WITTEN INDEX OF SUPERSYMMETRIC 3D THEORIES REVISITED

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based on
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JHEP 1205:103(2012) [arXiv:1202.6566]

MOTIVATION

- Widely known : Maldacena duality

  $AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ 4d SYM

- New duality (Bagger + Lambert, 07; Aharony + Bergman + Jafferis + Maldacena, 08)

$AdS_4 \times S^7$ or $AdS_4 \times CP^3 \leftrightarrow$
$\mathcal{N} = 8$ or $\mathcal{N} = 6$ 3d SCS

DYNAMICS ?

THE SIMPLEST VARIANT

N = 1 3d SYM + CS

$$S = \frac{1}{g^2}\mathrm{Tr}\int d^3x \left\{-\frac{1}{2}F^2_{\mu\nu} + i\bar{\lambda}\not{D}\lambda\right\} +$$
$$\kappa \mathrm{Tr}\int d^3x \left\{\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) - \bar{\lambda}\lambda\right\}$$

$\lambda_{1,2}$ - Majorana fermion.

- A chiral gauge theory.
- mass scale $m = \kappa g^2$.
- Gauge invariance dictates for $k = 4\pi\kappa$ to be integer (the level).

Vacuum dynamics ?

Witten index

$$I_W = \mathrm{Tr}\left\{(-1)^F e^{-\beta H}\right\} = ?$$

The result (Witten, 99)

$$I(k, N) \;=\; (\mathrm{sgn}(k))^{N-1} \begin{pmatrix} |k| + N/2 - 1 \\ N-1 \end{pmatrix}$$

- for $SU(N)$ group

if $|k| \geq N/2$.

- $I(k, N) = 0$ if $|k| < N/2$.
- Spontaneous SUSY breaking at $|k| < N/2$.

Witten's derivation

• Consider the theory in a LARGE spatial box, $g^2 L \gg 1$

• Integrate mentally over fermions. k is renormalized. For positive k,

$$k \;\rightarrow\; k - \frac{N}{2}\,.$$

• The coupling g^2 is also renormalized. New higher derivative terms appear. Irrelevant at large volume !

• We obtain a pure CS theory with renormalized coupling.

• Topological theory, a finite number of states.

$$\#_{\text{of vacuum states in SYMCS}}(k) \;=\; \#_{\text{of states in CS}}(k-N/2)\,.$$

Clever people can calculate the # of states in pure CS using the correspondence

pure CS $\leftrightarrow$ 2d WZNW $\leftrightarrow$ 2d conformal theories

• Number of states on the left equals number of conformal blocks on the right.

• This gives $(k > 0)$

$$\#_{CS} = \begin{pmatrix} N+k-1 \\ N-1 \end{pmatrix} .$$

canonical quantization

(Elitzur + Moore + Schwimmer + Seiberg, 89; Labastida + Ramallo, 89)

pure CS Lagrangian

$$\mathcal{L}_{CS} = -\kappa\epsilon_{jk}\left[\mathrm{Tr}\{A_j\dot{A}_k\} + \mathrm{Tr}\{A_0F_{jk}\}\right]$$

Canonical momenta

$$\Pi_j^a = \frac{\kappa}{2}\epsilon_{jk}A_k^a\ .$$

$\Pi_j^a \to -i\delta/(\delta A_j^a)$ as usual.

$G_j^a = \Pi_j^a - (\kappa/2)\epsilon_{jk}A_k^a = 0$ are second class constraints.

Only a half of them can be implemented

$$(\hat{G}_1^a + i\hat{G}_2^a)\Psi[A] = 0 \quad \text{or} \quad (\hat{G}_1^a - i\hat{G}_2^a)\Psi[A] = 0$$

• WE calculate the index via the effective Hamiltonian in small finite volume, $g^2 L \ll 1$. Similar to the index calculation for 4d theories (Witten, 82).

• Consider the $4d$ pure SYM theory.
• Small volume $g^2(L) \ll 1$
• Periodic b.c. $A_j(x+L, y, z) = A_j(x, y, z)$, etc.
• Expand in modes.

Born–Oppenheimer approach

slow variables: $C_j^{a=1,\ldots,r} \equiv A_j^{\mathrm{Cartan}(\vec{0})}$ and $\lambda_\alpha^{a=1,\ldots,r} \equiv \lambda_\alpha^{\mathrm{Cartan}(\vec{0})}$.

fast variables: all the rest

to leading order

$$H^{\text{eff}} = \frac{1}{2}(P_j^a)^2, \quad a = 1, \dots, r \text{ (rank of the group)}$$

• for SU(2), just 3 bosonic (C_j) and 2 holomorphic fermion (λ_α) variables.

One sees four vacuum states: $C, C\lambda_\alpha, C\lambda_\alpha\lambda^\alpha$.

• But original wave functions are gauge invariant.

Ergo effective wave functions are Weyl invariant

(rotation by π around 2-nd color axis),

$$\Psi(-C_j, -\lambda_\alpha) = \Psi(C_j, \lambda_\alpha)$$

• Two bosonic Weyl invariant functions: C and $C\lambda_\alpha\lambda^\alpha$

•$I_W = 2$. $I_W = N$ for SU(N).

SYMCS

assume $g^2 L \ll 1$

Slow variables: $C_{j=1,2}^{a=1,\dots,r} \equiv A_j^{\text{Cartan}(\mathbf{0})}$ and $\lambda^{a=1,\dots,r} \equiv \lambda_{1-i2}^{\text{Cartan}(\mathbf{0})}$.

Leading order $(N = 2)$

$$H^{\text{eff}} \;=\; \frac{g^2}{2L^2}\left(P_j - \frac{\kappa L^2}{2}\epsilon_{jk} C_k\right)^2 + \frac{\kappa g^2}{2}(\lambda\bar\lambda - \bar\lambda\lambda)\;.$$

- Landau problem on a finite (dual !) torus, $C_{j=1,2} \in (0, 4\pi/L)$
- Index = magnetic flux = $2k$. (Dubrovin, Krichever, Novikov, 76)

$$SU(N): \quad H \;=\; \frac{g^2}{L^2}\left[\frac{(P_j^a + \mathcal{A}_j^a)^2}{2} + \frac{1}{2}\mathcal{B}^{ab}(\lambda^a\bar\lambda^b - \bar\lambda^b\lambda^a)\right]\;,$$

where $\mathcal{B}^{ab} = \epsilon_{jk}\partial_j^a \mathcal{A}_k^b$.

- At the tree level,

$$\mathcal{A}_j^a = -\frac{\kappa L^2}{2}\epsilon_{jk}C_k^a \ ,$$

implying

$$\mathcal{B}^{ab} = \kappa L^2 \delta^{ab} \ .$$

- The index

$$I = \frac{1}{(2\pi)^r}\int_{T\times T}\prod_{ja} dA_j^a \ \det\|\mathcal{B}^{ab}\| = Nk^{N-1} \ .$$

(Cecotti + Girardello, 82).

Vacuum wave functions, SU(2)

- boundary conditions

$$\Psi(x+1,y) = e^{-2\pi i k y}\Psi(x,y)$$
$$\Psi(x,y+1) = e^{2\pi i k x}\Psi(x,y)$$

$(x = C_1 L/(4\pi),\ y = C_2 L/(4\pi))$.

$$\Psi_m \sim e^{-\pi k \bar{z} z} e^{\pi k \bar{z}^2} Q_m^{2k}(\bar{z})$$

- $\bar{z} = x - iy,$

$$Q_m^{2k}(\bar{z}) = \sum_{n=-\infty}^{\infty} \exp\left\{-2\pi k \left(n + \frac{m}{2k}\right)^2 + 4\pi i k \bar{z} \left(n + \frac{m}{2k}\right)\right\},$$

$m = 0, \ldots, 2k-1.$

- Weyl-invariant combinations

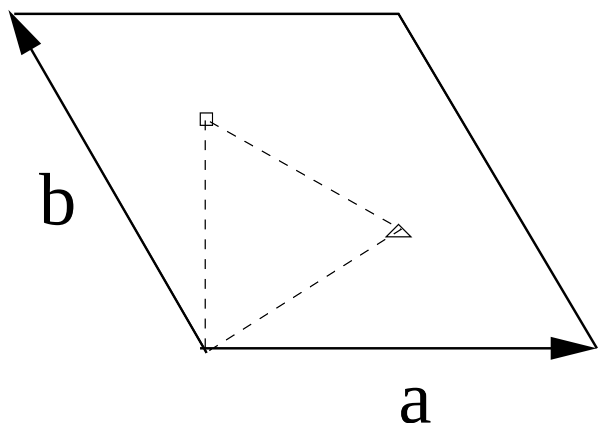

Figure 1: Maximal torus and Weyl alcove for $SU(3)$. **a** and **b** - simple coroots. The points $\Box$ and $\triangle$ - fundamental coweights.

Vacuum wave functions, SU(N=3)

- Motion on $T \times T$, T being the maximal torus formed by simple coroots $\mathbf{a} = (1, 0)$ and $\mathbf{b} = (-1/2, \sqrt{3}/2)$.

They satisfy $\exp\{4\pi i a^a t^a\} \equiv \exp\{iLC^a t^a\} = 1$).

Shifts along **a** or **b** are gauge transformations.

- boundary conditions

$$
\begin{aligned}
\Psi(\mathbf{x}+\mathbf{a},\mathbf{y}) &= e^{-2\pi i k \mathbf{a}\mathbf{y}}\Psi(\mathbf{x},\mathbf{y}) \ , \\
\Psi(\mathbf{x}+\mathbf{b},\mathbf{y}) &= e^{-2\pi i k \mathbf{b}\mathbf{y}}\Psi(\mathbf{x},\mathbf{y}) \ , \\
\Psi(\mathbf{x},\mathbf{y}+\mathbf{a}) &= e^{2\pi i k \mathbf{a}\mathbf{x}}\Psi(\mathbf{x},\mathbf{y}) \ , \\
\Psi(\mathbf{x},\mathbf{y}+\mathbf{b}) &= e^{2\pi i k \mathbf{b}\mathbf{x}}\Psi(\mathbf{x},\mathbf{y}) \ ,
\end{aligned}
$$

- zero energy eigenfunctions

$$
\Psi_w = \sum_{\mathbf{n}} \exp\left\{-2\pi k(\mathbf{n}+\mathbf{y}+\mathbf{w}_n)^2 - 2\pi i k\mathbf{x}\mathbf{y} - 4\pi i k\mathbf{x}(\mathbf{n}+\mathbf{w}_n)\right\} \ ,
$$

with $\mathbf{w}_n\mathbf{a}$ and $\mathbf{w}_n\mathbf{b}$ being integer multiples of $1/(2k)$.

- The # of Weyl invariant states is counted as the # of the points $\mathbf{w}_n$ in the Weyl alcove.

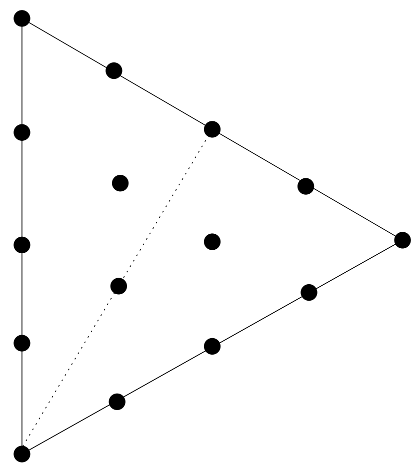

Figure 2: $SU(3)$: 15 vacuum states for $k = 4$. The dotted line marks the boundary of the Weyl alcove for G_2 (9 states).

This gives

$$I^{\text{tree}}_{SU(3)}(k > 0) \; = \; \sum_{m=1}^{k+1} m \; = \; \frac{(k+1)(k+2)}{2} \; = \; \left(\begin{array}{c} k+2 \\ 2 \end{array} \right) \; .$$

For any N, k,

$$I^{\text{tree}}_{SU(N)}(\text{any } k) \;=\; [\text{sgn}(k)]^{(N-1)} \begin{pmatrix} N+|k|-1 \\ N-1 \end{pmatrix} .$$

Symplectic groups:

$$I^{\text{tree}}_{\text{Sp}(2r)} \;=\; (-1)^r \begin{pmatrix} |k|+r \\ r \end{pmatrix} .$$

G_2:

$$I^{\text{tree}}_{G_2}(k) \;=\; \left\{ \begin{array}{cc} \frac{(|k|+2)^2}{4} & \text{for even } k \\ \frac{(|k|+1)(|k|+3)}{4} & \text{for odd } k \end{array} \right\} .$$

Coincides with the state counting in pure CS theory !

LOOP CORRECTIONS

(specific for $3d$!)

1-loop renormalization

$k(>0) \to k - N/2$ (fermions) $+ N$ (gluons)

• known in infinite volume (Pisarski + Rao, 85; Kao + Lee + Lee, 96).

• True also in finite volume.

• No renormalization beyond one loop.

• substituting $k \to k - N/2$ in the tree level index (taking into account only fermion corrections) gives Witten's result.

??? What about gluon loop corrections ???

Resolution of the paradox

Philosophical level

the shift $k \to k + N$ due to gluon loops is an immanent feature of pure CS (shows up in Wilson loop expectation values, etc.) and should not be counted twice.

Scientific level

- Take $N = 2$.
- Extra fluxes brought about by fermion and gluon loops are concentrated in the corners of the dual torus,

$$C_j = (0,0); \quad C_j = (2\pi/L, 0);$$
$$C_j = (0, 2\pi/L); \quad C_j = (2\pi/L, 2\pi/L) \ ,$$

where the BO approximation breaks down.

- in the massless limit $\to$ flux lines .
- Effective corner theory: non-Abelian SQM.

$H_{\text{corner}} =$

Accurate analysis:

matching the vacuum wave functions of two H_{eff}: *(i)* on the Abelian valley and *(ii)* in a corner.

• Dictates the valley wave functions that are not singular at the corners,

$$\Psi_m \propto \prod_{np} \left(\frac{z + n/2 + ip/2}{\bar{z} + n/2 - ip/2} \right)^{1/4} Q_m^{2k-2}(\bar{z}) \sqrt{Q_3^4(\bar{z}) - Q_1^4(\bar{z})}$$

• The argument of the square root has 4 zeros in 4 corners.

• branchings in the corners are cancelled.

• $m = 1, \ldots, 2k - 2$ giving finally k (rather than $k + 1$) Weyl invariant states.