BRST-BV treatment of Vasiliev's four-dimensional higher-spin gravity

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Based on
arXiv:1205.3339 with N. Boulanger and N. Colombo
and arXiv:1102.2219 with N. B.
and too a large extent also
arXiv:1103.2360 with E. Sezgin
arXiv:1107.1217 with C. Iazeolla
arXiv:1012.0813 with N. C.

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Outline

- Abstract and motivation
- Poisson sigma models on-shell
- Poisson sigma models off-shell
- BRST quantization
- Adaptation to Vasiliev's 4D higher-spin gravity
- Conclusions

Abstract

Vasiliev's 4D higher-spin gravities (HSGRA) are provided with a Batalin – Vilkovisky (BV) master action via an adaptation of the Alexandrov – Kontsevich – Schwarz – Zaboronsky (AKSZ) formalism to differential algebras on non-commutative manifolds.

- Vasiliev's equations for 4D HSGRAs can be derived (perturbatively) using the variational principle applied to a class of Poisson sigma models on non-commutative manifolds (NCPSM).
- (Standard) AKSZ procedure maps (classical) PSMs on commutative manifolds (CPSM) into (minimal) BV master actions.
- Thus, we have generalized the AKSZ procedure to the NCPSMs describing 4D HSGRAs.

Related series of works by Cattaneo, Felder; Grigoriev, Damgaard; Park; Hofman, Ma; Ikeda; Roytenberg; Zabzine; . . .
and in particular G., Barnich (how topological systems may contain local degrees of freedom) and Kotov, Strobl (geometries beyond fiber bundles).

Why HSGRAs?

We expect that existence of a (new) class of non-trivial gauge theories is of physical importance once its (quantum) dynamics is interpreted properly.

Vasiliev's equations set a benchmark for HSGRAs:

- Weak/weak coupling AdS/CFT correspondences
- New windows to (stringy?) de Sitter physics and cosmology
- New perspectives on the cosmological constant and dark matter
- Twistor formulation of (ordinary?) QFTs (if combined with HSSB)
- Generally covariant QFTs based on unfolded dynamics and PSMs

Classical PSM: I. Quasi-free differential algebras

- Vasiliev's HSGRAs are diffeomorphic invariant field theories containing local degrees of freedom.
- Spacetimes and twistor spaces arise on various submanifolds of a correspondence space (a Poisson manifold).
- The fundamental fields are differential forms, X^α.
- On-shell, the X^α together with a set of central elements J^c (of positive form degree) generate a quasi-free-differential algebra with differential d and associative product $\star$, *i.e.* there exists $\star$-functions $Q^\alpha(X;J)$ such that

$$R^\alpha \ := \ dX^\alpha + Q^\alpha(X;J) \ \approx \ 0 \ , \quad dJ^c \ \equiv \ 0 \ ,$$

$$(Q^\alpha \partial_\alpha) \star Q^\beta \ \equiv \ 0 \quad \text{modulo} \quad [J^c, X^\alpha]_\star \ \equiv \ 0 \ \equiv \ [J^c, J^{c'}]_\star \ ,$$

where $\approx$ refers to equations that hold on-shell.

Classical PSM: II. Cartan integrability

- X^α defined locally (in charts) up to gauge transformations

$$\delta_\epsilon X^\alpha = V_\epsilon^\alpha \ , \quad V_\epsilon^\alpha := d\epsilon^\alpha - (\epsilon^\beta \partial_\beta) \star Q^\alpha \ ,$$

for gauge parameters with compact support.

- The locally-defined solution spaces consist of gauge orbits over (non-linear) spaces of zero-form integration constants C and discrete moduli θ:

$$X^\alpha \approx X^\alpha_{C,\theta;\lambda} := \left[\exp V_\lambda^\beta \partial_\beta\right] \star X^\alpha \Big|_{X^\alpha = X^\alpha_{C,\theta}} \ ,$$

  - gauge functions λ^α have non-trivial values at chart boundaries
  - particular solutions $X^\alpha_{C,\theta}$ obey $X^\alpha_{C,\theta}|_p = C^\alpha$ at a base point p
  - zero-form integration constants C^α belong to non-linear cells (surrounded by walls of "critical field strengths")
  - θ-moduli related to non-trivial flat connections on noncommutative submanifolds, *e.g.* projectors

Classical PSM: III. Fiber-bundle-like systems

- Choices of boundary conditions and transitions (between charts) $\rightsquigarrow$ physically inequivalent globally-defined formulations
- In the case of CPSMs, fiber-bundle-like geometries arise by gluing together the X^α across overlaps using gauge transitions with parameters t^α obeying the compatibility condition

$$(t^\alpha \partial_\alpha)\, \partial_\beta \partial_\gamma Q^\delta \;\equiv\; 0\;,$$

  defining a generalized structure algebra.
- Similar geometries exist for NCPSMs containing 4D HSGRAs.
- Moduli spaces coordinatized by classical observables $\mathcal{O}$ obeying

$$\delta_t \mathcal{O} \;\equiv\; 0\;, \quad \delta_\epsilon \mathcal{O} \;\approx\; 0\;.$$

- *N.B.* In general, a given observable need only be well-defined in a subspace (of moduli space), defining a super-selection sector corresponding to some specific choice of representations for the gauge-transitions $\exp V_t^\alpha \partial_\alpha$ and data at ∂B.

Classical PSM: IV. Zero-form charges

Perturbative analysis in zero-form sector:

- The integration constant C^α, which parameterize masses, charges etc, belong to linearized representations of the Cartan gauge algebra (in degree zero).
- Letting Φ^α denote the zero-forms, zero-form charges

$$\mathcal{O} \;:=\; \mathcal{I}[\Phi] \;=\; \oint_\Sigma \mathcal{J}[\Phi; J]\;, \quad d\mathcal{J} \;\approx\; 0$$

  where Σ are suitable cycles (on which J have support) and $\oint_C$ acts as a graded cyclic trace operation.
- On-shell, $\mathcal{I}[\Phi; J] \approx \mathcal{I}[C; J]$ are gauge-equivariant (non-linearly invariant) functions of C^α which do not break any Cartan gauge symmetries.
- Off-shell formulation $\rightsquigarrow$ deformed on-shell actions given by sums of suitable $\mathcal{I}$ thus interpretable as semi-classical contributions to entropy function in unbroken phases.

Classical PSM: V. Abelian charges

- Generalized soldering forms E^α of degrees $\geqslant 1$ defined by

$$\delta_t E^\alpha \;\equiv\; -\left(t^\beta \partial_\beta\right) \star Q^\alpha$$

- Abelian charges

$$\mathcal{O} \;=\; \mathcal{Q}[E,\Phi] \;:=\; \oint_\Sigma \mathcal{J}(E,\Phi;J)$$

  for globally-defined $\mathcal{J}$ obeying $d\mathcal{J} \approx 0$.
- Abelian charges break some Cartan gauge symmetries off-shell.
- Broken symmetries re-emerge on-shell with parameters forming sections $\rightsquigarrow$ $\mathcal{Q}$ depend on $\lambda^\alpha|_{\partial B}$ modulo shifts by $t^\alpha|_{\partial B}$.
- Off-shell formulation $\rightsquigarrow$ suitable $\mathcal{Q}$ can be interpretable as contributions to entropy function in broken phases.

Off-shell CPSM: I. Classical action

Total action given by bulk piece plus deformations:

$$S^{\rm cl}_{\rm tot} \ = \ \int_B (\vartheta - \mathcal{H}(X,P)) + \sum_i \mu_i \oint_{\Sigma_i} \mathcal{V}^i(X,dX)$$

- Pre-symplectic form $\vartheta := P_\alpha dX^\alpha$
- Canonical momenta P_α (non-linear Lagrange multipliers)

$$\deg(P_\alpha) \ := \ \hat{p} - \deg(X^\alpha) \ , \quad \hat{p} \ := \ \dim(B) - 1 \ .$$

- Equations of motion:

$$\mathcal{R}^\alpha \ := \ dX^\alpha + \mathcal{Q}^\alpha \ \approx \ 0 \ , \qquad \mathcal{R}_\alpha \ := \ dP_\alpha + \mathcal{Q}_\alpha \ \approx \ 0 \ ,$$

$$\mathcal{Q}^\alpha \ := \ (-1)^{\hat{p}(\alpha+1)+1} \partial^\alpha \mathcal{H} \ , \qquad \mathcal{Q}_\alpha \ := \ (-1)^\alpha \partial_\alpha \mathcal{H} \ .$$

Off-shell CPSM: II. Structure equation

The generalized Hamiltonian is assumed to obey

$$\{\mathcal{H},\mathcal{H}\}_{\rm P.B.} \equiv (-1)^{\alpha}\partial_{\alpha}\mathcal{H}\wedge\partial^{\alpha}\mathcal{H} \equiv 0$$

Power-series expansion

$$\mathcal{H}=\sum_r P_{\alpha_1}\cdots P_{\alpha_r}\Pi^{\alpha_1\cdots\alpha_r}(X)$$

$\rightsquigarrow$ rank-n poly-vector fields $\Pi_{(n)}$ in target space of degrees $1+(1-n)\hat{p}$ whose mutual Schouten brackets vanish, *viz.*

$$\{\Pi_{(n_1)},\Pi_{(n_2)}\}_{\rm S.B.} \equiv 0 \quad \text{for all } n_1,n_2\geqslant 0\,.$$

Off-shell CPSM: III. Fiber-bundle-type models

The gauge variation of the bulk Lagrangian reads

$$\delta_\varepsilon \mathcal{L}^{\rm cl}_{\rm bulk} \equiv dK_\varepsilon \ ,$$

$$K_\varepsilon := (-1)^{\hat{p}(\alpha+1)}\eta_\alpha \mathcal{R}^\alpha + \left((\overrightarrow{P}-1)\overrightarrow{\epsilon} + \overrightarrow{P}\,\overrightarrow{\eta}\right)\mathcal{H} \ ,$$

where $\overrightarrow{P} := P_\alpha \frac{\partial}{\partial P_\alpha}$, $\overrightarrow{\epsilon} := \epsilon^\alpha \frac{\partial}{\partial X^\alpha}$ and $\overrightarrow{\eta} := \eta_\alpha \frac{\partial}{\partial P_\alpha}$.

Globally-defined formulations of fiber-bundle type requires:

- transition functions with parameters t^α obeying

$$(\overrightarrow{P}-1)\,\overrightarrow{t}\,\mathcal{H} = 0 \quad \Leftrightarrow \quad \overrightarrow{t}\,\Pi_{(n)} = 0 \quad \text{for } n \neq 1 \, ,$$

- boundary conditions

$$K_\varepsilon|_{\partial B} \equiv 0 \ ,$$

  which can be implemented by the following Dirichlet conditions:

$$\eta_\alpha|_{\partial B} \equiv 0 \ , \qquad P_\alpha|_{\partial B} \equiv 0 \ ,$$

  provided that the "classical anomaly" $\Pi_{(0)} \equiv \mathcal{H}|_{P_\alpha=0} \equiv 0$.

Off-shell CPSM: IV. Summary of target-space structures

Globally-defined formulations of fiber-bundle type in target spaces $T^*[\hat{p}]N$ over $\mathbb{N}$-graded manifolds N equipped with Schouten-integrable structures:

(i) a vector field $Q := \Pi_{(1)} \equiv Q^\alpha \partial_\alpha$ of degree 1 that is nilpotent in the sense that $\mathcal{L}_Q Q = 2\{Q, Q\} \equiv 0$, referred to as the Q-structure;

(ii) a tower of generalized Poisson structures $\Pi_{(n)}$ with $n \geqslant 2$ that are compatible with Q in the sense that $\mathcal{L}_Q \Pi_{(n)} \equiv 0$;

(iii) if in addition $\Pi_{(n)} = 0$ for $n \geqslant 3$ then $\Pi_{(2)}$ is a Poisson structure equipping N with a Poisson bracket of intrinsic degree $1 - \hat{p}$, referred to together with its compatible Q-structure as a QP-structure.

Off-shell CPSM: V. Topological vertex operators

Perturb bulk action by topological vertex operators obeying

$$\delta\mathcal{V}^i(X,dX) \;\equiv\; \delta X^\alpha M^i_{\alpha\beta}(X,dX)R^\beta + d(\delta X^\alpha \mathcal{P}^i_\alpha(X,dX)) \;,$$

for some matrices $M^i_{\alpha\beta}$ (that need not be invertible):

- $\delta S^{\rm cl}_{\rm tot}$ thus consists of bulk terms which impose $\mathcal{R}^\alpha \approx 0 \approx \mathcal{R}_\alpha$ plus boundary terms that vanish on-shell (since $P_\alpha|_{\partial B} \equiv 0$, which holds off-shell, implies $R^\alpha|_{\partial B} \equiv \mathcal{R}^\alpha|_{\partial B} \approx 0$).
- Hence $\delta \int_{\Sigma_i} \mathcal{V}^i \approx 0$ and the on-shell values

$$\mathcal{O}^i[X|\Sigma_i] \;:=\; \int_{\Sigma_i} \mathcal{J}^i(X) \;, \qquad \mathcal{J}^i \;:=\; \mathcal{V}^i(X^\alpha, -Q^\alpha) \;,$$

are classical observables that are intrinsic in the sense that if δ_{Σ_i} denotes a small variation of Σ_i then

$$d\mathcal{J}^i \;\approx\; 0 \qquad \Rightarrow \qquad \delta_{\Sigma_i}\mathcal{O}^i \;\approx\; 0 \;.$$

Off-shell CPSM: VI. Ensembles and entropies

- The couplings μ_r are chemical potentials of a grand canonical ensemble with partition function

$$Z(\mu_r; w) \; = \; \left\langle \prod_i e^{\frac{i\mu_r}{\hbar} \int_{\Sigma_i} \mathcal{V}^i} \right\rangle \; ,$$

  where w denotes the moduli hidden in the transition functions.

- Micro-canonical ensembles with fixed $\int_{\Sigma_i} \mathcal{V}^i = q^i$ have partition functions given by path integrals with fixed boundary observables, *viz.*

$$\widetilde{Z}(q; w) \; = \; \prod_i \int \frac{d\mu_i}{2\pi} e^{-\frac{iq^i \mu_i}{\hbar}} Z\{\mu; w\} \; = \; \left\langle \prod_i \delta\left(\int_{\Sigma_i} \mathcal{V}^i - q^i \right) \right\rangle \; .$$

- *N.B.* "Regularized" closed PSMs can be obtained by "filling in" the boundary components.

BV formalism: I. Towers of ghosts

The first step in the gauge-fixing procedure is to exhibit all gauge-for-gauge symmetries:

- extend the classical fields

$$(X^{\alpha}, P_{\alpha}) \equiv (X^{\alpha,\langle 0\rangle}_{[p_\alpha]}, P^{\langle 0\rangle}_{\alpha,[\hat{p}-p_\alpha]})$$

  with finite towers of ghosts, ghost-for-ghosts and so on:

$$(X^{\alpha,\langle q\rangle}_{[p_\alpha-q]}, P^{\langle q'\rangle}_{\alpha,[\hat{p}-p_\alpha-q']})$$

  with ghost numbers $q = 1, \dots, p_\alpha$ and $q' = 1, \dots, \hat{p} - p_\alpha$
- Recuperate original spectrum of classical observables as a cohomology group of a suitable BRST differential

$$s\phi^i \ = \ \delta_\varepsilon \phi^i|_{\varepsilon\to\mathrm{ghosts}} + \cdots \ , \qquad \mathrm{gh}(s) \ := \ 1 \ ,$$

  where the "fields" ϕ^i comprise the classical fields as well as ghost towers, and ε^i comprise all levels of gauge parameters.

BV formalism: II. Fields/anti-fields and BV bracket

Off-shell non-closure of gauge symmetries (as for PSMs) $\rightsquigarrow$ natural to identify the BRST differential as the adjoint action generated by a "minimal" master action using a suitable bracket:

$$s\phi^i \ := \ (S,\phi^i)\ , \quad S[\phi^i,\phi^+_i] \ := \ S_{\rm cl} + \int_B \phi^+_i \delta_{\rm ghosts}\phi^i + \cdots$$

- "anti-fields" ϕ^+_i obey

$$\mathrm{gh}(\phi^i) + \mathrm{gh}(\phi^+_i) \ = \ -1\ , \qquad \deg(\phi^i) + \deg(\phi^+_i) \ = \ \hat{p} + 1$$

- BV bracket

$$(A, A') \ := \ \int_{p\in B} (-1)^{\sigma_i} \delta_i(p) A\, \delta^i_+(p) A'\ , \qquad \mathrm{gh}(\cdot,\cdot) \ = \ 1$$

where $\delta_i(p)$ denotes the functional derivative with respect to ϕ^i at the point p *idem* $\delta^i_+(p)$ and σ_i is a suitable phase.

BV formalism: III. Classical and quantum master equation

- Gauge-fixing amounts to projecting to Lagrangian submanifold by eliminating ϕ_i^+ by means of a canonical transformation (CT).
- Demand gauge-fixed path-integral to be independent of the CT $\Rightarrow$

$$(S, S) + \frac{i}{2}\hbar\Delta S \equiv 0 ,$$

where BV Laplacian Δ is slightly singular operator defined by

$$\Delta := \int_{p\in\mathcal{M}} \delta_i(p)\delta_\star^i(p) , \qquad \mathrm{gh}(\Delta) = 1 .$$

- Δ is formally nilpotent but does not act as a differential; rather

$$\Delta(AA') - \Delta(A)A' - (-1)^A A\Delta(A') \equiv (-1)^A (A, A') .$$

BV formalism: IV. BRST differential and BRST current

- The BRST differential is generated by a current only if $\Delta S = 0$, *i.e.* if S obeys both the classical and quantum BV master equations.
- *N.B.* The latter is equivalent to that the BRST transformation is a CT, *i.e.* a formally manifest symmetry of the path-integral measure:

$$\delta_{\mathcal{E}}\phi^i \ := \ (\mathcal{E},\phi^i) \ , \quad \delta_{\mathcal{E}}\phi^*_i \ := \ (\mathcal{E},\phi^*_i) \ ,$$

$$\mathrm{gh}(\mathcal{E}) \ = \ -1 \ , \qquad \Delta\mathcal{E} \ = \ 0 \ ,$$

namely for $\mathcal{E}_{\mathrm{BRST}} = \epsilon S$ with $\mathrm{gh}(\epsilon) = -1$ and $d\epsilon = 0$.

AKSZ formalism: I. Vectorial superfields

"Minimal" set of fields and anti-fields of a PSM can be arranged into (unconstrained) vectorial superfields:

$$\mathbf{X}^\alpha := \underbrace{X^{\alpha\,\langle p_\alpha\rangle}_{[0]} + X^{\alpha\,\langle p_\alpha-1\rangle}_{[1]} + \ldots + X^{\alpha\,\langle 0\rangle}_{[p_\alpha]}}_{fields} + \underbrace{P^{\alpha\,\langle -1\rangle}_{[p_\alpha+1]} + P^{\alpha\,\langle -2\rangle}_{[p_\alpha+2]} + \ldots + P^{\alpha\,\langle p_\alpha-\hat{p}-1\rangle}_{[\hat{p}+1]}}_{antifields} \quad ,$$

$$\mathbf{P}_\alpha := \underbrace{P^{\langle \hat{p}-p_\alpha\rangle}_{\alpha\ [0]} + P^{\langle \hat{p}-p_\alpha-1\rangle}_{\alpha\ [1]} + \ldots + P^{\langle 0\rangle}_{\alpha\ [\hat{p}-p_\alpha]}}_{fields} + \underbrace{X^{\langle -1\rangle}_{\alpha\ [\hat{p}-p_\alpha+1]} + X^{\langle -2\rangle}_{\alpha\ [\hat{p}-p_\alpha+2]} + \ldots + X^{\langle -p_\alpha-1\rangle}_{\alpha\ [\hat{p}+1]}}_{antifields} \quad ,$$

of fixed total degree $|\cdot| := \mathrm{deg}(\cdot) + \mathrm{gh}(\cdot)$ *viz.*

$$|\mathbf{X}^\alpha| = p_\alpha \ , \qquad |\mathbf{P}_\alpha| = \hat{p} - p_\alpha \ .$$

AKSZ formalism: II. Master action

The AKSZ master action is given by the superfunctional

$$\mathbf{S}_{\rm bulk} \ := \ \int_B \mathbf{L} \ , \quad \mathbf{L} \ := \ d\mathbf{X}^\alpha \mathbf{P}_\alpha - \mathcal{H}(\mathbf{X}, \mathbf{P}) \ ,$$

where $\int_B(\cdot)$ projects onto form degree $\hat{p}+1$ such that

$$\mathrm{gh}(\mathbf{S}) \ = \ 0 \ , \quad \mathbf{S}_{\rm bulk}|_{\mathbf{X}=X,\mathbf{P}=P} \ = \ S^{\rm cl}_{\rm bulk}$$

As in the classical case, $\delta_\varepsilon \mathbf{S}_{\rm bulk} \equiv \int_B d\mathbf{K}_\varepsilon$ with

$$\mathbf{K}_\varepsilon \ = \ (-1)^{\hat{p}(\alpha+1)} \boldsymbol{\eta}_\alpha \mathbf{R}^\alpha + \left((\overrightarrow{\mathbf{P}} - 1)\overrightarrow{\epsilon} + \overrightarrow{\mathbf{P}}\overrightarrow{\eta} \right) \mathcal{H} \ .$$

Thus $\mathbf{S}_{\rm bulk}$ is globally-defined in fiber-bundle type geometries where

- gauge transition parameters $\mathbf{t}^\alpha$ obey

$$(\overrightarrow{\mathbf{P}} - 1)\overrightarrow{\mathbf{t}}\mathcal{H} \ \equiv \ 0$$

- fields and gauge parameters obey Dirichlet boundary conditions

$$\boldsymbol{\eta}_\alpha|_{\partial B} \ = \ 0 \ , \qquad \mathbf{P}_\alpha|_{\partial B} \ = \ 0$$

AKSZ formalism: III. Classical master equation

- Ultra-local super-functionals $\mathbf{F} := F(\mathbf{X}, \mathbf{P})$ *idem* F' obey

$$\left(\int_B \mathbf{P}_\alpha d\mathbf{X}^\alpha, \mathbf{F}\right) \equiv d\mathbf{F}\,, \quad \left(\int_B \mathbf{F}, \mathbf{F}'\right) \equiv \{F, F'\}_{\rm P.B.}\big|_{(X,P)\to(\mathbf{X},\mathbf{P})}\,.$$

- It follows that

$$(\mathbf{S}_{\rm bulk}, \mathbf{S}_{\rm bulk}) = (-1)^{\hat{p}} \int_B d\left(\mathbf{R}^\alpha \mathbf{P}_\alpha - 2\mathbf{L}\right) = 0$$

where the former equality follows from the structure equation $\{\mathcal{H}, \mathcal{H}\}_{\rm P.B.} \equiv 0$, while the latter follows from
  - $\mathbf{P}_\alpha|_{\partial B} = 0$ and $\mathcal{H}_{P_\alpha = 0} = 0$ which imply that boundary terms cancel
  - $\delta_{\mathbf{t}}\mathbf{L} \equiv \mathbf{K}_{\mathbf{t}} \equiv 0$ and

$$\delta_{\mathbf{t}}\mathbf{P}_\alpha = -(-1)^\alpha \overrightarrow{\mathbf{t}}\, \partial_\alpha \mathcal{H}\,, \quad \delta_{\mathbf{t}}\mathbf{R}^\alpha = (-1)^{\hat{p}(\alpha+1)} \overrightarrow{\mathbf{R}}_X \overrightarrow{\mathbf{t}}\, \partial^\alpha \mathcal{H}$$

with $\overrightarrow{\mathbf{R}}_X := \mathbf{R}^\alpha \partial_\alpha$, which imply that

$$\delta_{\mathbf{t}}(\mathbf{R}^\alpha \mathbf{P}_\alpha) \equiv \overrightarrow{\mathbf{R}}_X \overrightarrow{\mathbf{t}} (\overrightarrow{\mathbf{P}} - 1)\mathcal{H} \equiv 0\,,$$

such that contributions from chart boundaries cancel.

AKSZ formalism: IV. BRST current and operator algebra

- If $\mathbf{L} = L(X, P; dX, dP)$ is an ultra-local superfunctional then

$$\Delta \int_B \mathbf{L} \equiv 0 .$$

- In particular, it follows that

$$\Delta \mathbf{S}_{\rm bulk} \equiv 0 .$$

- $\rightsquigarrow$ Existence of a BRST current and hence a BRST operator
- $\rightsquigarrow$ Suitable (perturbatively defined) correlation functions yield homotopy-associative operator algebras $A_\infty[\hat{p}]$ (with n-ary products led by $\Pi_{(n)}$)

AKSZ formalism: V. Deformed action

- BRST cohomology at $\mathrm{gh} = 0$ consists of classical observables $\mathcal{O}$.
- Super-field formalism lead to off-shell extensions

$$\widehat{\mathbf{O}} \; := \; \mathcal{O}[\mathbf{X},\mathbf{P}] + \int_{\Sigma} (\mathbf{R}^{\alpha}\mathbf{L}_{\alpha} + \mathbf{R}_{\alpha}\mathbf{L}^{\alpha}) \; \equiv \; \mathbf{O} + s\left(\int_{\Sigma} (\mathbf{X}^{\alpha}\mathbf{L}_{\alpha} + \mathbf{P}_{\alpha}\mathbf{L}^{\alpha})\right) ,$$

where it is assumed that

$$s\mathbf{L}_{\alpha} \; = \; 0 \; = \; s\mathbf{L}^{\alpha} \; .$$

- Assuming also $(\widehat{\mathbf{O}}^{r}, \widehat{\mathbf{O}}^{r}) \equiv 0$, one has a total quantum master action

$$\mathbf{S}_{\rm tot} \; := \; \mathbf{S}_{\rm bulk} + \sum_{i} \mu_i \widehat{\mathbf{O}}^{i} \; .$$

- Compatibility between boundary conditions off-shell (classical master equation) and on-shell (classical variational principle) yields

$$\mathbf{S}_{\rm tot} \; = \; \mathbf{S}_{\rm bulk} + \sum_{i} \mu_i \int_{\Sigma_i} \mathbf{V}^{i} \; , \quad \mathbf{V}^{i} \; := \; \mathcal{V}^{i}(\mathbf{X}, d\mathbf{X}) \; .$$

4D HSGRA: I. Classical action

- Fields live on correspondence space $C = B \times T$ where
  - B is a universal noncommutative base manifold
  - T is a noncommutative twistor space (supporting central elements)
- Bulk action

$$S^{\rm cl}_{\rm bulk} = \int_C \left[U \star DB + V \star \left(F + \mathcal{F}(B;J) + \widetilde{\mathcal{F}}(U;J) \right) \right] ,$$

$$DB := dB + [A,B]_\star , \quad F := dA + A \star A ,$$

  - graded-cyclic trace operation $\int_C$ non-degenerate on subspace of $\Omega(C)$ such that "zero-modes" $\leftrightarrow$ fiber coordinates
- $\dim(B) \equiv 2n+1$ ($\hat{p} = 2n+4$) $\rightsquigarrow$ form degrees

$$A = A_{[1]} + A_{[3]} + \cdots + A_{[2n-1]} , \qquad B = B_{[0]} + B_{[2]} + \cdots + B_{[2n-2]} ,$$

$$U = U_{[2]} + U_{[4]} + \cdots + U_{[2n]} , \qquad V = V_{[1]} + V_{[3]} + \cdots + V_{[2n-1]} .$$

4D HSGRA: II. Structure equation

- General variation

$$\delta S^{\rm cl}_{\rm bulk} \;=\; \int_C \delta Z^i \star \mathcal{R}^j \mathcal{M}_{ij} + \int_C d\left(U \star \delta B - V \star \delta A\right) \;,$$

where $\mathcal{M}_{ij}$ is a constant non-degenerate matrix and

$$\mathcal{R}^A \;=\; F + \mathcal{F} + \widetilde{\mathcal{F}} \;, \qquad \mathcal{R}^B \;=\; DB + (V\partial_U) \star \widetilde{\mathcal{F}} \;,$$

$$\mathcal{R}^U \;=\; DU - (V\partial_B) \star \mathcal{F} \;, \qquad \mathcal{R}^V \;=\; DV + [B, U]_\star \;.$$

- On-shell Cartan integrability for

$$\text{bilinear } Q\text{-structure} \;:\; \mathcal{F} \;=\; B \star J \;, \qquad J \;=\; J_{[2]} + J_{[4]} \;,$$

$$\text{bilinear } P\text{-structure} \;:\; \widetilde{\mathcal{F}} \;=\; U \star J' \;, \qquad J' \;=\; J'_{[2]} + J'_{[4]} \;,$$

- *N.B.* Exist more general cases of which some are of interest for 3D HSGRAs

4D HSGRA: III. Gauge invariance

- On-shell Cartan gauge transformations remain symmetries off shell modulo boundary terms, *viz.*

$$\delta_{\epsilon,\eta} S^{\rm cl}_{\rm bulk} \ \equiv \ \int_C dK_\eta \ ,$$

$$K_\eta \ := \ \eta^U \star DB + \eta^V \star (F + \mathcal{F} + (1 - U\partial_U) \star \widetilde{\mathcal{F}}) \ .$$

- Gauge transitions with parameters (t^A, t^B) whose action on (η^U, η^V) reads

$$\delta_t \eta^U \ = \ -[t^A, \eta^U] - (t^B \partial_B) \star (\eta^V \partial_B) \mathcal{F} \ , \quad \delta_t \eta^V \ = \ -[t^A, \eta^V] + \{\eta^U, t^B\}$$

$\rightsquigarrow \delta_t K_\eta = 0$, *i.e.* the contributions to $\delta_{\epsilon,\eta} S^{\rm cl}_{\rm bulk}$ from interior chart boundaries cancel.

- At the boundaries of C *i.e.* of B one imposes

$$\eta_\alpha|_{\partial B} \ = \ 0 \ , \quad P_\alpha|_{\partial B} \ = \ 0 \ .$$

4D HSGRA: IV. Fiber-bundle compatibility conditions

- The compatibility conditions on $\{t^A, t^B\}$ read as follows:

$$\overrightarrow{\mathcal{R}} \star [\overrightarrow{t}, \overrightarrow{\epsilon}]_\star \star \mathcal{Q}^i \ = \ 0 \qquad \text{for all } i,\ \overrightarrow{\mathcal{R}} \text{ and } \overrightarrow{\epsilon}\ ,$$

  c.f. the commutative case where $(t^\alpha \partial_\alpha)\,\partial_\beta \partial_\gamma Q^\delta \equiv 0$.
- Conditions on t^A hold for all $\mathcal{F}$.
- Those for t^B hold only if $\mathcal{F}$ is at most bi-linear.
- Thus, if $\mathcal{F}$ is at least tri-linear then t^B-transitions must be discarded.

HSGRA: V. Adapting BV to non-commutative setting

- $\star$-functional differentiation defined naturally via

$$\delta F[X,P] \equiv F[X+\delta X, P+\delta P] =: \int_{\mathcal{M}} (\delta X^\alpha \star \delta_\alpha F + \delta P_\alpha \star \delta^\alpha F) \ .$$

- *N.B.* The $\star$-functional derivatives act as differentials on ultra-local functionals and cyclic derivatives on local functionals.
- $\rightsquigarrow$ non-commutative generalization of the Dirac delta function (which in a certain sense is less singular than the commuting ditto).
- BV gauge-fixing procedure $\rightsquigarrow$ BV Laplacian given by double $\star$-functional derivatives.

HSGRA: VI. AKSZ master action for 4D HSGRA

- AKSZ master action

$$\mathbf{S} \;=\; S^{\rm cl}_{\rm bulk}[\mathbf{A},\mathbf{B};\mathbf{U},\mathbf{V}]\Big|^{\langle 0\rangle} \;\equiv\; \int_C \mathbf{L}\Big|^{\langle 0\rangle} \;,$$

$$\mathbf{L} \;=\; \mathbf{U}\star \mathbf{D}\mathbf{B}+\mathbf{V}\star\left(\mathbf{F}+\mathcal{F}(\mathbf{B};J^r)+\widetilde{\mathcal{F}}(\mathbf{U};J^r)\right) \;.$$

- Using $(\mathbf{S},\mathbf{X}^\alpha)\equiv \mathbf{R}^\alpha := \mathrm{d}\mathbf{X}^\alpha+\mathbf{Q}^\alpha$ with $\mathbf{Q}^\alpha := \mathcal{Q}^\alpha(\mathbf{A},\mathbf{B};\mathbf{U},\mathbf{V})$ *idem* $\mathbf{P}_\alpha$, one has

$$(\mathbf{S},\mathbf{S}) \equiv -\int_C d\left(\mathbf{U}\star\mathbf{D}\mathbf{B}+\mathbf{V}\star\left(\mathbf{F}+\mathcal{F}(\mathbf{B};J)+(1-\mathbf{U}\partial_{\mathbf{U}})\star\widetilde{\mathcal{F}}(\mathbf{U};J)\right)\right)$$

i.e. the natural generalization of the noncommutative result, *viz.*

$$(\mathbf{S},\mathbf{S}) \;\equiv\; (-1)^{\hat p}\int_C d\,[(\mathbf{R}^\alpha\star\mathbf{P}_\alpha - 2\mathbf{L}] \;.$$

HSGRA: VII. Boundary conditions and gauge transitions

The classical BV master equation is thus obeyed provided that

$$\mathbf{P}_{\boldsymbol{\alpha}}|_{\partial B} \;=\; 0 \;,$$

and that gauge transitions between charts act as follows:

$$\begin{aligned}
\delta_{\mathbf{t}}\mathbf{A} &= \mathbf{D}\mathbf{t}^{A} - (\mathbf{t}^{B}\partial_B)\star\mathcal{F} \;,\\
\delta_{\mathbf{t}}\mathbf{B} &= \mathbf{D}\mathbf{t}^{B} - [\mathbf{t}^{A}, B]_\star \;,\\
\delta_{\mathbf{t}}\mathbf{U} &= -[\mathbf{t}^{A}, \mathbf{U}]_\star + (\mathbf{t}^{B}\partial_B)\star(\mathbf{V}\partial_B)\star\mathcal{F} \;,\\
\delta_{\mathbf{t}}\mathbf{V} &= -[\mathbf{t}^{A}, \mathbf{V}]_\star - [\mathbf{t}^{B}, \mathbf{U}]_\star \;,
\end{aligned}$$

with parameters $(\mathbf{t}^{A}, \mathbf{t}^{B})$ obeying the natural super-field extension of the fiber-bundle compatibility conditions.

Conclusions

Aspects of quantizing 4D HS fields using higher-dimensional PSMs:

- Bulk Lagrangian contains two types of couplings:
  - Q-structures providing the correct classical limit
  - Generalized Poisson structures providing quantum corrections
- Fiber-bundle-like geometries arise for generalized Poisson structures obeying additional compatibility conditions.
- We have also found a number of topological vertex operators that can be inserted on submanifolds as to yield on-shell actions.
- Main frontiers at this point:
  - Existence of topological vertex-operator four-form that yields holographic correlators coinciding with the O(N) vector model.
  - Existence of more general NCPSMs applicable to 3D HSGRAs and how to make these models globally defined (need to go beyond fiber-bundle-type geometries)
  - Generalization to quasi-free differential algebras of homotopy-associative type.