Ginzburg Conference on Physics
Moscow 2012

Vortex Filaments and Velocity Statistics in Hydrodynamic Turbulence.

V.A. Sirota & K.P. Zybin

Lebedev Physical Institute, Moscow

Stationary, locally isotropic and homogenous turbulence in incompressible liquid; Inertial range: $\eta \ll l \ll L$

Euler velocity structure functions

Longitudinal:

$$S_n^{\parallel}(l) = \left\langle \left| \left(\vec{V}(\vec{r} + \vec{l}) - \vec{V}(\vec{r}) \right) \frac{\vec{l}}{l} \right|^n \right\rangle$$

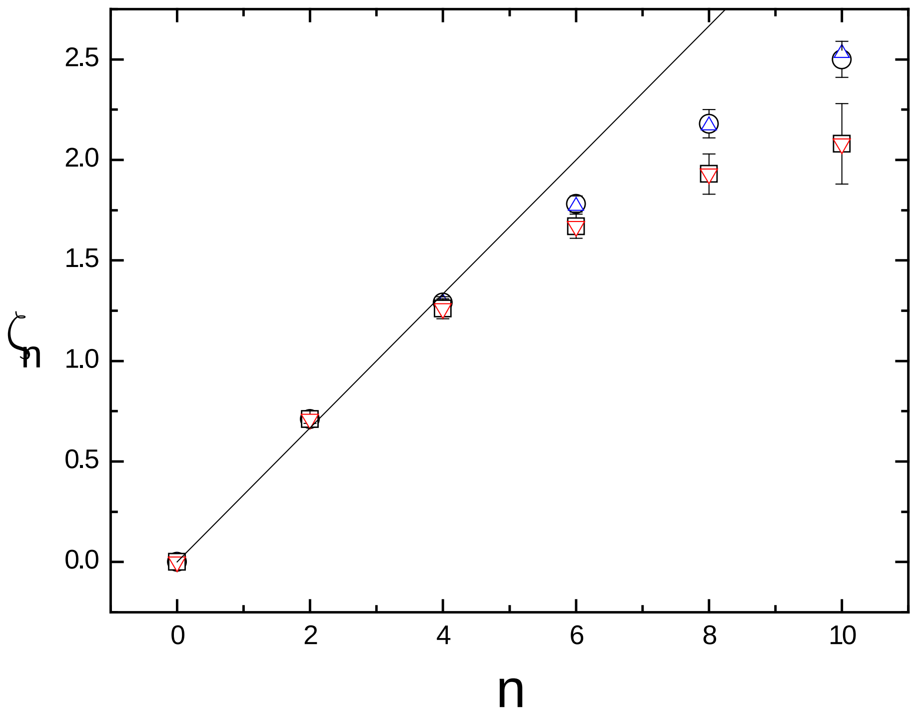
Transverse:

$$S_n^{\perp}(l) = \left\langle \left| \left(\vec{V}(\vec{r} + \vec{l}) - \vec{V}(\vec{r}) \right) \times \frac{\vec{l}}{l} \right|^n \right\rangle$$

$$S_n(l) \propto l^{\zeta_n}$$

$$S_n(l) \propto l^{\zeta_n}$$

K41 (Kolmogorov): $\zeta_n = n/3$



Benzi et al. 2010: $\triangle$ $(\parallel)$, $\triangledown$ $(\perp)$;
Gotoh et al. 2002 : $\circ$ $(\parallel)$, $\square$ $(\perp)$

Intermittency :

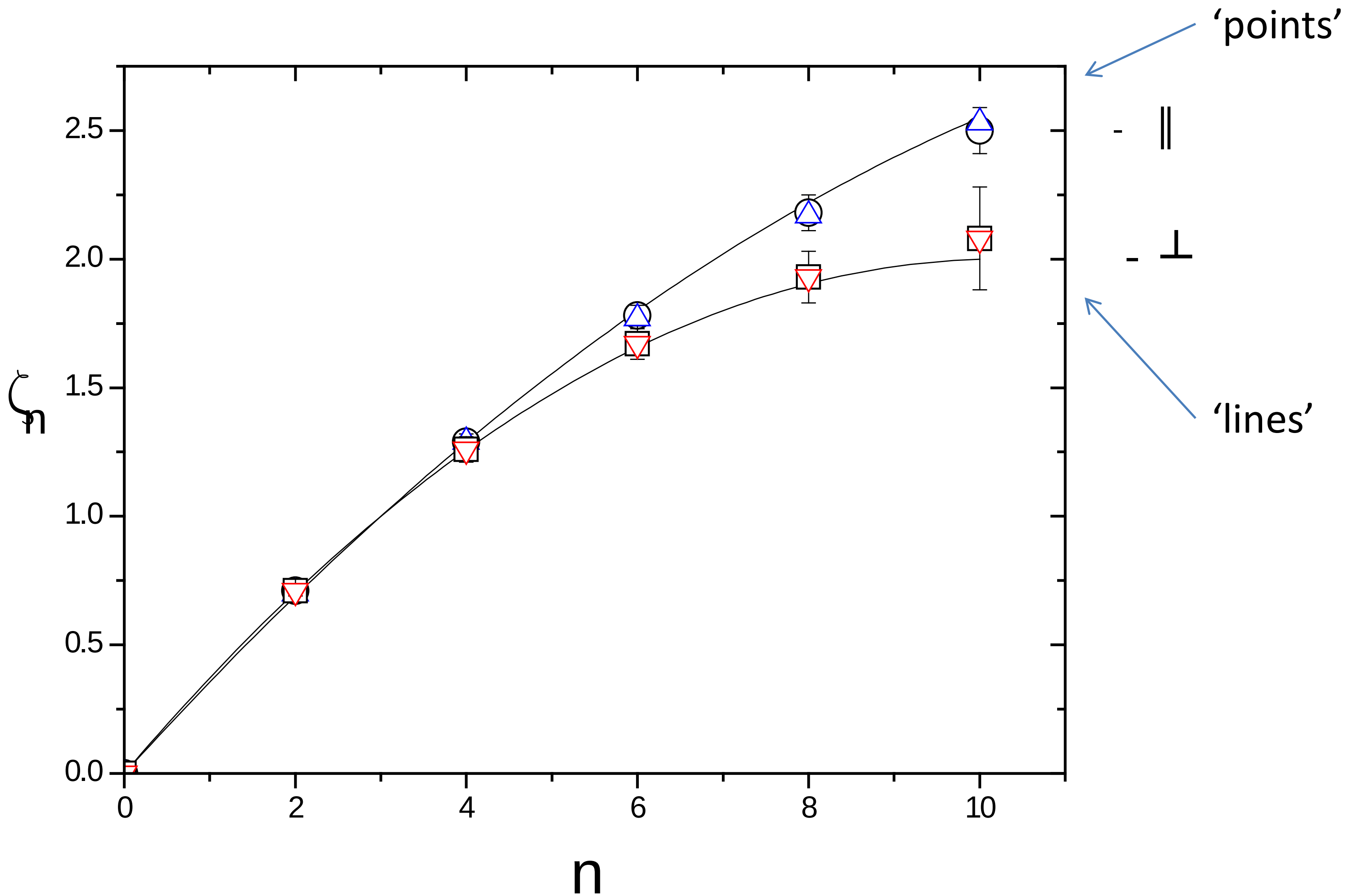
$$\frac{S_{2n}}{S_n^{\;2}} \underset{l\to 0}{\longrightarrow} \infty \quad \Rightarrow \quad \zeta_{2n} < 2\zeta_n$$

Phenomenological methods:

diagram (Belinicher,L'vov 1987),
multifractal (Parisi, Frisch 1985)

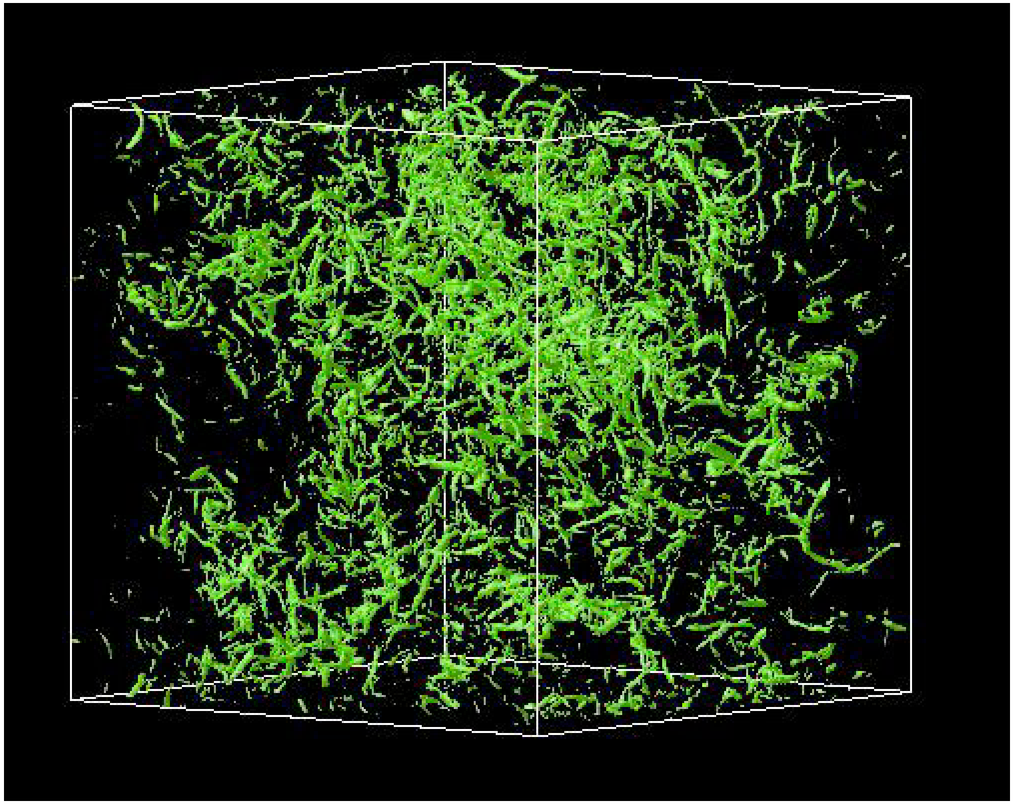
Vortex filament model
(Zybin, Sirota, Ilyin, Gurevich 2007)

Vortex Filament model + Multifractal model:

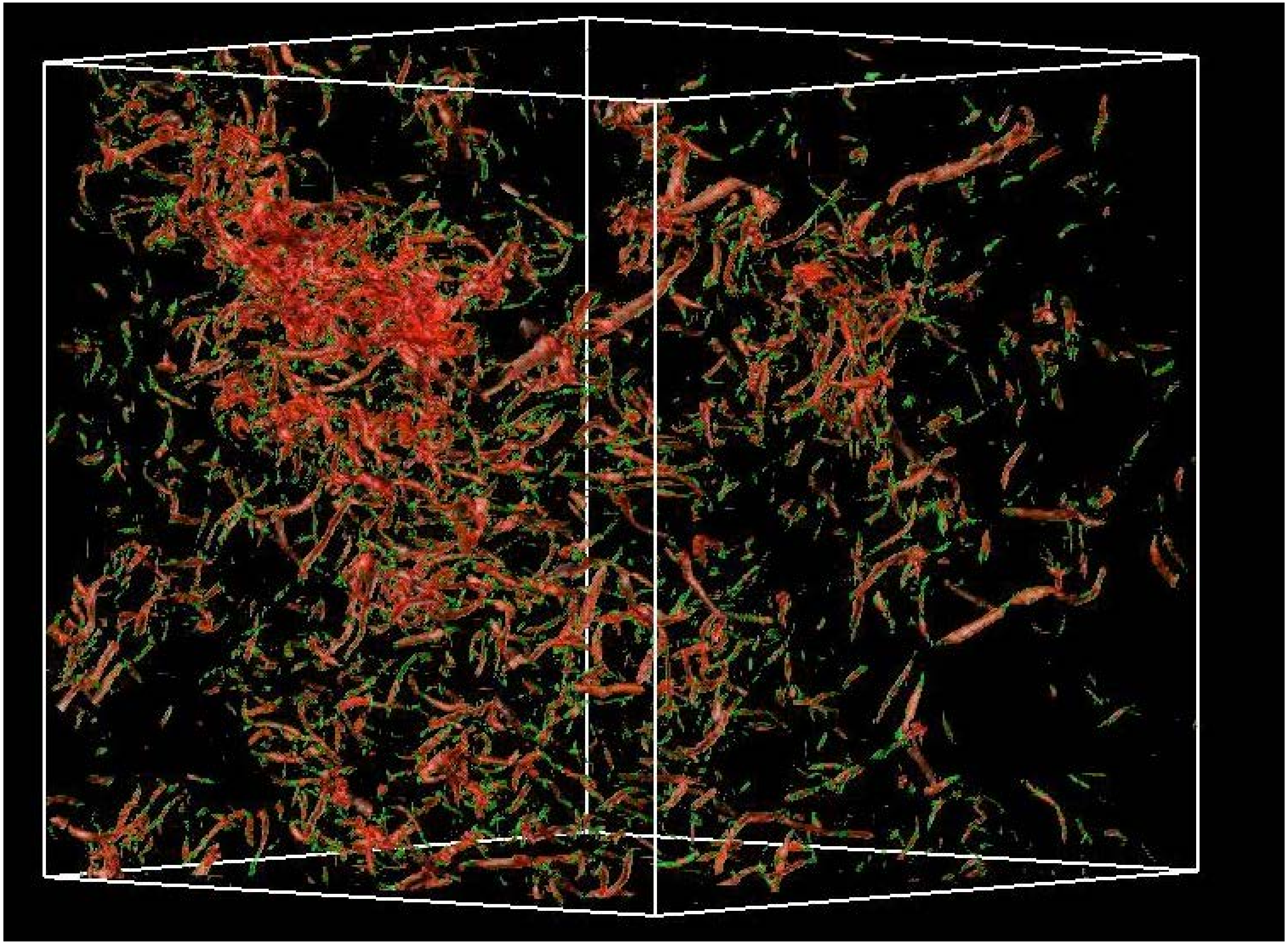


Zybin, Sirota arXiv:1204.1465v1; PRE **85**, 056317 (2012)

Regions $\omega > \Omega$:



Okamoto, Yohsimatsu, Schneider, Farge and Kaneda, 2010



Okamoto et al., 2007 *Phys. Fluids,* ***19(11)***

Coherent structures: long-living vortices

Crow, Champagne 71; Siggia 81; She,Jackson,Orszag 90, …

DNS, wavelet method:

99% of total energy, 80% of enstrophy

in coherent structures *Okamoto et al. 2007*

Kolmogorov (K41): Richardson cascade

Eddy turnover time is the maximal characteristic time

Multifractal approach: dimensional consideration, no information on geometry of a flow

-No account of structures;

-No theory based on the Navier-Stokes equation.

Vortex filament model

•Vortex filaments ($\omega \to \infty$)

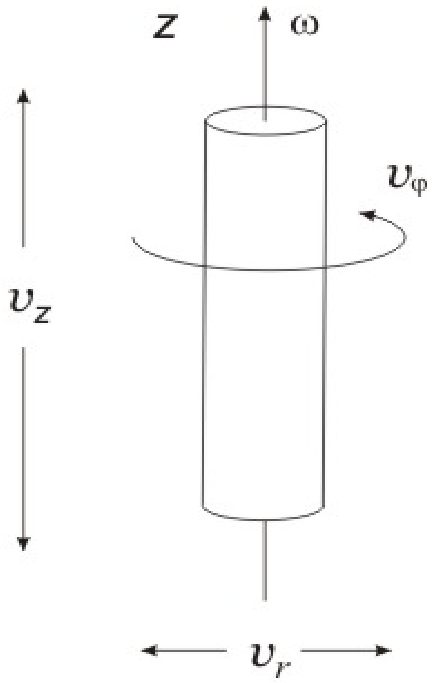
The appearance of filaments is random, but the flow inside a filament must have regular features ($\to$ *stability*)

•Large-scale fluctuations act as an external force to increase vorticity; no need of additional random forces

•Quasi-Lagrangian reference frame:

linear equation for vorticity $\omega = rot\ V$

Example: Cylindrical vortex filament



$$V_\varphi = \omega r \ , \quad V_r = a(t) r \ , \quad V_z = b(t) z$$

$$p(r,z,t) = \frac{1}{2} P_1(t) r^2 + \frac{1}{2} P_2(t) z^2$$

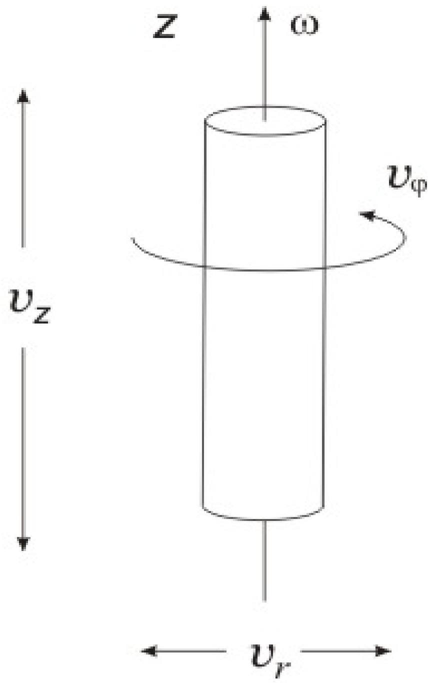
$$\left.\begin{aligned} & 2a + b = 0 \\ & \dot{a} + a^2 - \omega^2 = -P_1 \\ & \dot{\omega} + 2a\omega = 0 \\ & \dot{b} + b^2 = -P_2 \end{aligned}\right\} \quad \ddot{\omega} = -P_2(t)\omega$$

P_2 acts as “external” large-scale force

As $P_2(t) > 0$ the function ω oscillates, the oscillation amplitude changes weakly. On the contrary, as $P_2(t) < 0$,

the function ω grows exponentially, the cylinder stretches.

Example: Cylindrical vortex filament



$$V_\varphi = \omega\, r \;, \quad V_r = a(t) r \;, \quad V_z = b(t) z$$

$$\dot{\omega} + 2a\omega = 0 \qquad \ddot{\omega} = -P_2(t)\omega$$

$$S_n^{\perp} \propto \left\langle \omega^n \right\rangle l^n \;, \quad S_n^{\parallel} \propto \left\langle a^n \right\rangle l^n$$

$$S_n^{\perp} \gg S_n^{\parallel}$$

General case

$$\nabla_i V_j = \frac{1}{2}\varepsilon_{ijk}\omega_k + b_{ij} \ , \qquad b_{ij} = b_{ji} \quad , \qquad \vec{\omega} = \nabla \times \vec{V}$$

Euler equation :

$$\dot{b}_{ij} + (\vec{V}\nabla)b_{ij} + \frac{1}{4}\left(\omega_i\omega_j - \omega^2\delta_{ij}\right) + b_{ik}b_{kj} + \nabla_i\nabla_j p = 0$$

$$\dot{\omega}_i + (\vec{V}\nabla)\omega_i = b_{ik}\omega_k$$

Quasi-Lagrangian reference frame: (Belinicher, L'vov 87)

$$\vec{r}\,' = \vec{r} - \vec{\xi}(t) \ , \quad \vec{V}\,' = \vec{V} - \dot{\vec{\xi}} \ , \quad \ddot{\vec{\xi}} = -\nabla p\Big|_{\vec{r}=\vec{\xi}(t)}$$

Along the trajectory $\vec{V}\,'\Big|_{\vec{r}\,'=0} = 0$; $\dot{\omega}_i = b_{ik}\omega_k$

$$\boxed{\ddot{\omega}_i = -\rho_{ik}\omega_k} \qquad \rho_{ik} = \nabla_i\nabla_k p\Big|_{r'=0}$$

(Ohkitani 93)

$$\ddot{\omega}_i = -\rho_{ik}\omega_k$$

Euler equation: 15-10=5 free functions

$$\rho_{ij} = \tilde{\rho}_{ij} + \rho_{ij}^{\perp} \ , \quad \rho_{ij}^{\perp}\omega_j = 0$$

We assume that $\tilde{\rho}_{ij}$ is determined by **large-scale** pulsations and

Does not depend on ω inside the vortex filament.

Introducing probability

Suppose $\tilde{\rho}_{ij}$ is a stationary Gaussian random process.

$$\left\langle \tilde{\rho}_{ij}(t)\tilde{\rho}_{kl}(t')\right\rangle = D(t-t') \ , \quad D(\delta t > \tau_g) = 0$$

τ_g : correlation time

Probability density function:

$$f\left(t,\vec{\omega},\vec{v}\right)=\left\langle \delta\left(\vec{\omega}-\vec{\omega}\left(t\right)\right)\delta\left(\vec{v}-\vec{v}\left(t\right)\right)\right\rangle$$

$t \gg \tau_g$ δ – correlated process

$$\vec{v} \equiv \dot{\vec{\omega}}$$

One-dimensional case: $f\left(t,\omega,v\right)$

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial \omega}=\omega^2\frac{\partial^2 f}{\partial v^2}$$

$$t \to \infty\ , \quad v \to z = v^3/(3\omega^3)\ , \ \ \omega>0, \ \ v>0$$

$$\omega\frac{\partial f}{\partial \omega}=(3z+2)\frac{\partial f}{\partial z}+3z^2\frac{\partial^2 f}{\partial z^2}$$

$$f(\omega,z)=\sum_\alpha \omega^{-\alpha}F(z)$$

$$zF''+(z+2/3)F'+(\alpha/3)F=0$$

Probability density function

$$F_1(z) = e^{-z}\Phi(\tfrac{2-\alpha}{3}, \tfrac{2}{3}, z)$$

$$F_2(z) = (-z)^{1/3} e^{-z}\Phi(\tfrac{3-\alpha}{3}, \tfrac{4}{3}, z)$$

$$\Phi(a,b,z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \dots$$

$$J_\omega\big|_{\omega=0} = J_\nu\big|_{\nu=0} = 0 \quad \Rightarrow \quad \nu f(0,\nu) = -\omega^2 \partial f/\partial \nu\big|_{\nu=0} = 0$$

$$\Rightarrow \quad f(\omega,\nu) = \sum_n C_n \omega^{-(3n+2)} e^{-z}\Phi(-n, \tfrac{2}{3}, z)$$

$$P(\omega) = \int f d\nu = \frac{C_1}{\omega^4} + \frac{C_2}{\omega^8} + \dots$$

Probability density function:

$$f\left(t,\vec{\omega},\vec{v}\right)=\left\langle \delta\left(\vec{\omega}-\vec{\omega}\left(t\right)\right)\delta\left(\vec{v}-\vec{v}\left(t\right)\right)\right\rangle$$

$t \gg \tau_g$ $\quad$ δ – correlated process $\quad$ $\vec{v}\equiv\dot{\vec{\omega}}$

$$\frac{\partial f}{\partial t}+\vec{v}\,\frac{\partial f}{\partial \vec{\omega}}=\omega^2\,\frac{\partial^2 f}{\partial v^2}+\left(\vec{\omega}\frac{\partial}{\partial \vec{v}}\right)^2 f$$

Asymptotic stationary solution $\quad t\to\infty\ ,\ \ \omega\to\infty:$

$$P(\omega,t)=\left\langle \delta\left(\omega-\omega\left(t\right)\right)\right\rangle \propto \omega^{-4}$$

Multifractal model

(Parisi, Frisch 1985)

- generalization of K41

The Euler equation is invariant relative to

$$r \to \gamma r', \quad V \to \gamma^h V', \quad t \to \gamma^{1-h} t'$$

⟹ Classes of solutions with different *h*

Assumption: determinative contribution to velocity structure functions is given by the power-law solutions:

$$\Delta V(l) \equiv \left| \vec{V}(\vec{r} + \vec{l}) - \vec{V}(\vec{r}) \right| \sim l^{h(\vec{r})}$$

(K41: *h*=1/3)

Large deviations theory:

$$P(l,h) = l^{3-D(h)}$$

D(h) $\approx$ dimension of the set *h* ; e.g., $D = 0 \Rightarrow P \propto l^3$

$$\left\langle \Delta V^n \right\rangle = \int \Delta V(l,h) P(h) d\mu(h)$$

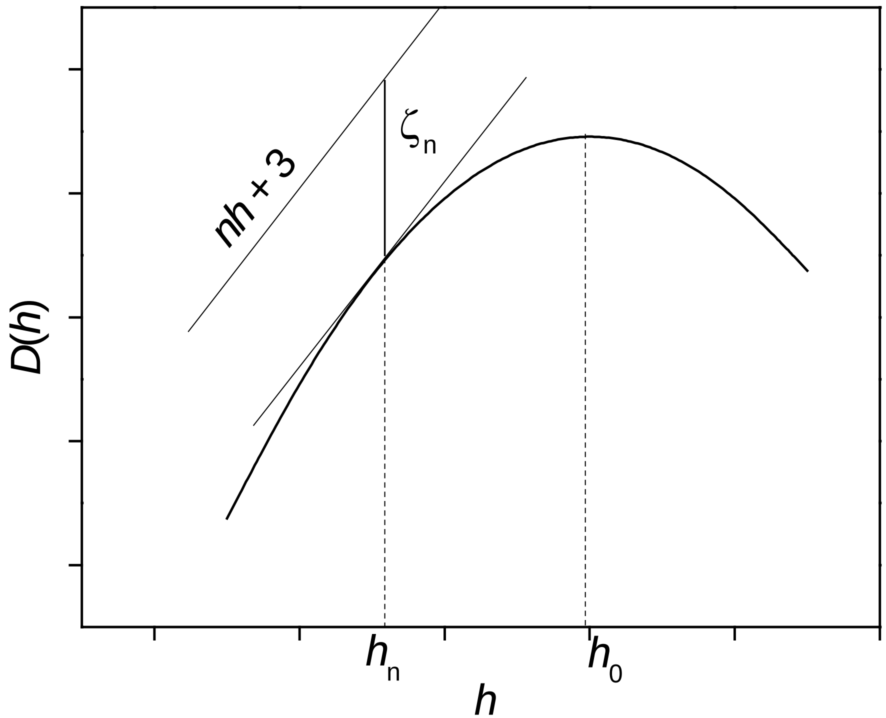
$$= \int l^{nh} l^{3-D(h)} d\mu(h)$$

$$S_n(l) \propto l^{\zeta_n}, \qquad \zeta_n = \lim_{l \to 0} \frac{\ln \left\langle \Delta V^n \right\rangle (l)}{\ln l}$$

$$\boxed{\zeta_n = \min_n \left(nh + 3 - D(h) \right)}$$

- Legendre transformation

$$\zeta_n = \min_n \left(nh + 3 - D(h) \right)$$



Without loss of generality
D''(h)<0;

$$\zeta_0 = 0 \quad \Rightarrow \quad D\left(h_0\right) = 3$$

$$\zeta_n'' < 0 \ , \quad \zeta_n > 0 \ \Rightarrow \ \zeta_n' \geq 0 \ \Rightarrow \ h_{\min} \geq 0$$

$$n \geq n_* = D'(h_{\min}) \ \Rightarrow \quad \zeta_n = nh_{\min} + 3 - D(h_{\min})$$

Combination with Multifractal model

$$P(\omega) \equiv \left\langle \delta\left(\omega - \Omega_\gamma\right)\right\rangle \propto \frac{1}{\omega^4}$$

Ergodic theorem:

$$P(\omega) = \frac{1}{V}\int \delta\left(\omega - \Omega(\vec{r})\right)\, d^3\vec{r}$$

Local axial symmetry:

$$P(\omega) = \frac{2\pi L}{V}\sum_i \int \delta\left(\omega - \Omega_i(r)\right) r\,dr = \frac{2\pi L}{V}\sum_i \frac{r_i(\omega)}{\left|\Omega_{i\,r}^{'}\right|}$$

$$\Rightarrow \quad c\sum_i \frac{r_i(\omega)\ \omega^4}{\left|\Omega_{i\,r}^{'}(\omega)\right|} = 1$$

Combination with Multifractal model

$$P(\omega) \propto \frac{1}{\omega^4} \quad \Longrightarrow \quad c\sum_i \frac{r_i(\omega)\ \omega^4}{\left|\Omega'_{i\,r}(\omega)\right|} = 1$$

Parametrize by h, α:

$$\omega \sim \Delta V / l \sim l^{h-1},\ \ \Omega'_r \sim l^{h-2},\ \ r \sim l$$

$$\sum_\alpha c_\alpha \int l^{3-D(h)} \frac{r\,\Omega^4}{\left|\Omega'_r\right|}\, dh \ = \sum_\alpha \tilde{c}_\alpha \int l^{3-D(h)+3h-1} dh = 1$$

$$l \to 0 \quad \Rightarrow \qquad \min_h \left(3h + 2 - D(h)\right) = 0$$

$$\zeta_n = \min_n \left(nh + 3 - D(h)\right) \quad \Rightarrow \qquad \boxed{\zeta_3 = 1}$$

-NO assumptions on *D(h)* !

Combination with Multifractal model

$h_{\min}$=?

$$\zeta_n' \geq 0 \;\Rightarrow\; h_{\min} \geq 0$$

Small *h* = large *n* = most rare events;

$$h_{\min} = \lim_{n\to\infty}\left(\zeta_n / n\right) \;, \quad \left\langle \Delta V^n \right\rangle^{1/n} \propto l^{\zeta_n/n} \underset{n\to\infty}{\longrightarrow} l^{h_{\min}}$$

The steeper $\Delta V(l)$ for a given vortex, the less is *h* to which it contributes.

$$\boxed{h_{\min} = 0}$$

- extreme filament: $\Delta V \sim l^0$

$$V \propto \left[\vec{e}_z, \vec{r}/r\right]$$

(This is the limit; for any h>0 there is a smooth solution.)

Combination with Multifractal model

$D(0) = ?$ *extreme filament:* $\Delta V \sim l^0$

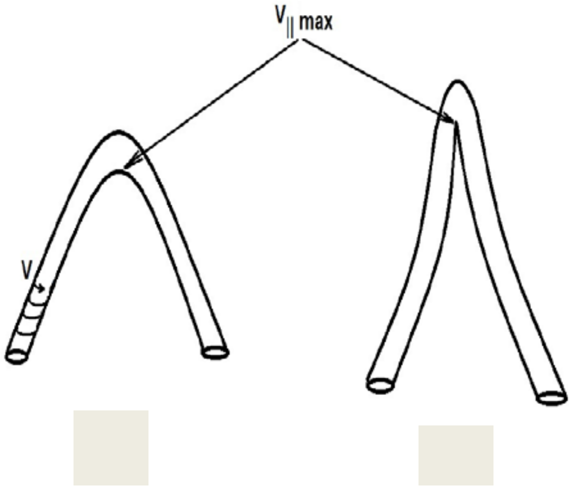
$$V \propto \left[\vec{e}_z, \vec{r}/r\right]$$

-cylindric extreme filament

$$\left\langle \left| \Delta V \times \vec{l}/l \right|^n \right\rangle \propto \frac{2^n}{n} l^2 \Leftrightarrow P(l) \propto l^{3-D} \propto l^2 \Rightarrow D^{\perp}(0) = 1$$

$$\left\langle \left| \Delta V \cdot \vec{l}/l \right|^n \right\rangle \propto n^{-5/2} l^2$$

-small contribution to $S_n^{\parallel}$



'point-like' extreme filament:

$$\vec{V}(\theta, \phi) \qquad \vec{V} = \left(V_r(\theta), V_\theta(\theta), 0\right)$$

$$P(l) \propto l^{3-D} \propto l^3 \Rightarrow D^{\parallel}(0) = 0$$

$$D(h) = 3 - b(h - h_0)^2 \quad ;$$

$$\zeta_3 = \min_h \left(3h + 3 - D(h)\right) = 1 \quad ;$$

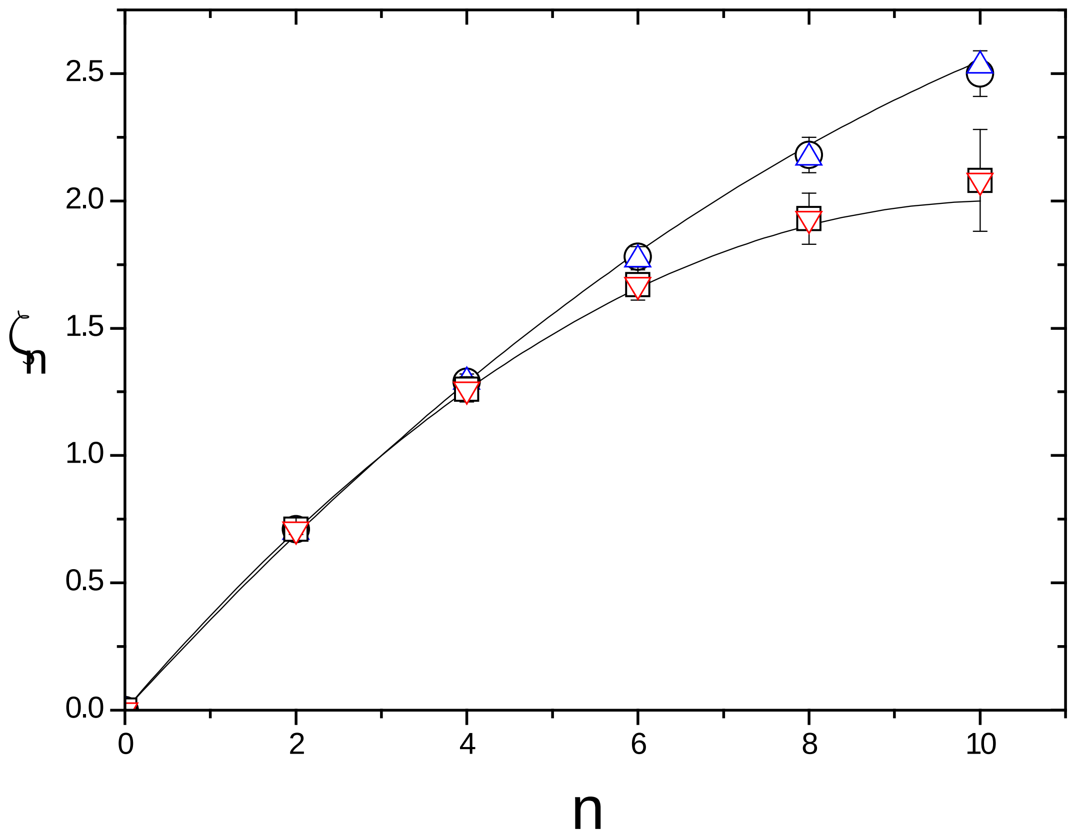
$$h_{\min} = 0 \quad ; \quad D^{\parallel}(h_{\min}) = 0 \quad , \quad D^{\perp}(h_{\min}) = 1$$

$$\zeta_n = nh_0 - n^2/4b \ , \quad n \le n_* = 2bh_0 \quad \begin{cases} D^{\parallel}: \ b = 3/h_0^{\parallel 2}, & h_0^{\parallel} - \frac{1}{4} h_0^{\parallel 2} = \frac{1}{3} \\ D^{\perp}: \ b = 2/h_0^{\perp 2}, & h_0^{\perp} - \frac{3}{8} h_0^{\perp 2} = \frac{1}{3} \end{cases}$$

$$\zeta_n^{\parallel} = \begin{cases} 0.367n - 1.12 \cdot 10^{-2} n^2, & n \le 16.3 \\ 3, & n > 16.3 \end{cases}$$

$$\zeta_n^{\perp} = \begin{cases} 0.391n - 1.91 \cdot 10^{-2} n^2, & n \le 10.2 \\ 2, & n > 10.2 \end{cases}$$

Small corrections may change the intermediate regime of approaching the constant at n~10-15



$$D(h) = 3 - b(h - h_0)^2 \quad ;$$

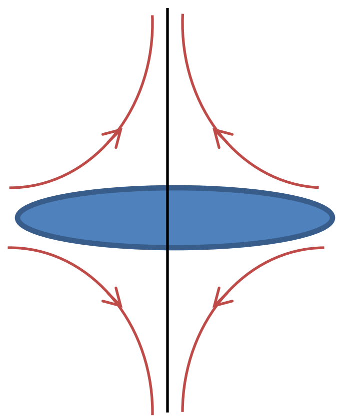
$$\zeta_3 = 1 \;\; ; \quad h_{\min} = 0 \;\; ; \quad D^{\parallel}(h_{\min}) = 0 \;\; , \quad D^{\perp}(h_{\min}) = 1$$

Conclusion

- We combine the Vortex filament model with Multifractal model. This allows to derive the third-order velocity scaling exponent independently on Kolmogorov's assumptions.
- We calculate transverse and longitudinal Euler velocity scaling exponents. The results coincide with the DNS inside the error boxes. NO fitting parameters are used.
- We explain the difference between transverse and longitudinal exponents by the difference of contributing filaments: roughly cylindric for $\zeta_n^{\perp}$, strongly curved for $\zeta_n^{\parallel}$.

Example: a bunch of filaments

Large-scale flow:



$$V_r = a(t) r / R \,, \quad V_z = b(t) z / L$$

$$p = \frac{1}{2} p_1(t) \; (r/R)^2 + \frac{1}{2} p_2(t) \; (z/Z)^2$$

$$\left.\begin{aligned} &2a/R + b/L = 0 \\ &R\dot{a} + a^2 = -p_1 \\ &L\dot{b} + b^2 = -p_2 \end{aligned}\right\}$$

Adding small-scale pulsations:

$$\vec{V}(t,r) = \vec{e}_r a(t) \frac{r}{R} + \vec{e}_z b(t) \frac{z}{L} + \vec{u} \qquad \nabla \cdot \vec{u} = 0, \quad \nabla \times \vec{u} = \vec{\omega}$$

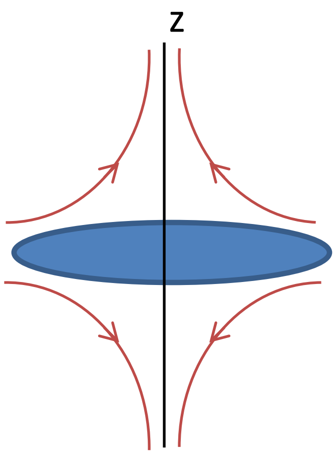
$$u_z = 0, \quad \frac{\partial \vec{u}}{\partial z} = 0$$

$$\frac{d\vec{u}}{dt} + \frac{a}{R}\vec{u} + \nabla \tilde{p} = 0, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{ar}{R}(\vec{e}_r \cdot \nabla) + (\vec{u} \cdot \nabla)$$

$$\Rightarrow \qquad \frac{d^2 \omega}{dt^2} = -p_2(t)\omega$$

p_2 acts as “external” large-scale force independent on $\vec{u}$

Example: a bunch of filaments



$$\vec{V}(t,r) = \vec{e}_r a(t)\frac{r}{R} + \vec{e}_z b(t)\frac{z}{L} + \vec{u}$$

$$\nabla \cdot \vec{u} = 0, \quad \nabla \times \vec{u} = \vec{\omega}$$

$$\Rightarrow \quad \frac{d^2\omega}{dt^2} = -p_2(t)\omega$$

p_2 acts as “external” large-scale force independent on $\vec{u}$

As $P_2(t) > 0$ the function ω oscillates, the oscillation amplitude changes weakly. On the contrary, as $P_2(t) < 0$,

the function ω grows exponentially, the filament stretches.

Statistics: moments of ω

$$\ddot{\omega}_i = -\rho_{ik}\omega_k$$

$$D(t) = \delta(t):$$

Exact solution:

$$M(n,m,k) = \left\langle \omega^{2n}\dot{\omega}^{2m}(\omega\cdot\dot{\omega})^k \right\rangle$$

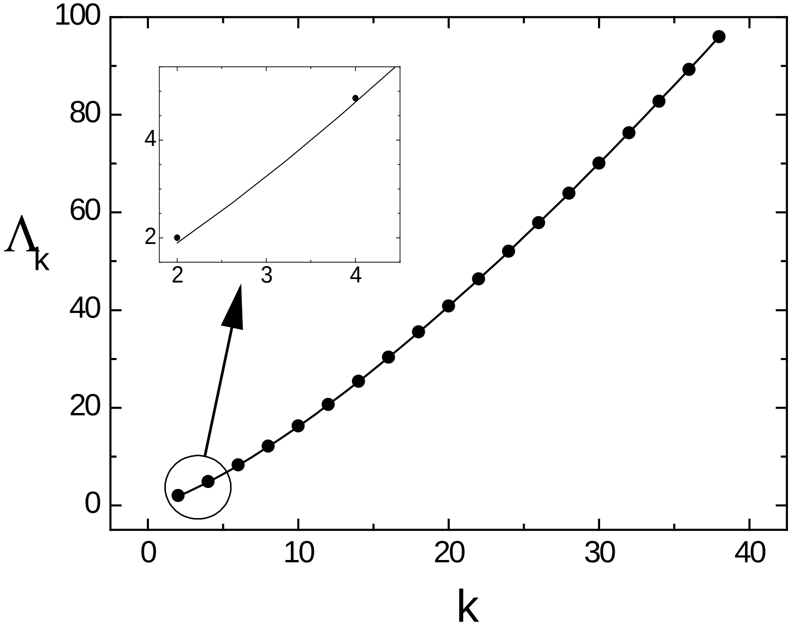
$$\frac{d}{dt}M(n,m,k) = 2nM(n-1,m,k+1) + 2m(4k+2m+2)M(n+1,m-1,k) + \ldots$$

$$\left\langle \omega^{2n} \right\rangle \propto e^{\Lambda_{2n}t}$$

$$\ddot{\omega}_i = -\rho_{ik}\omega_k$$

Statistics: moments of ω

$D(t) = \delta(t):$



Approximate analytic solution:

WKB: $$\omega(t) \sim e^{\int \sqrt{\lambda(t')}dt'}$$

$$\left\langle \omega^n(t) \right\rangle = \int \prod_{\tau \le t} \prod_{i \le j} d\rho_{ij}(\tau) e^{-\frac{1}{2}\int_0^t \rho_{ij}^2(t')dt'} \omega^k[t, \rho(\tau)]$$

saddle point method

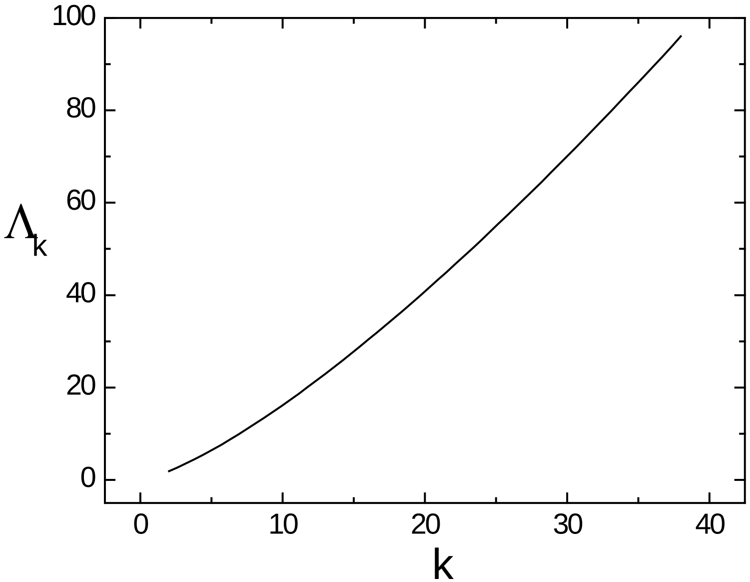
$$\left\langle \omega^n \right\rangle \propto e^{\Lambda_n t}, \quad \Lambda_n = \alpha n^{4/3} \quad \forall n$$

Arbitrary *D(t)*: $\left\langle \omega^n \right\rangle \propto e^{\Lambda_n t \cdot \Phi(t)}, \quad \Phi(t) < 1, \ \ \Phi(t) \underset{t \to \infty}{\longrightarrow} 1$ - rescaling of *t*

Bounded ρ: $\lambda = \sqrt{\rho_{\max}} \qquad \Lambda_n = \lambda n, \quad n > N$

Statistics: moments of ω

For any *D(t)* $\left\langle \omega^{2n} \right\rangle \propto e^{\Lambda_{2n} t}$ $\Lambda_n \propto n^{4/3}$



$\Lambda_{2n} > n\Lambda_2$ - Intermittency

(higher moments grow faster than lower ones)

Bounded ρ : $\Lambda_n = \lambda n, \quad n > N$ Intermittency breaks at large n

$$\lambda = \sqrt{\rho_{\max}}$$

Transverse Euler structure functions

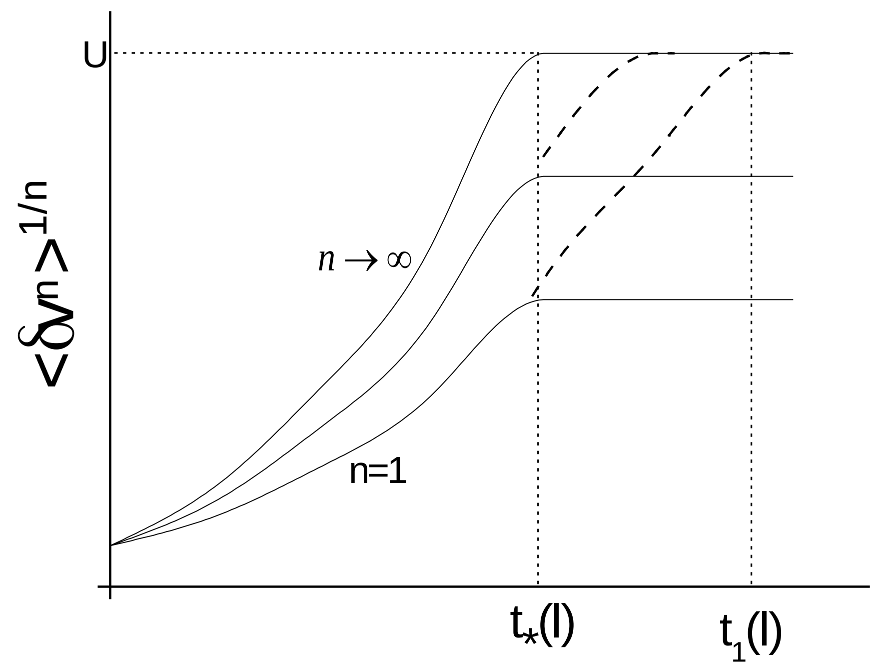
$$S_n^{\perp}(l) = \left\langle \left| \vec{V}_{\perp}(\vec{r}+\vec{l}) - \vec{V}_{\perp}(\vec{r}) \right|^n \right\rangle \qquad \Delta V_{\perp}^n \approx l^n \omega^n \qquad \left\langle \omega^n \right\rangle \propto e^{\Lambda_n t}$$

The vorticity growth is limited by nonlinearity; “energetic-like” considerations are needed

$$\Delta V < U \quad \Rightarrow \left\langle \Delta V^m \right\rangle \approx l^m \left\langle \omega^m \right\rangle$$

$$\approx l^m T^{-m} e^{\Lambda_m t} < U^m \quad \forall m$$

$$t_* = \inf_m \left(\frac{m}{\Lambda_m} \ln \frac{U}{l} \right) = \frac{1}{\lambda} \ln \frac{UT}{l}$$



Another way to obtain t*

Karman-Howarth-Monin, $\nu \to 0$

$$\frac{\partial}{\partial t}\left\langle \vec{V}(\vec{r})\vec{V}(\vec{r}+\vec{l}\,)\right\rangle = \frac{1}{2}\nabla_{\vec{l}}\left\langle |\,\delta\vec{V}(\vec{l}\,)\,|^2\,\delta\vec{V}(\vec{l}\,)\right\rangle$$

$$\Longrightarrow \quad \frac{1}{l}\left\langle \delta\vec{V}(\vec{l}\,)^3\right\rangle \sim l^2 T^{-3} e^{\Lambda_3 t_3} \sim U^2/T \quad \Longrightarrow \quad e^{\Lambda_3 t_3} \sim L^2/l^2$$

"Karman-Howarth-Monin" for higher orders:

$$\frac{\partial}{\partial t}\left\langle \left(\vec{V}(\vec{r})\vec{V}(\vec{r}+\vec{l}\,)\right)^n\right\rangle = a_0\nabla_{\vec{l}}\left\langle |\,\delta\vec{V}(\vec{l}\,)\,|^{2n}\,\delta\vec{V}(\vec{l}\,)\right\rangle + \ldots$$

$$\Longrightarrow \quad \frac{1}{l}\left\langle \delta\vec{V}^{\,2n+1}\right\rangle \sim l^{2n} T^{-(2n+1)} e^{\Lambda_{2n+1} t_{2n+1}} \sim U^{2n}/T \quad \Longrightarrow \quad e^{\Lambda_{2n+1} t_{2n+1}} \sim \left(L/l\right)^{2n}$$

$$t_* = \inf_m t_m = \inf_m \left(\frac{m}{\Lambda_m}\ln\frac{L}{l}\right) = \frac{1}{\lambda}\ln\frac{L}{l} \qquad \lambda = \lim_{n\to\infty}\frac{\Lambda_n}{n}$$

Transverse Euler structure functions:

$$S_n^{\perp}(l) \propto l^n e^{\Lambda_n t_*} \propto l^{\zeta_n}, \qquad \zeta_n = n - \frac{\Lambda_n}{\lambda}$$

$$\lambda = \lim_{n\to\infty} \frac{\Lambda_n}{n}$$

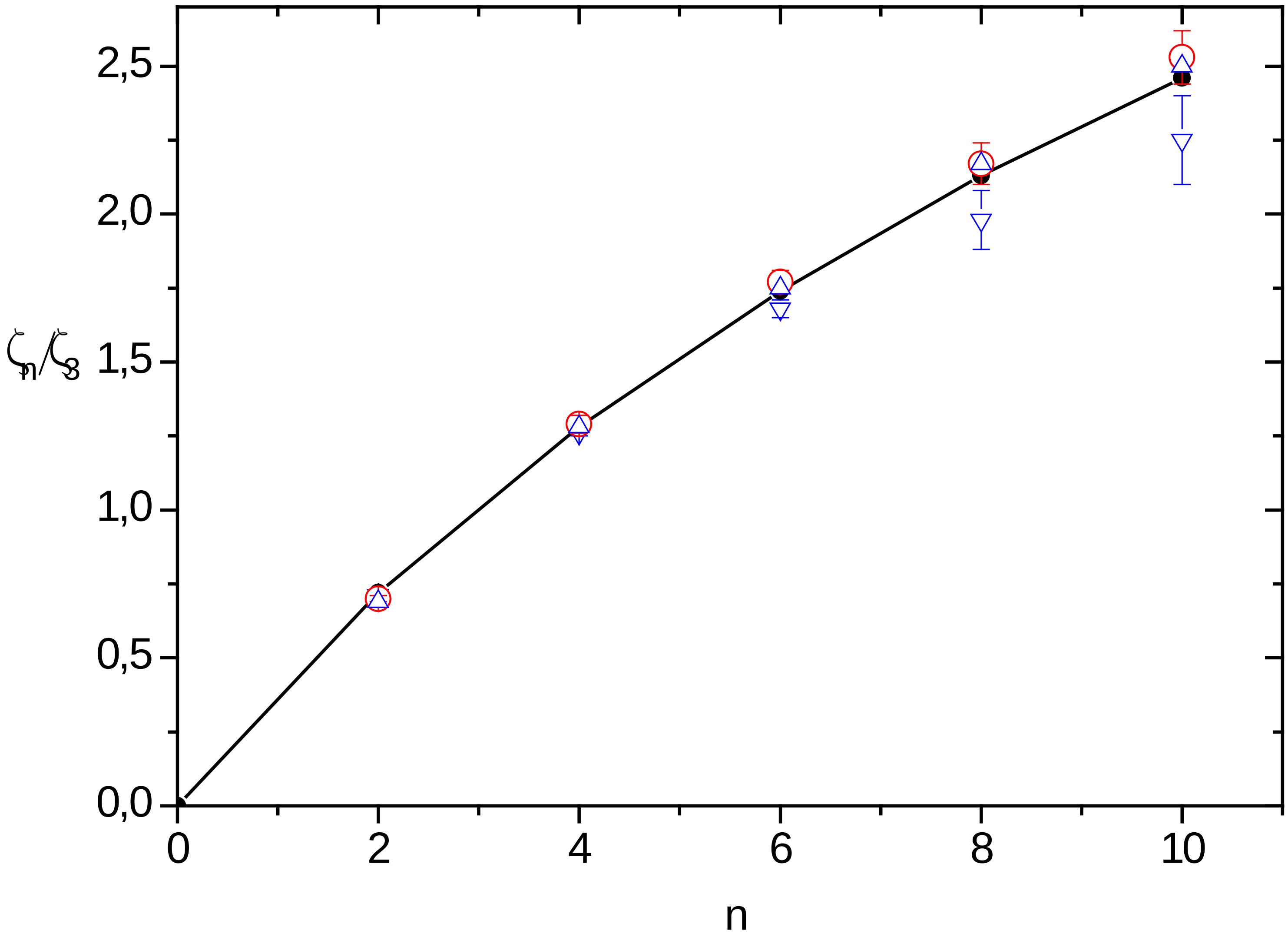
Lagrangian structure functions:

$$S_n^L(\tau) = \left\langle \left| \vec{V}(t+\tau) - \vec{V}(t) \right|^n \right\rangle \qquad \Delta V \approx r_0 \omega^2 \tau$$

$$\Delta V < U \quad \Rightarrow \quad \left\langle \Delta V^m \right\rangle \approx \tau^m \left\langle \omega^{2m} \right\rangle \approx \tau^m T^{-2m} e^{\Lambda_{2m} t_*^L} < U/r_0 \quad \forall m$$

$$t_*^L = \inf_m \left(\frac{m}{2\Lambda_m} \ln \frac{UT^2/r_0}{\tau} \right)$$

$$S_n^L(l) \propto \tau^n e^{\Lambda_{2n} t_*^L} \propto l^{\xi_n}, \qquad \xi_n = n - \frac{\Lambda_{2n}}{2\lambda}$$



Theory: ● +line; Benzi et al. 2010: ▲ (∥), ▼ (⊥); Gotoh et al. 2002 : ○ (∥)

Comparison with DNS

DNS: *R. Benzi et al. J. Fluid Mech.* ***653****,221 (2010)*

n	ξ_n / ξ_2 (Lagrangian)		ζ_n / ζ_3 (Euler)		
	DNS	Theory	DNS $\perp$	DNS $\parallel$	Theory
2	1	1	0.71 ± 0.01	0.69 ± 0.005	0.72
4	1.66 ± 0.02	1.66	1.26 ± 0.01	1.28 ± 0.01	1.28
6	2.10 ± 0.10	2.14	1.68 ± 0.03	1.75 ± 0.01	1.74
8	2.33 ± 0.17	2.45	1.98 ± 0.1	2.17 ± 0.03	2.13
10	2.45 ± 0.35	2.64	2.25 ± 0.15	2.5 ± 0.05	2.46

Comparison with DNS

$$\lambda = \frac{\Lambda_{2n} - \Lambda_4(\zeta_n / \zeta_2)}{2n - 4\ \zeta_n / \zeta_2} = \frac{\Lambda_n - \Lambda_3(\xi_n / \xi_3)}{n - 3\ \xi_n / \xi_3}$$

The values λ calculated from scaling exponents of different orders

(by data of *R. Benzi et al. J. Fluid Mech.* ***653****,221 (2010)*)

n	λ	
	Lagrange	Euler
4	3.0 ± 0.1	3.0 ± 0.3
6	2.9 ± 0.2	2.8 ± 0.25
8	2.8 ± 0.2	2.7 ± 0.25
10	2.8 ± 0.3	2.7 ± 0.25

Kolmogorov approach

Energy flux:

$$\varepsilon = \mathrm{const}$$

Dimension theory

Nonlinear cascade

Our approach

Stochastic vortex instability

Vortex filaments

Linear vortex stretching

Balance across the vortex filament

process

nonlinearity

$$\frac{\mathrm{dv}}{\mathrm{dt}} = -\frac{\nabla p}{\rho} + f$$

Stochastic equation

$$\frac{\mathrm{d}^2\omega_i}{dt^2} = -\rho_{ik}\omega_k$$

Velocity structure functions can be determined without use of energy flux or dimensional consideration

Main points

$$\frac{d^2\omega_i}{dt^2} = -\rho_{ik}\omega_k$$

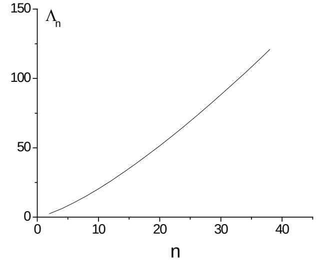
-the main equation, consequence of the NSE

$$\rho_{ik} = \nabla_i \nabla_k p\Big|_{r'=0}$$

The longitudinal part is determined by large-scale pulsations and act as an “external” random force. The equation becomes a linear stochastic equation.

$$\left\langle \omega^{2n} \right\rangle \propto e^{\Lambda_{2n}\, t}$$

$$\Lambda_{2n} > n\Lambda_2$$



-solution to the main equation;

Intermittency breaks at large *n* ;

Approximate analytic solution for any statistics of ρ

$$t_* = \inf_m \left(\frac{m}{\Lambda_m} \ln \frac{L}{l} \right) = \frac{1}{\lambda} \ln \frac{L}{l}$$

-nonlinearity breaks the exponential growth

$$S_n^{\perp}(l) \propto l^n e^{\Lambda_n t_*} \propto l^{\zeta_n}, \quad \zeta_n = n - \frac{\Lambda_n}{\lambda}$$

$$S_n^{L}(l) \propto l^{\xi_n}, \quad \xi_n = n - \frac{\Lambda_{2n}}{2\lambda}$$

Conclusion

1.Lagrangian and Eulerian transverse velocity structure functions of high orders are derived in the inertial range.
2. Scaling exponents are determined.
3.The comparison of the theory predictions with experimental results shows a very good agreement.
4. The results do not depend on the correlation time of the large-scale fluctuations and depend only weakly on the Gaussianity of their statistics.

All the obtained relations are the consequences of the equation derived directly from the Navier-Stokes equation in the inertial range. The power-law dependence of the structure functions was not suggested in the model.
Only one fitting parameter was used to determine all the scaling exponents. The result could, in principle, differ by many times from the experiments. The more inspiring is the coincidence.