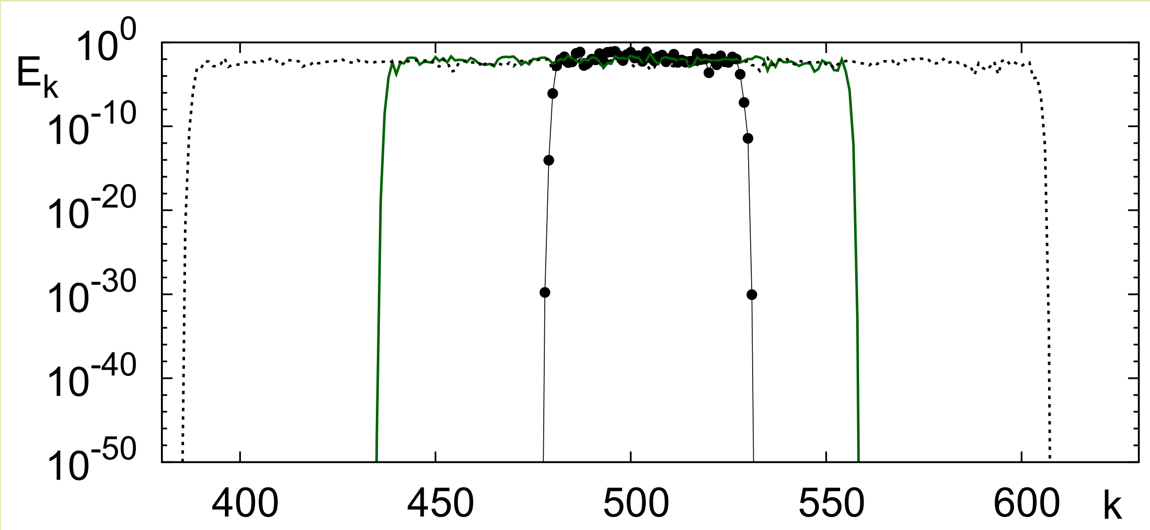
to a strongly nonlinear lattice

$$H=\sum_k \frac{p_k^2+\omega_k^2 q_k^2}{2}+\frac{(q_{k+1}-q_k)^4}{4}$$

Nonlinearly coupled disordered linear oscillators, the total energy E is the only parameter

Strong compactness of the spreading field

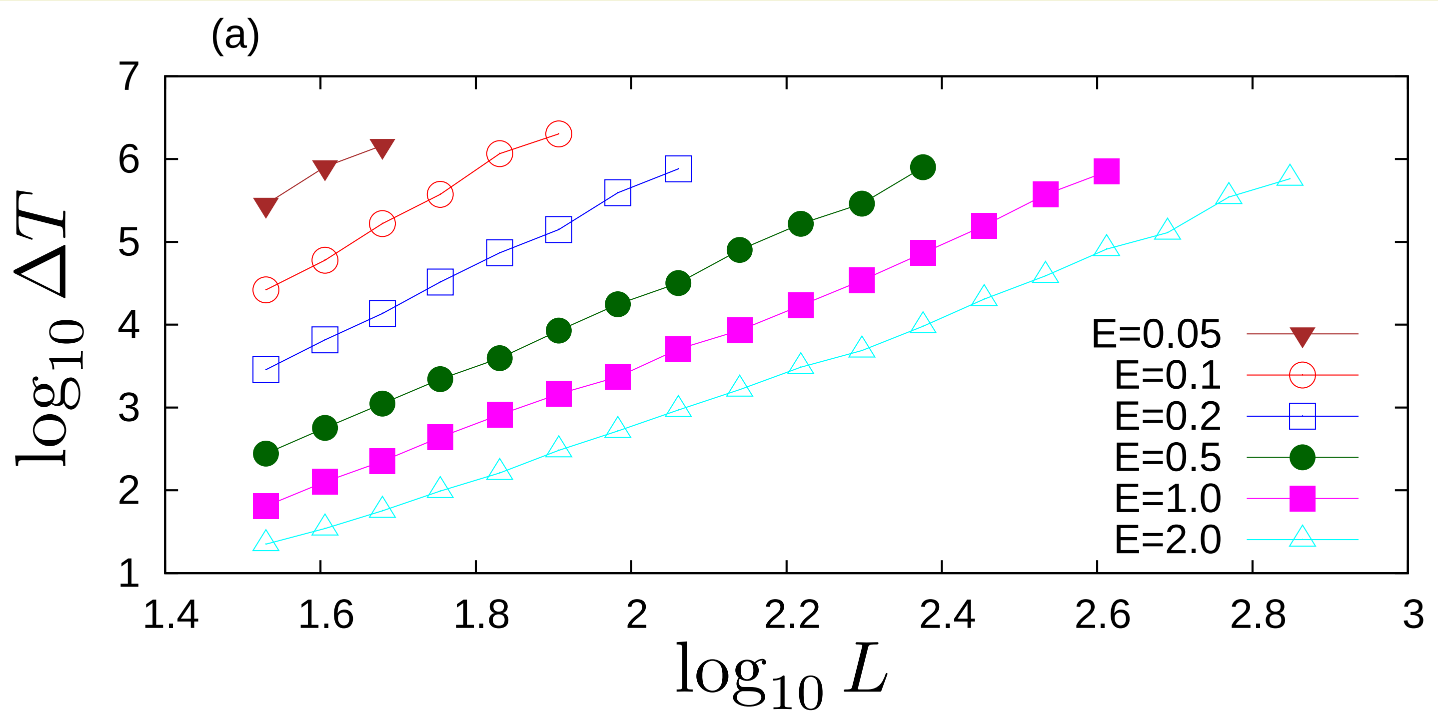
Here ”Anderson modes” are one site oscillators -> no exponential tails, numerics is efficient and the packet width is well-defined



One can easily calculate propagation time $\Delta T \approx \frac{\mathrm{d}t}{\mathrm{d}X}$

Spreading in a lattice of nonlinearly coupled linear oscillators

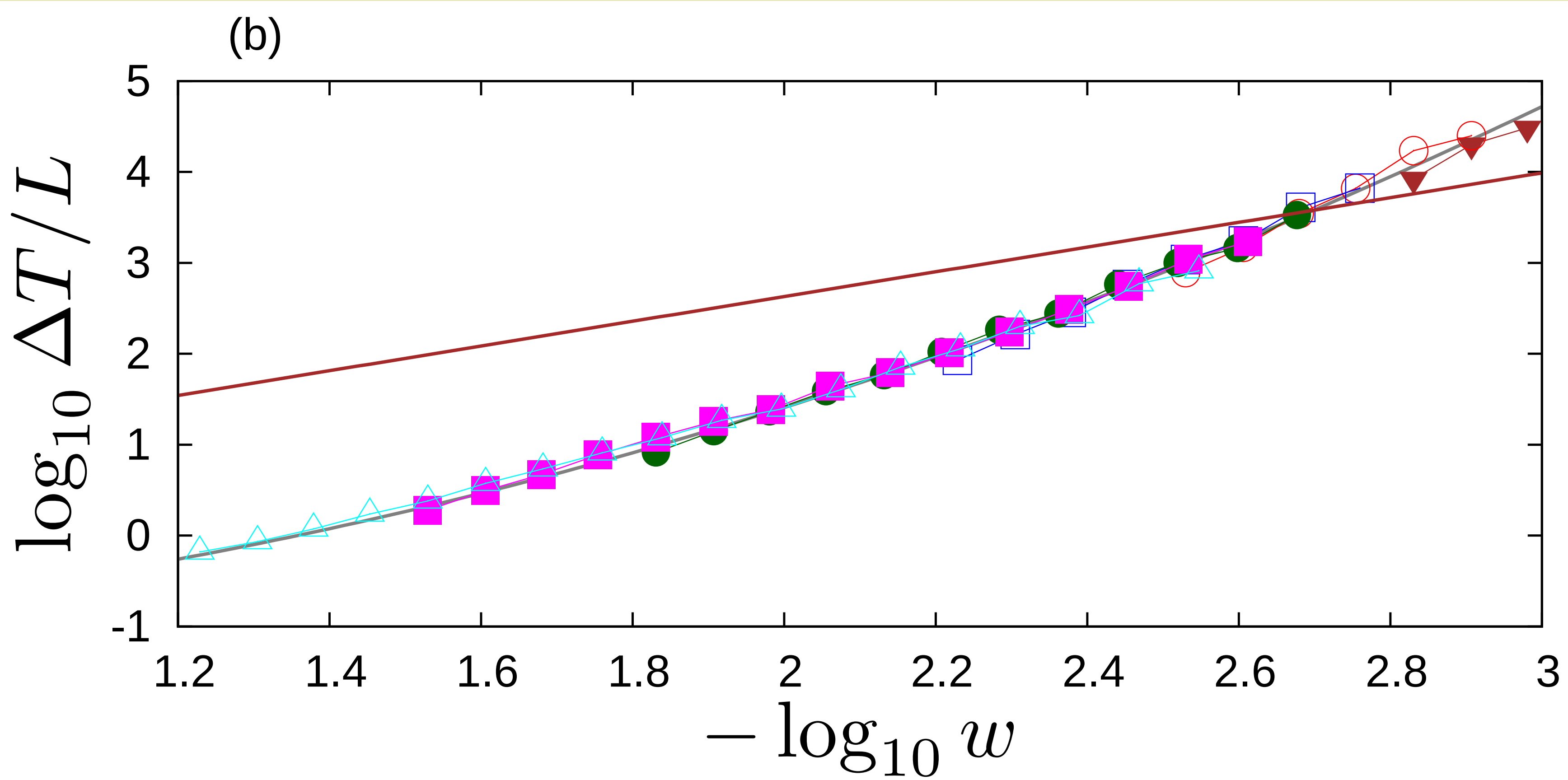
$$H = \sum_k \frac{p_k^2 + \omega_k^2 q_k^2}{2} + \frac{(q_{k+1} - q_k)^4}{4}$$



Spreading for different energies

Verification of scaling

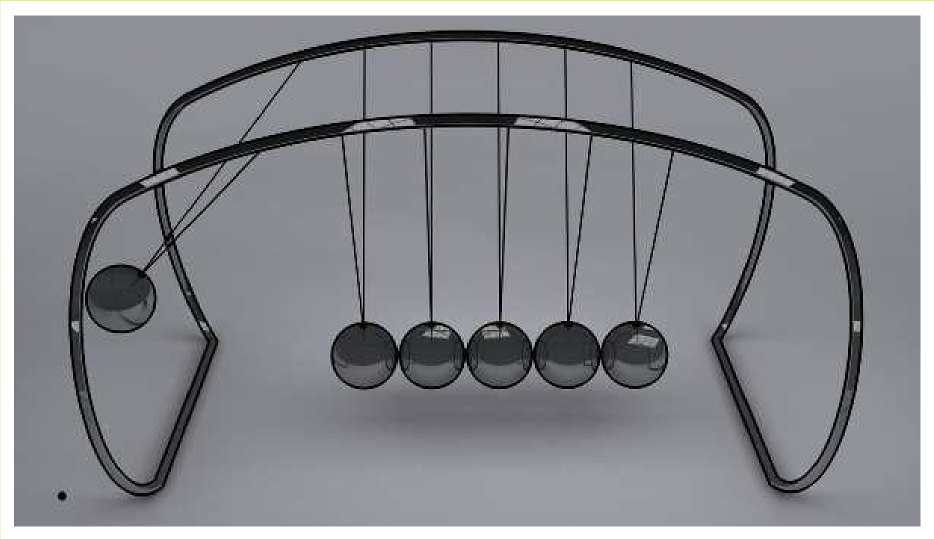
Rescale coordinates to achieve collapse on one curve

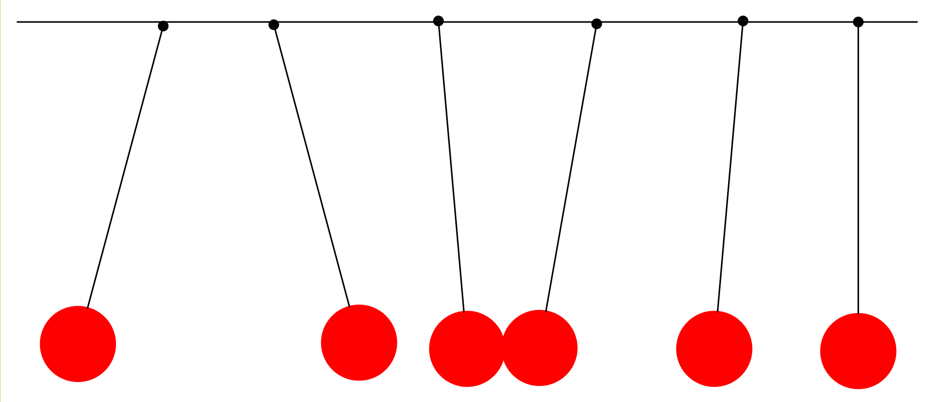


Empirical fit for density-dependence of index: $a(w) \approx -0.3 - 1.5 \log_{10} w$

Toy model: Ding-Dong lattice

This is a strongly nonlinear lattice that is easy to model numerically





Ding-Dong model (Prosen, Robnik, 92) is a chain of linear oscillators with elastic collisions

Ding-Dong dynamics

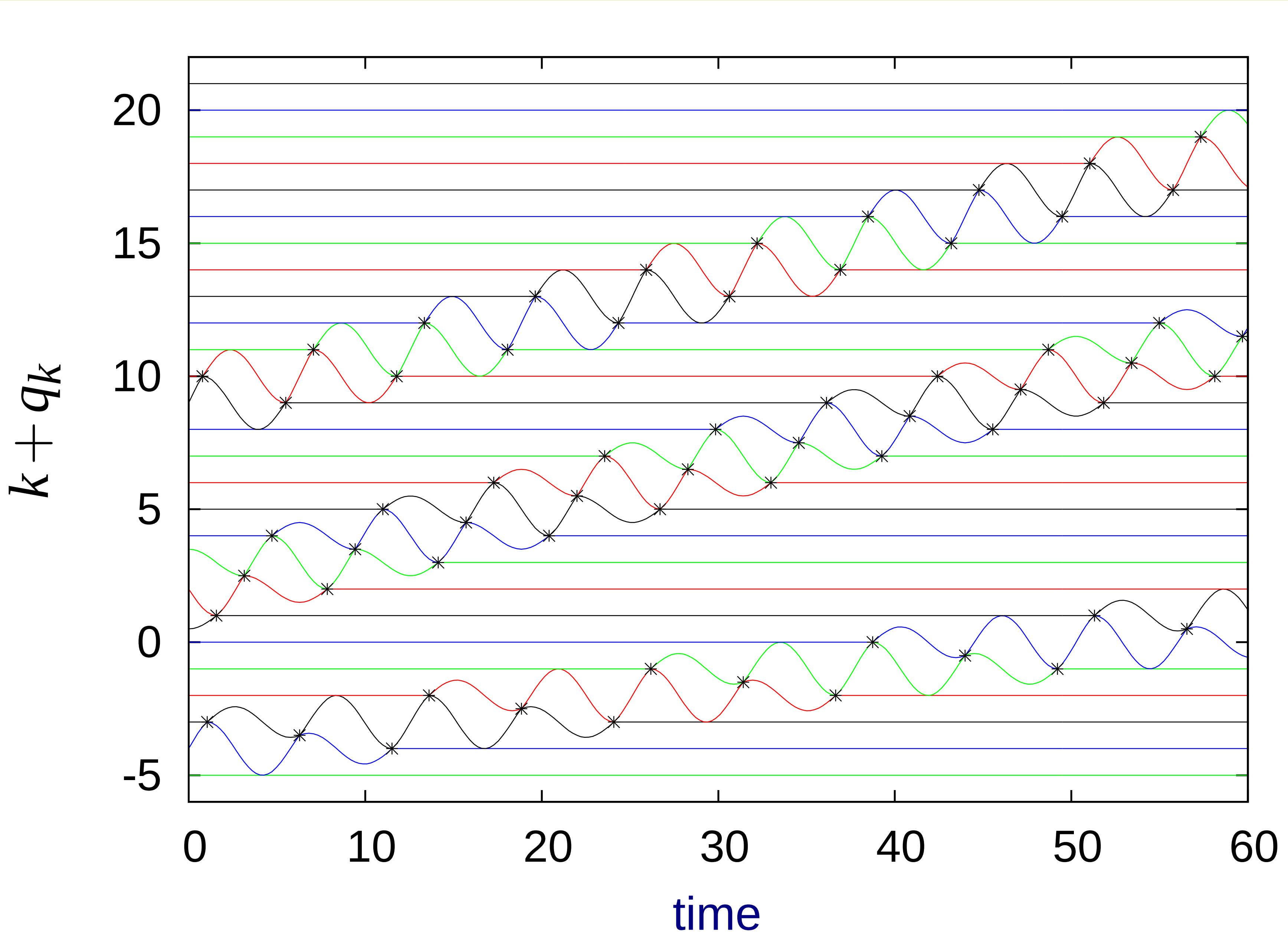
Hamiltonian and collision condition

$$H = \sum_k \frac{p_k^2 + q_k^2}{2} \qquad \text{when } q_k - q_{k+1} = 1 \text{ then } p_k \to p_{k+1},\ p_{k+1} \to p_k$$

Effective calculation of the collision times – simulation on very long times pissible

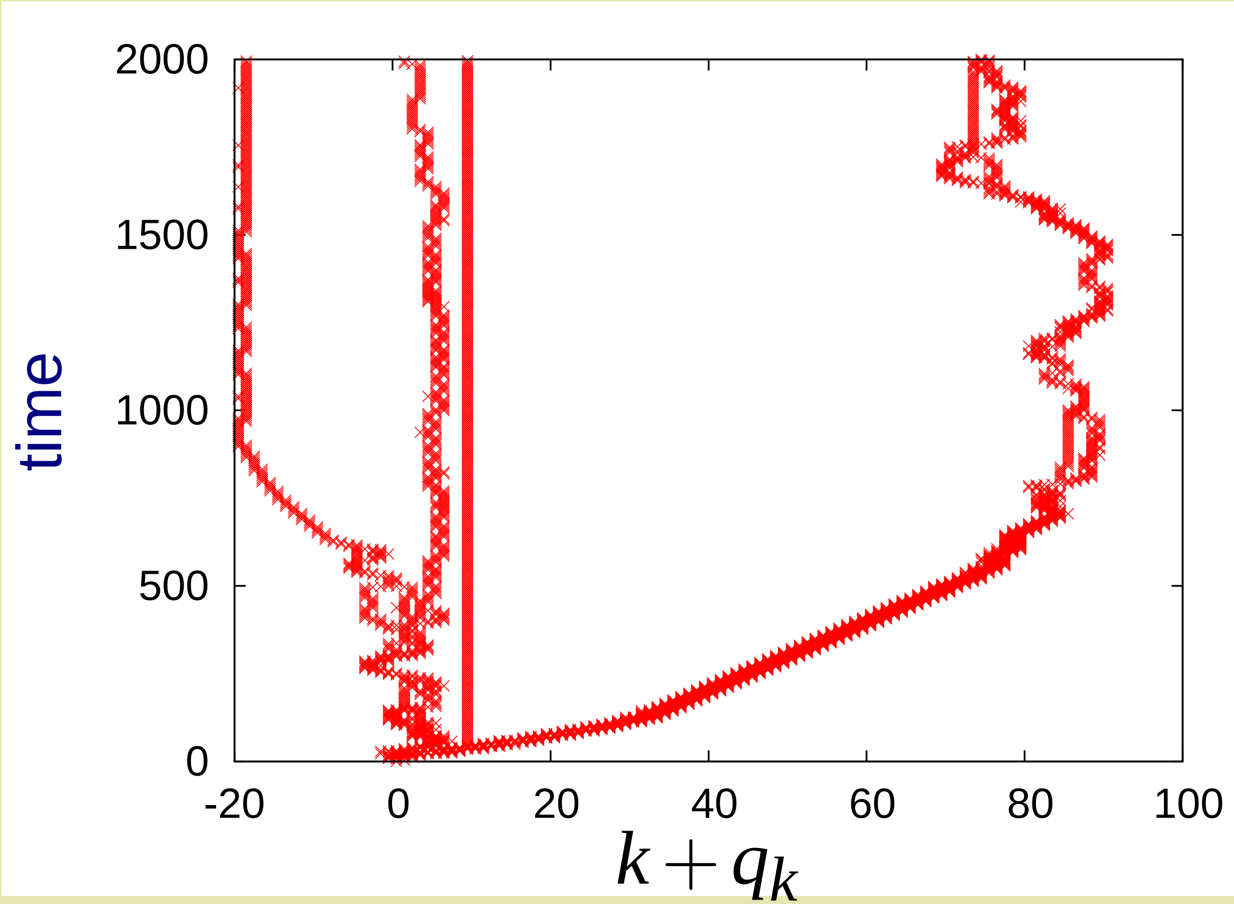
Strongly nonlinear lattice: no linear waves, no phonons, all propagating perturbations are nonlinear

Compactons in a homogeneous lattice

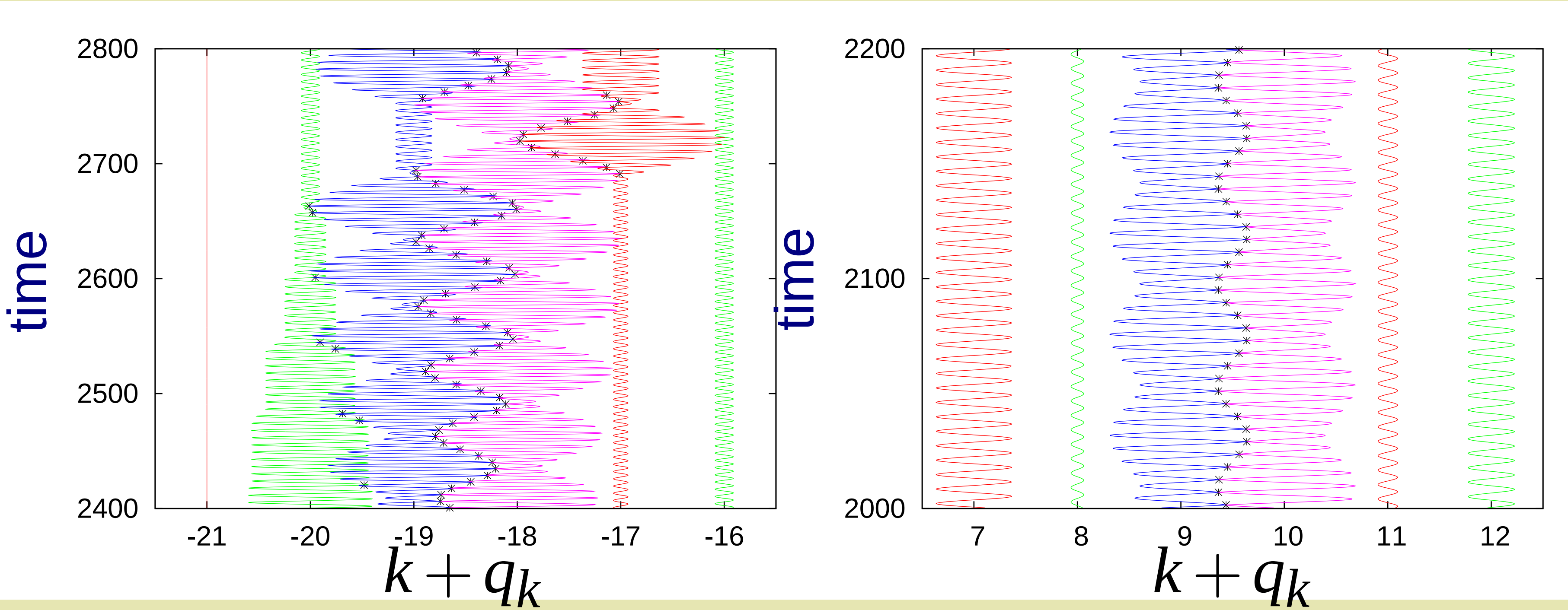


Spreading in a homogeneous lattice

From random initial conditions: chaos, breathers, and (almost)compactons appear



Examples of chaos and breathers

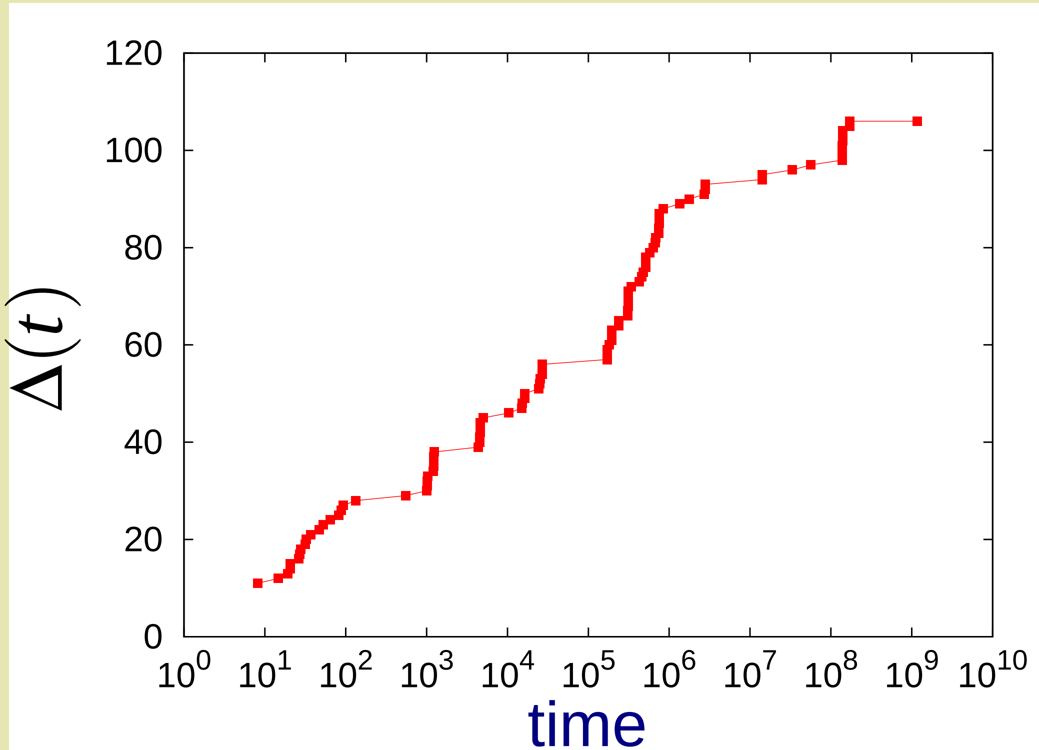


Spreading in a disordered lattice

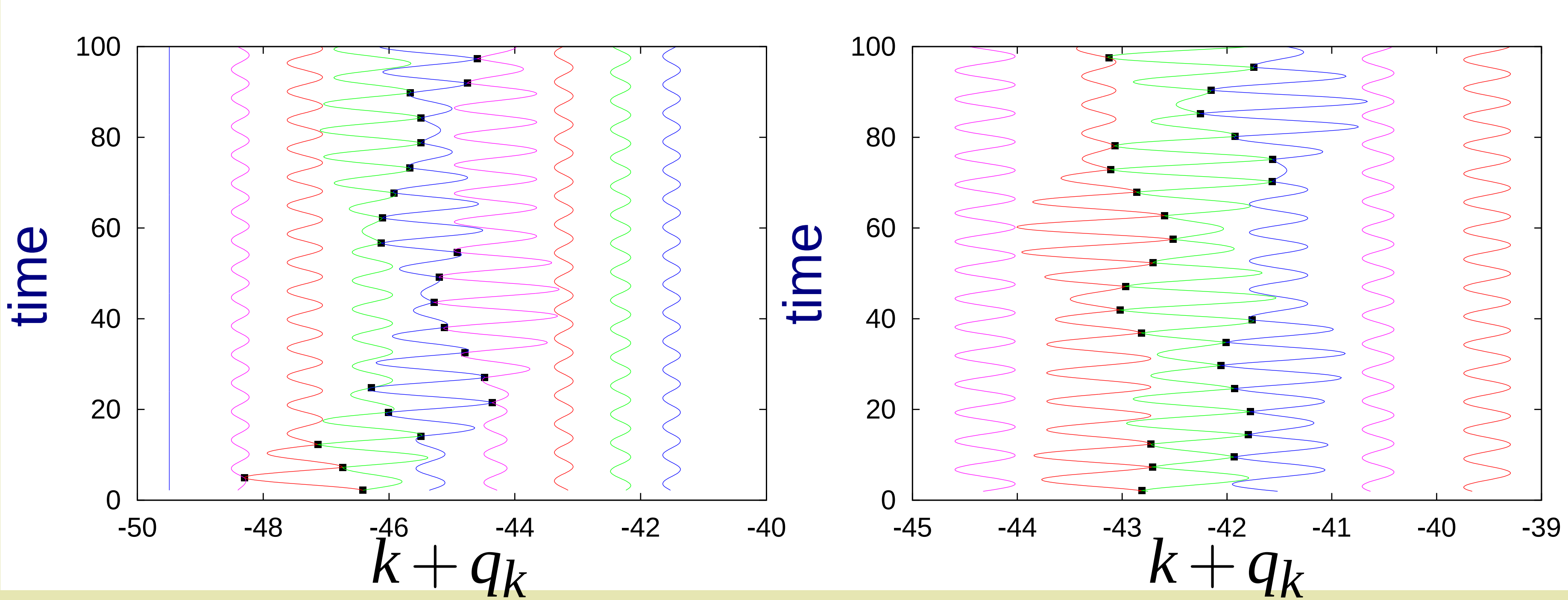
Disorder in distances or masses destroys compactons

Spreading effectively stops: no spreading events for time interval 10^{10},

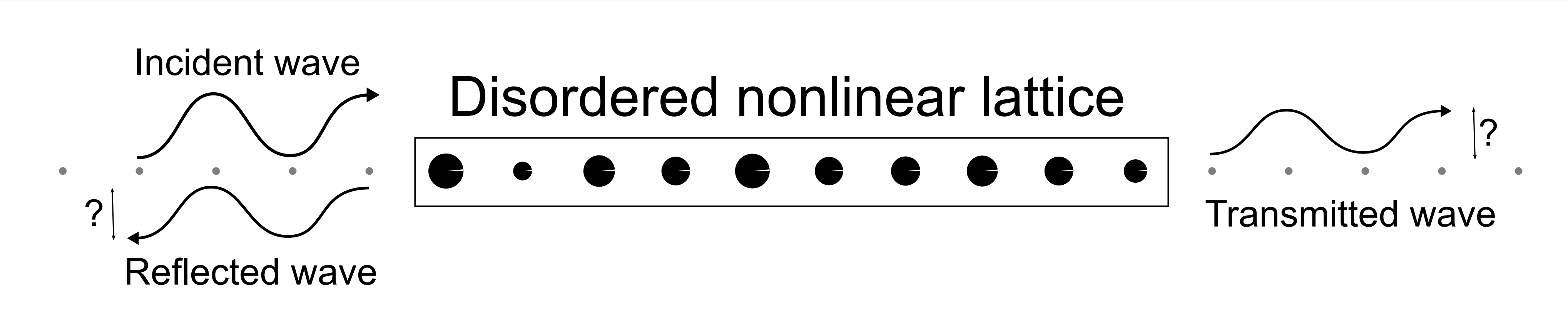
a few chaotic spots appear



Chaotic spot



Scattering problem



- a finite nonlinear disordered layer embedded in a linear regular environment
- a regular incident wave
- **what are transmission/reflection properties ?**

Scattering problem for an NSE

Discrete Anderson nonlinear Schrödinger equation:

$$i\frac{\partial\psi_n}{\partial t} = E_n\psi_n + \beta|\ \psi_n\ |^2\psi_n + \psi_{n+1} + \psi_{n-1} \qquad 1 \le n \le L$$

β characterizes nonlinearity

energies E_n are independent random variables distributed uniformly in the range $-W/2 < E_n < W/2$ (we take $W = 4$ in this study)

Model boundary conditions:

$$\psi_0 = 2A - i\psi_1\,, \qquad \psi_{L+1} = -i\psi_L\,.$$

A is the amplitude of the incident wave (hereafter $A = 1$)

Scattering problem as a dissipative dynamical system

System

$$i\frac{\partial \psi_n}{\partial t} = E_n \psi_n + \beta |\psi_n|^2 \psi_n + \psi_{n+1} + \psi_{n-1} \qquad 1 \le n \le L$$

$$\psi_0 = 2 - i\psi_1 \,, \qquad \psi_{L+1} = -i\psi_L \,.$$

is a **dissipative forced** nonlinear dynamical system with L degrees of freedom

The problem is in finding **attractors** and characterizing their properties

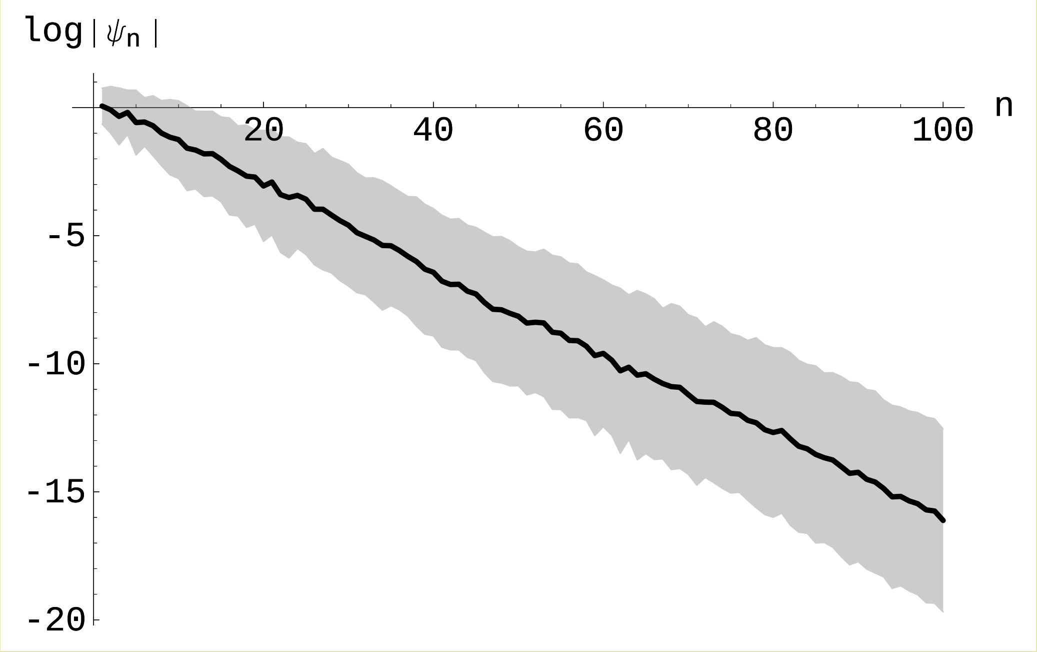
Linear scattering problem

$$i\frac{\partial\psi_n}{\partial t} = E_n\psi_n + \psi_{n+1} + \psi_{n-1} \qquad 1 \leq n \leq L$$

$$\psi_0 = 2 - i\psi_1\,, \qquad \psi_{L+1} = -i\psi_L\,.$$

This linear dissipative system has a unique stable attractor – an equilibrium point. This equilibrium corresponds to an exponentially localized field:

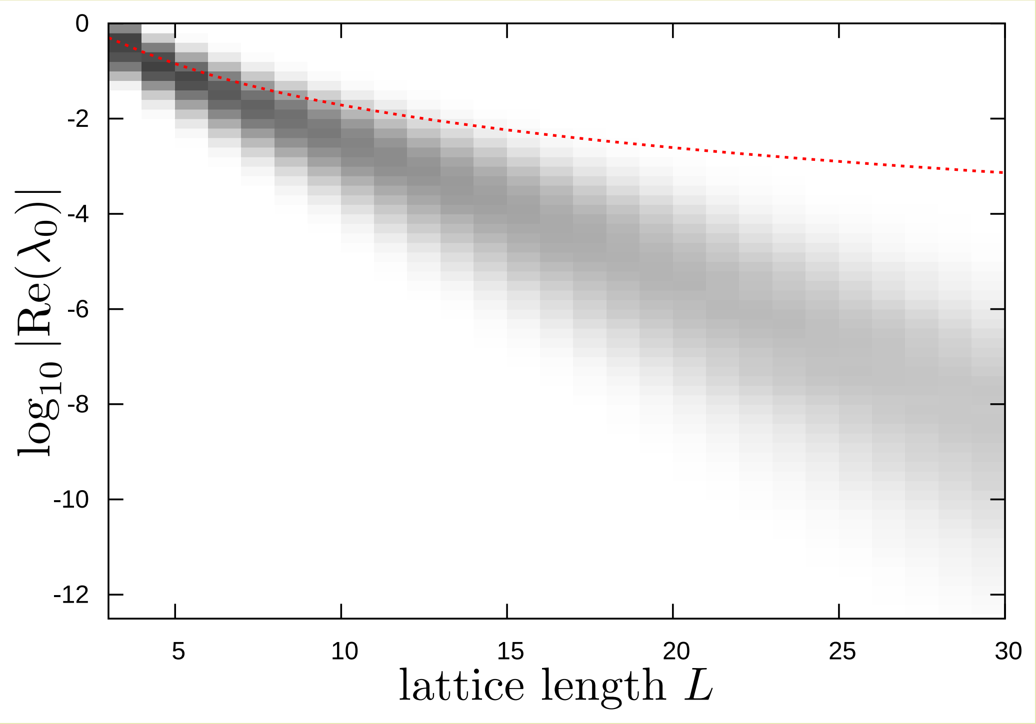
$$\psi_L \propto \exp[-L/\lambda]$$



Stability of the linear solution

For $\beta = 0$ the linear solution is stable, however real parts of some eigenvalues are very close to zero

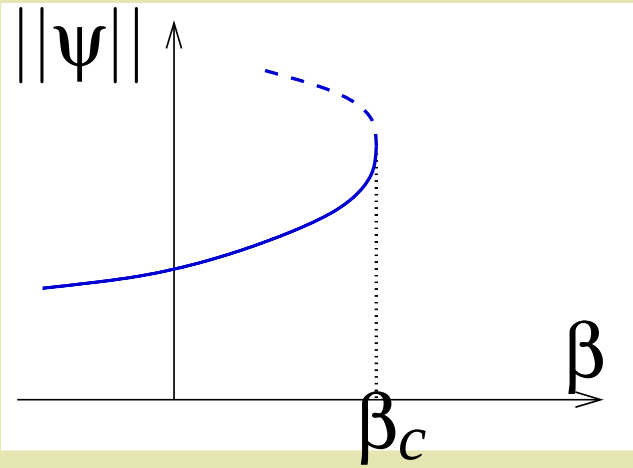
This is due to localized modes at the middle of the lattice that are weakly coupled to the ends

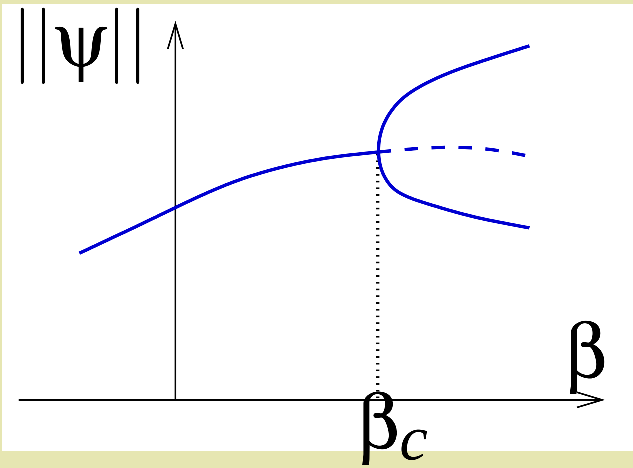


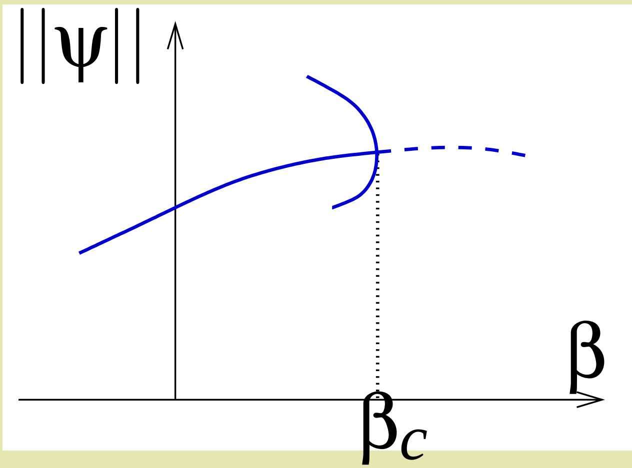
Nonlinear continuation

The linear solution can be continued into $\beta \neq 0$, until it bifurcates

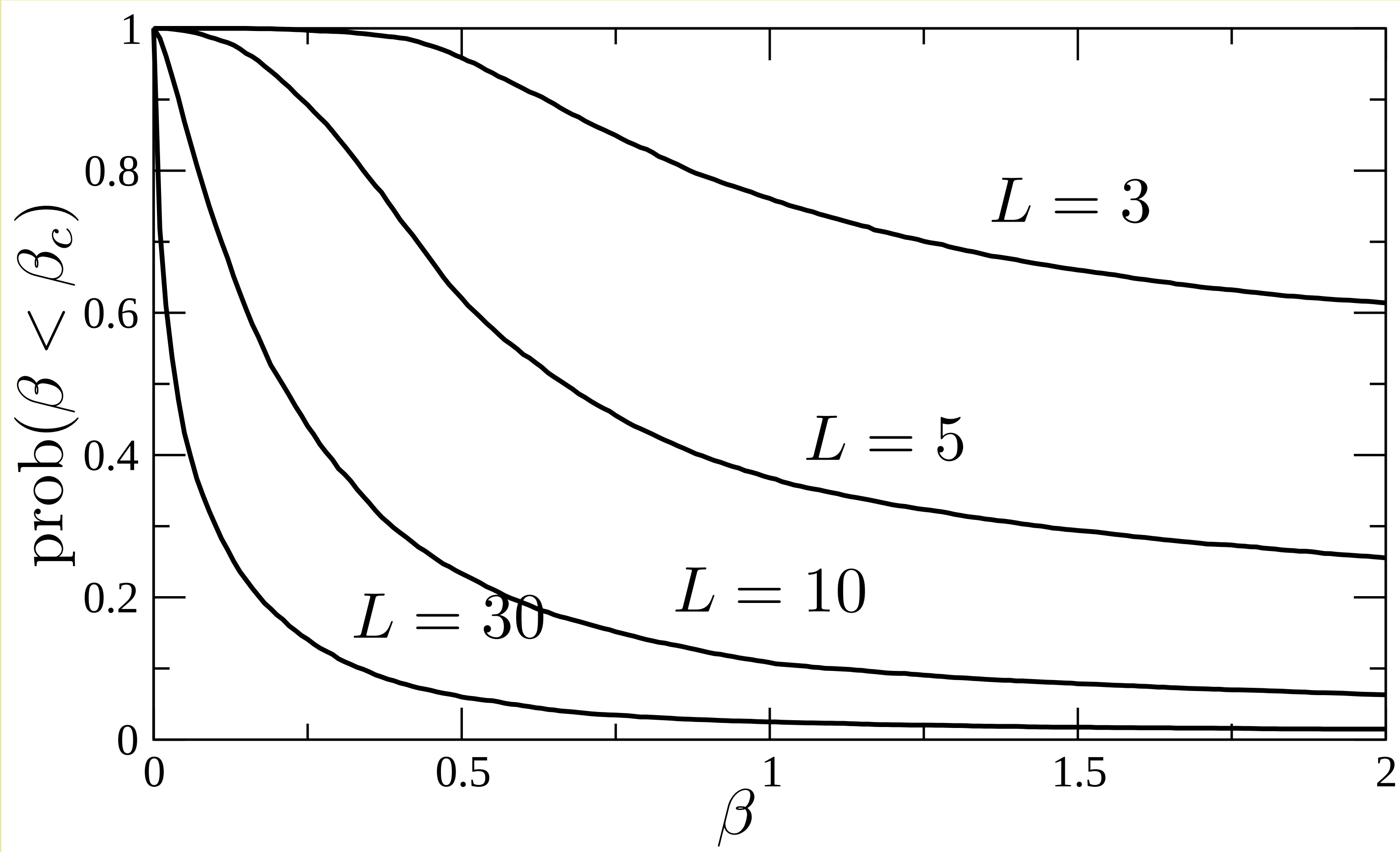
saddle-node/fold bifurcation	supercritical Hopf bifurcation	subcritical Hopf bifurcation





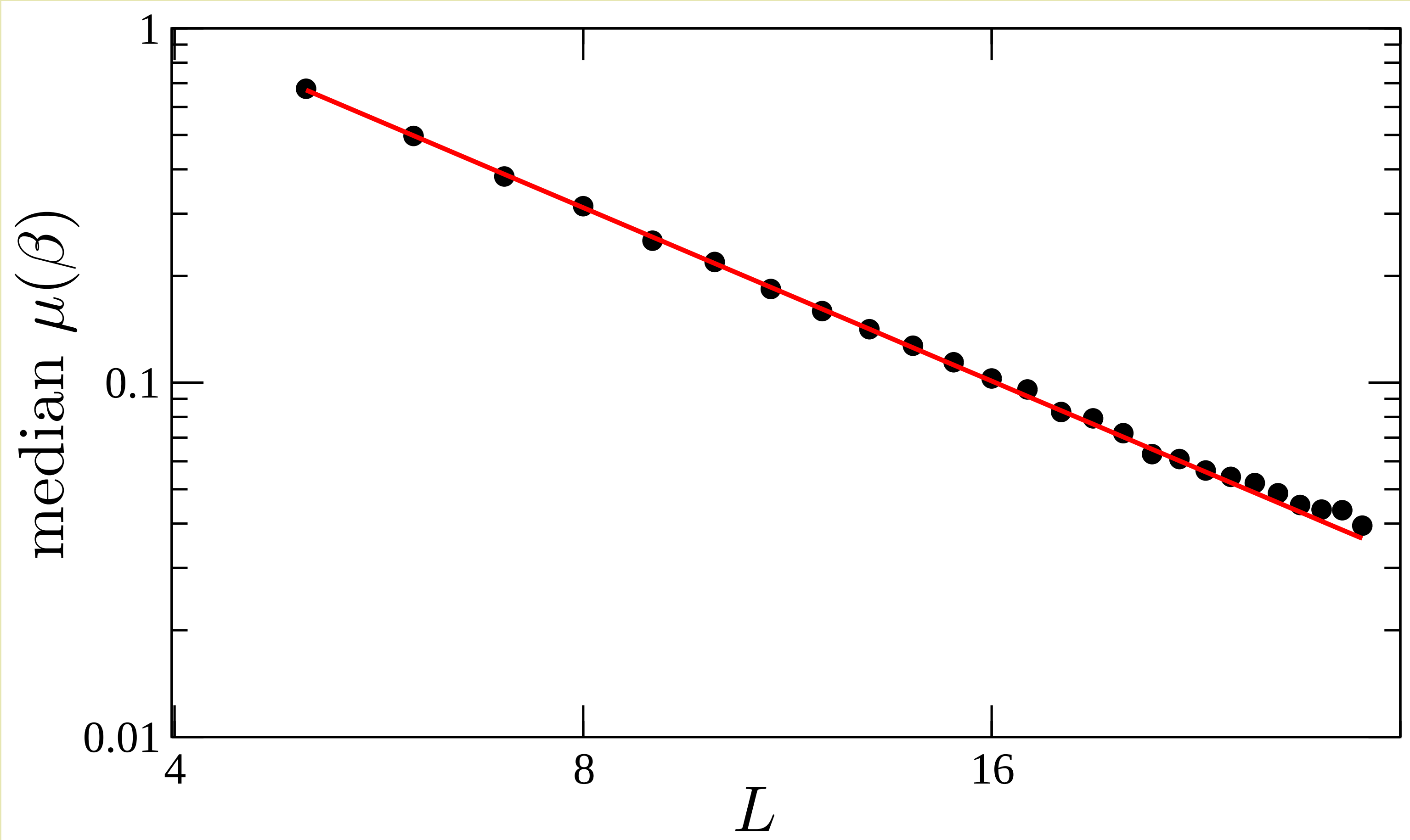


Statistical bifurcation analysis



Cumulative distributions of the critical nonlinearity values, the curves show the probability that $\beta_c > \beta$

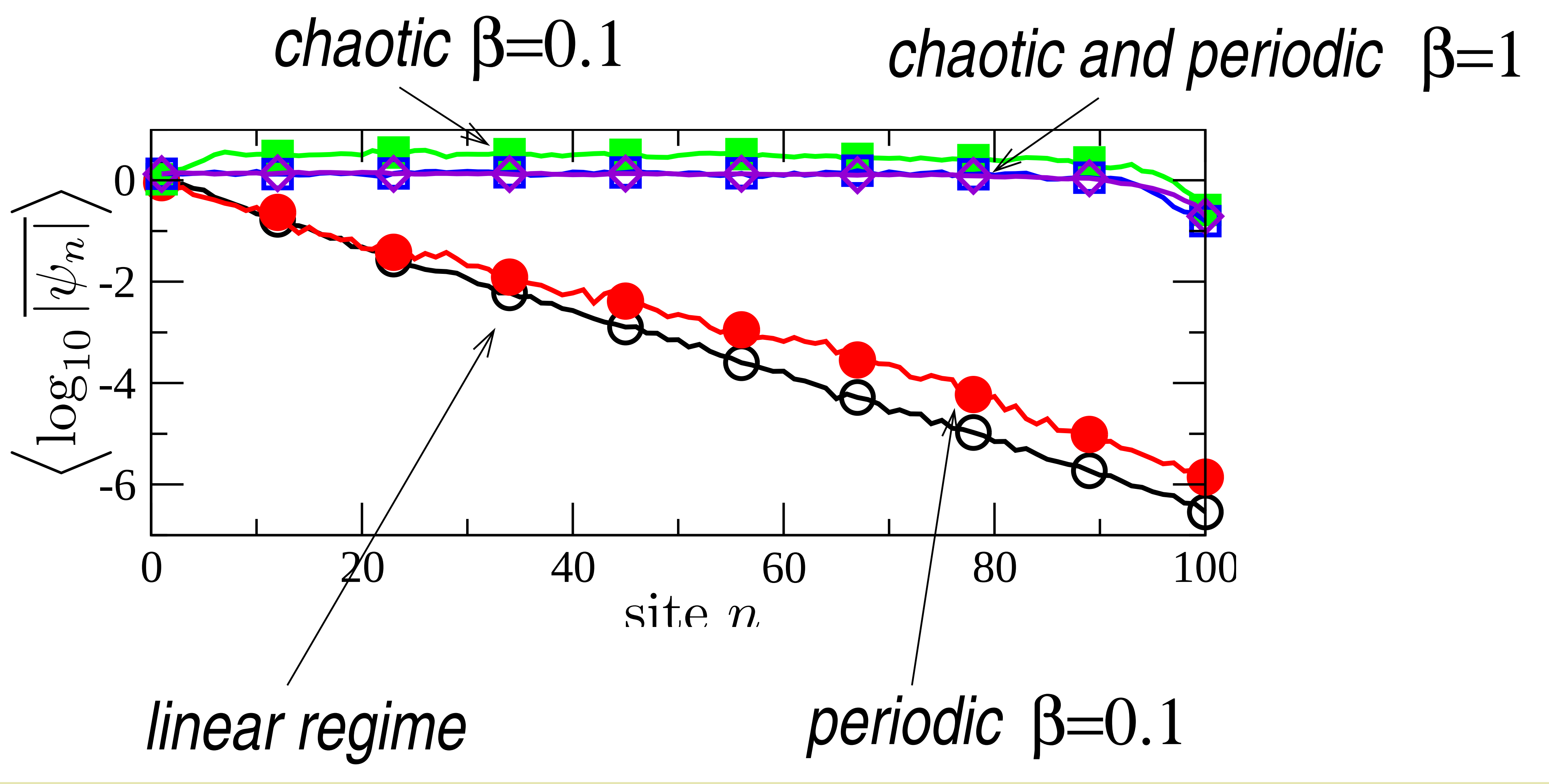
Scaling of the critical nonlinearity



Dependence of the median $\mu(\beta)$ of the distribution (defined as prob$[\beta_c > \mu(\beta)] = 1/2$) vs. lattice length L

The line is the power law $\mu(\beta) \approx 9.2 \cdot L^{-1.6}$

Long lattices: direct numerical simulations



Conclusion

- Nonlinearity destroys Anderson localization
- Subdiffusive spreading of initially localized wave packets
  width $\sim$ time$^{0.17}$
- Linear regime of scattering on a nonlinear layer is destroyed via a bifurcation at $\beta_c \sim L^{-1.6}$,
  leading to chaos-induced transparency
- Scaling approach based on a nonlinear diffusion equations captures well the scaling of spreading in a strongly nonlinear lattice
- A toy strongly nonlinear lattice – Ding-Dong model – shows stop of spreading, final state consists of a few chaotic spots + background

Publications

Collaborators: Dima Shepelyansky, Steffen Tietsche, Mario Mulansky, Karsten Ahnert, Sthitadhi Roy

EPL, **84** n. 1, 10006 (2008)

Phys. Rev. Lett **100**, 094101 (2008)

Phys. Rev. E, **80**, 056212 (2009)

EPL, **90**, 10015 (2010)

Phys. Rev. E, **83**, 026205 (2011)

Phys. Rev. E, **83**, 025201(R) (2011)

J. Stat. Phys., **145** 1256 (2011)

arXiv:1111.7128v1 [nlin.CD] (2011)

Other groups: S. Aubry (Paris), S. Flach (Dresden), S. Fishman (Technion), Ts. Kottos (Wesleyan), B. Shapiro (Technion)

Review “The Nonlinear Schroedinger Equation with a random potential: Results and Puzzles” Fishman et al., arXiv:1110.3024