Spacetime Mechanical Pictures of Higher Spin Gauge Interactions

... still work in progress ...

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Construction of consistent self-interactions for higher spin gauge fields

Two main avenues of approach:

GAUGING approach

A priori global but non-Abelian H.S. gauge algebra. FDA's. General covariance built in from the beginning.

Initiated by Vasiliev and Fradkin, developed by M. Vasiliev. Many more authors (Sezgin, Sundell, ...)

DEFORMATION approach

No apriori gauge algebra - local but abelian gauge symmetries. Treat all spins equally - no preference to spin 2

Proposed by Fang and Fronsdal in 1978 as "Gupta program for higher spin".

Self-interactions for integer spin gauge fields

$s = 1$: Lie group gauge theories on fixed space-time backgrounds. Yang-Mills theory ...

$s = 2$: Gauge theory of spacetime itself. General Relativity, Einstein gravity ...

$s \geq 3$: What more can there possibly be?

- Interactions for $s = 3 \Rightarrow \forall s : s \geq 3$

- Non-polynomial

- Increasing numbers of derivatives in vertices

  Spin s, n-point vertex: at least $(n-2)s - 2n + 6$ derivatives.

Weinberg 1964: *high spin massless particles cannot produce macroscopic fields*

But we already knew that, didn't we (even then)? Otherwise they'd been seen? So where are they?

Philosophy

Study Higher Spins as free-standing theoretical constructs.

The problem of interactions is characterized by high *COMPLEXITY*

My approach to deformation

- Simplify
- Abstract
- Look for a physical picture

... while keeping what's characteristic of the problem.

Quite natural ideas

Pack all higher spin fields away in some

$\mathcal{M}$aster $\mathcal{F}$ield $\mathbf{\Phi}$

Fronsdal (1979): Cotangent symplectic structure (x^μ, π_ν) over spacetime

$\mathbf{\Phi}(\pi, x) = \sum_n f(n) \phi^{\mu_1 \mu_2 \cdots \mu_n}(x) \pi_{\mu_1} \pi_{\mu_2} \cdots \pi_{\mu_n}$

I don't know how far this carried, haven't found any real reference. [There is a conference paper from 1979]

Derivative basis:

$\mathbf{\Phi} = \sum_s \phi^{\mu_1 \mu_2 \cdots \mu_s} \partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_s}$

Jet bundle formulation seems natural.[Worked on this with I. Bengtsson]

Quite natural ideas

BRST: String field theory inspiration [Ouvry and Stern, Bengtsson, 1986] [Independently rediscovered over and over]

$|\mathbf{\Phi}\rangle = \sum_n \phi^{\mu_1\mu_2\cdots\mu_n}\alpha^\dagger_{\mu_1}\alpha^\dagger_{\mu_2}\cdots\alpha^\dagger_{\mu_n}|0\rangle$

$+$ auxiliary ghost terms.

This actually works!

Action and gauge transformations

$$A = \langle\mathbf{\Phi}|Q|\mathbf{\Phi}\rangle$$

$$\delta A = 0 \quad \text{with} \quad \delta|\mathbf{\Phi}\rangle = Q|\mathbf{\Xi}\rangle$$

Reproduces all free field actions & gauge invariances.

Tracelessness: $T|\mathbf{\Phi}\rangle = T|\mathbf{\Xi}\rangle = 0$

Quite natural ideas

Vasiliev construction in AdS

$\omega(Y|x) = \sum_{l=0}^{\infty} \omega_{A_1 \ldots A_l, B_1 \ldots B_l} Y_1^{A_1} \ldots Y_1^{A_l} Y_1^{B_1} \ldots Y_1^{B_l}$

with oscillators $[Y_i^A, Y_j^B] = \epsilon_{ij} \eta^{AB}$.

Bilinears in Y_i^A span an O(D-2,2) algebra.

Polynomials of unbounded degree span an infinite dimensional extension of O(D-2,2).

ω is a one-form, general covariance is included from the start.

Abstraction (in Computer Science sense)

Focus on **what** we want to do, rather than **how** it is done in detail (cf Category Theory).

Computer Science "split thinking":

Interface $\longleftrightarrow$ Implementation

Syntax $\longleftrightarrow$ Semantics

! Pack the fields away, but don't worry about the details (Spacetime dimension, signature, multiplet structure, supersymmetry, ...)

! **Abstract** the free field theory

! **Abstract** interactions and gauge transformations

! A **semantic map** $\hookrightarrow$

from *syntax* to *semantics*,

or from *interface* to *implementation*

An abstract interface (1)

Pack away the fields into $\Phi(\sigma_i)$ (short Φ_i) where σ_i parametrizes all inner structure, as well gives us a handle to treat field configurations.

Possible support for the abstraction:

1. **Set theoretic**: Fields Φ belong to some set $\mathcal{H}$ (Hilbert space or vector space) ($\rightarrow$ sections of bundles ...)

2. **Type theoretic**: $\Phi :: \mathrm{Hs}$

A *type* is an abstract "template" for a variable or an object, specifying the allowed "values" and the supported "operations".

3. **Category theoretic**: Φ objects of a category **HSField**. There are various possibilities for the morphisms, some involving *operads*.

An abstract interface (2)

Let's go for type theoretic: $\mathbf{\Phi} :: \mathbf{Hs}$, supposing the type $\mathbf{Hs}$ supports the standard linear operations (more technical stuff, Grassman parities, symmetrisations).

"Inner product" is a map

$\mathbf{in}(\cdot, \cdot) :: \mathbf{Hs^2} \rightarrow \mathbf{k}$ (a number field)

where $\mathbf{Hs}^n$ is short for the *function type*

$\underbrace{\mathbf{Hs} \rightarrow \mathbf{Hs} \rightarrow \cdots \rightarrow \mathbf{Hs}}_{n}$

"Linear operators"

$K :: \mathbf{Hs} \rightarrow \mathbf{Hs}$

Free field theory action $A(\mathbf{\Phi}) = \mathbf{in}(\mathbf{\Phi}, K\mathbf{\Phi})$.

An abstract interface (3)

A product of n fields is a multilinear map

$\mathbf{pr} :: \mathbf{Hs}^n \rightarrow \mathbf{Hs}$

$\mathbf{pr}(\mathbf{\Phi}(\sigma_1), \mathbf{\Phi}(\sigma_2), \ldots, \mathbf{\Phi}(\sigma_n)) \rightarrow \mathbf{\Phi}(\sigma_{n+1})$.

A shorthand notation

$\mathbf{pr}(\mathbf{\Phi}^n) \equiv \mathbf{pr}(\mathbf{\Phi}(\sigma_1), \mathbf{\Phi}(\sigma_2), \ldots, \Phi(\sigma_n)) \equiv$

$\mathbf{pr}(\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n)$

Simplified notation for low indices $n = 0$ and $n = 1$

$\mathbf{pr}(\Phi^0) \equiv \mathbf{pr}() = 0$

where Φ^0 is defined to be a void argument, and

$\mathbf{pr}(\Phi) = K\Phi$

An abstract interface (4)

"n-point" interaction is a multilinear map $\mathbf{vx}$ typed by

$$\mathbf{vx} :: \mathbf{Hs}^n \to \mathbf{k}$$

and syntactically defined by

$$\mathbf{vx}(\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_n) \equiv \mathbf{in}(\mathbf{\Phi}_n, \mathbf{pr}(\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_{n-1}))$$

NOTE: In my way of looking at it, brackets ($[\cdot, \cdot]$ et cetera ...) belong to the (concrete calculational) implementation (i.e. the semantics).

Action and gauge invariance

$$A(\mathbf{\Phi}) = \sum_{i=2}^{\infty} \frac{g^{i-2}}{i!} \mathbf{vx}(\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_i) =$$

$$\mathbf{in}(\mathbf{\Phi}, K\mathbf{\Phi}) + \sum_{i=3}^{\infty} \frac{g^{i-2}}{i!} \mathbf{in}(\mathbf{\Phi}, \mathbf{pr}(\mathbf{\Phi}_{i-1}))$$

$$\delta_{\mathbf{\Xi}} \mathbf{\Phi} = \sum_{i=0}^{\infty} \frac{g^i}{i!} \mathbf{pr}(\mathbf{\Phi}^i, \mathbf{\Xi}) =$$

$$K\mathbf{\Xi} + \sum_{i=1}^{\infty} \frac{g^i}{i!} \mathbf{pr}(\mathbf{\Phi}^i, \mathbf{\Xi})$$

Gauge invariance to all orders forces

$$\sum_{\substack{k=0 \\ l=0}}^{k+l=n} \frac{1}{k!\,l!} \mathbf{pr}(\mathbf{\Phi}^k, \mathbf{pr}(\mathbf{\Phi}^l)) = 0$$

The **product identities** for a strongly homotopy Lie algebra (L_∞). [Stasheff, cf Zweibach closed string field theory, implicit(?) in BBvD(?)]

Note that this follows from the syntax alone using only equational reasoning!

Mechanical two-particle models

- *Discretized string* [Gershun and Pashnev, 1988]

  Regge trajectory of massive excitations

- *Rigid string* [Casalbuoni, Dominici and Longhi, 1975-76]

  Massless (and massive), single spin, excitations

- *Vibrating "spring"* Discussions with Bo Sundborg, 1989

  Infinite tower of massless excitations

  Also zero-slope limit, but there is still a dimensionful parameter in the theory as can be seen from

$$a = \frac{1}{\sqrt{2m\omega}}(\pi - im\omega\xi)$$

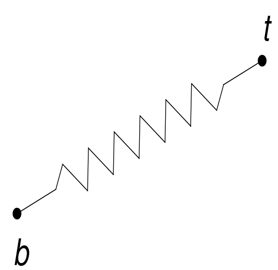
  So we can take

$$\alpha' \sim \frac{1}{m\omega} \sim \kappa^2$$

Mechanical two-particle models

Simplest possible setting: two point particles considered as a *physical* harmonic oscillator

Coordinates and momenta (t^μ, b^μ), (u_μ, d_μ)



Center of mass and relative coordinates

$$q^\mu = \frac{1}{2}(t^\mu + b^\mu), \quad \xi^\mu = \frac{1}{2}(t^\mu - b^\mu)$$

$$p^\mu = u^\mu + d^\mu, \quad \pi^\mu = u^\mu - d^\mu$$

The *internal harmonic motion phase space* (ξ^μ, π_ν) can alternatively be described by oscillators $(\alpha_\mu, \alpha_\nu^\dagger)$

$$\alpha_\mu = \frac{1}{\sqrt{2}}(\kappa\pi_\mu - \frac{i}{\kappa}\xi_\mu), \quad \alpha_\mu^\dagger = \frac{1}{\sqrt{2}}(\kappa\pi_\mu + \frac{i}{\kappa}\xi_\mu)$$

Mechanical two-particle models

Excluding explicit x, six different bilinears can be made out of these variables:

$$p^2, \quad \pi \cdot p, \quad \xi \cdot p$$

$$\pi \cdot \pi, \quad \xi \cdot \xi, \quad \pi \cdot \xi.$$

By chosing linear combinations of subsets of these bi-linears as constraints, we get various types of bi-local mechanical models, all describing arbitrary spin excitations/fields.

Supply conjugate ghost pairs $(c,b),(\gamma,\beta),(\widetilde{\gamma},\widetilde{\beta})$ and do BRST field theory

$$A = \int \Phi Q \Phi d(x,\xi,c,\gamma,\widetilde{\beta})$$

with

$$Q = -\frac{1}{2}cp^2 + \gamma\pi \cdot p + \widetilde{\gamma}\xi \cdot p - 2i\gamma\widetilde{\gamma}b,$$

then lift to BV field theory.

States [Following Casalbuoni et al]

Relative coordinate *Fock space*

$$|\mu_1, \ldots, \mu_n\rangle = \alpha^\dagger_{\mu_1} \cdots \alpha^\dagger_{\mu_n}|0\rangle$$

or *configuration space*

$$\langle\xi|\mu_1, \ldots, \mu_n\rangle = f_{\mu_1\ldots\mu_n}(\xi) =$$

$$f_{\mu_1\ldots\mu_s}(\xi) = N_s c_0 \exp[-\xi^2/2\kappa^2] H^s_{\mu_1\ldots\mu_s}(\xi/\kappa).$$

Generalized Hermite polynomials

$$H^s_{\mu_1\ldots\mu_s}(\xi/\kappa) = \kappa^s \frac{\partial^{(s)}}{\partial J^{\mu_1}\ldots\partial J^{\mu_s}} h(\xi, J)|_{J=0},$$

in terms of the generating function

$$h(\xi, J) = \exp[(-J^2 + 2J\cdot\xi)/\kappa^2].$$

Center of mass momentum states as usual

$$\langle x|p\rangle = \frac{1}{(2\pi)^2}\exp(ip\cdot x).$$

Fields

Fock space representation

$$|\Phi(x,\alpha^\dagger)\rangle = \sum_{n=0}^{\infty} \phi^{\mu_1\cdots\mu_n}(x)|\mu_1,\ldots,\mu_n\rangle.$$

Configuration space representation

$$\langle\xi|\Phi(x,\alpha^\dagger)\rangle = \Phi(x,\xi) =$$

$$\frac{1}{(2\pi)^2}\int d^4p \sum_{n=0}^{\infty} \phi^{\mu_1\cdots\mu_n}(p)e^{ip\cdot x} f_{\mu_1\ldots\mu_n}(\xi).$$

Assembling everything

$$\Phi(x,\xi) = \int d^4p\, \Phi(p)^{(s)} \cdot D_J^{(s)} f(x,p;\xi,J)|_{J=0}$$

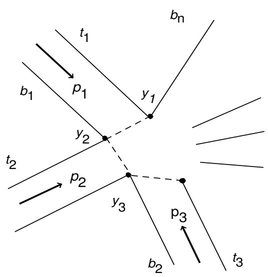
$$\Phi(p)^{(s)} \cdot D_J^{(s)} = \sum_{s=0}^{\infty} c_s \phi^{\mu_1\cdots\mu_s}(p)\frac{\partial^{(s)}}{\partial J^{\mu_1}\ldots\partial J^{\mu_s}}$$

$$f(x,p;\xi,J) =$$

$$\frac{c_0}{(2\pi)^2}e^{ip\cdot x}\,\mathrm{exp}[-\xi^2/2\kappa^2]\,\mathrm{exp}[(-J^2+2J\cdot\xi)/\kappa^2]$$

Implementing the abstract product (I)

Collisions with end-point coordinate overlap



Semantic map: $\mathbf{\Phi}_k \hookrightarrow \Phi(t_k, b_k)$

$$\mathbf{pr}(\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_{n-1}) \hookrightarrow$$

$$\int dy_2 \ldots d_{n-1} \Phi(y_1, y_2) \ldots \Phi(y_{n-1}, y_n) \qquad \text{for } n \geq 3$$

evaluates to a field $\Phi(y_1, y_n)$ ($\hookleftarrow \mathbf{\Phi}_n$)

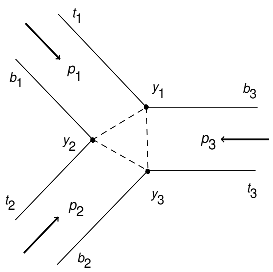
$$\mathbf{in}(\mathbf{\Phi}_1, \mathbf{\Phi}_2) \hookrightarrow$$

$$\int \mathbf{\Phi}(t_1, b_1)\mathbf{\Phi}(t_2, b_2)\delta(t_1 - b_2)\delta(b_1 - t_2) dt_1 dt b_1 dt_2 db_2$$

The n:th order interaction is

$$\sim \int dy_1 \ldots d_n \Phi(y_1, y_2) \ldots \Phi(y_n, y_1)$$

Cubic interaction as an example



Putting it all together (following Casalbuoni et al.) and calculating

$$\sim g\kappa^{-6}\int d^4x_1 d^4x_2 d^4x_3 \Phi(x_1,x_2)\Phi(x_2,x_3)\Phi(x_3,x_1)$$

we get a generating function for the vertex

$$V(J_i,p_i) \sim g\kappa^2 \exp[\frac{1}{3\kappa^2}(J_1^2+J_2^2+J_3^2)$$

$$-\frac{4}{3\kappa^2}(J_1J_2+J_2J_3+J_3J_1)$$

$$+\frac{2i}{3}(J_1(p_2-p_3)+J_2(p_3-p_1)+J_3(p_1-p_2))$$

$$-\frac{\kappa^2}{6}(p_1^2+p_2^2+p_3^2)]$$

Result

For spin 1 we calculate

$$\frac{\partial^3}{\partial J^{\mu_1}\partial J^{\mu_2}\partial J^{\mu_3}}V(J_i,p_i)$$

Supplying anti-symmetrisation with f_{abc} we get Yang-Mills cubic vertex

$$g f_{abc}[\eta_{\mu_1\mu_2}(p_1-p_2)_{\mu_3}+$$

$$\eta_{\mu_2\mu_3}(p_2-p_3)_{\mu_1}+\eta_{\mu_3\mu_1}(p_3-p_1)_{\mu_2}]$$

plus order p^3 terms.

Plus a factor $\exp[-\frac{\kappa^2}{6}(p_1^2+p_2^2+p_3^3)]$.

For higher order vertices and higher spin, you get qualitatively the right combinations of momenta (plus higher order terms).

Generic form - arbitrary vertex order

Q_- stands for momentum factor

N_{--} stands for metric factor

indexed by field spacetime indices

# derivatives	types of terms	comment
0	-	
1	Q_-	
2	N_{--}, Q_-Q_-	
3	$N_{--}Q_-, Q_-Q_-Q_-$	Y.M. cubic
4	$N_{--}N_{--}$	Y.M. quartic
$-''-$	$N_{--}Q_-Q_-$	-
$-''-$	$Q_-Q_-Q_-Q_-$	-
$\vdots$	$\vdots$	$\vdots$
6	$N_{--}N_{--}N_{--}$	-
$-''-$	$N_{--}N_{--}Q_-Q_-$	spin 2 cubic
$-''-$	$N_{--}Q_-Q_-Q_-Q_-$	-
$-''-$	$Q_-Q_-Q_-Q_-Q_-Q_-$	-
$\vdots$	$\vdots$	$\vdots$
9	$N_{--}N_{--}N_{--}Q_-Q_-Q_-$	spin 3 cubic

The ghost complex - BV fields (1)

Generating ghost states with sources $\tilde{\zeta}, \eta$:

$$g(\gamma, \tilde{\beta}; \tilde{\zeta}, \eta) = \exp[\frac{\varepsilon}{2\kappa^2}\gamma\tilde{\beta}] \exp[\kappa^2\tilde{\zeta}\eta + \varepsilon_1\tilde{\zeta}\gamma + \varepsilon_2\eta\tilde{\beta}]$$

$$\begin{aligned} g_{00} &= g|_{\tilde{\zeta},\eta=0} = 1 + \frac{\varepsilon}{2\kappa^2}\gamma\tilde{\beta} = \mathcal{D}_{00}g|_0 \\ g_{01} &= \frac{\partial g}{\partial\tilde{\zeta}}|_{\tilde{\zeta},\eta=0} = \varepsilon_1\gamma = \mathcal{D}_{01}g|_0, \\ g_{10} &= \frac{\partial g}{\partial\eta}|_{\tilde{\zeta},\eta=0} = \frac{\varepsilon_2}{\kappa^2}\tilde{\beta} = \mathcal{D}_{10}g|_0, \\ g_{11} &= \frac{\partial^2 g}{\partial\eta\partial\tilde{\zeta}}|_{\tilde{\zeta},\eta=0} = 1 - \frac{\varepsilon}{2\kappa^2}\gamma\tilde{\beta} = \mathcal{D}_{11}g|_0. \end{aligned}$$

Othonormal:

$$\begin{aligned} \kappa^2\int g_{00}^2 d\gamma d\tilde{\beta} &= -\varepsilon, \\ \kappa^2\int g_{01}g_{10}\gamma d\tilde{\beta} &= -\varepsilon, \\ \kappa^2\int g_{11}^2 d\gamma d\tilde{\beta} &= \varepsilon, \end{aligned} \tag{1}$$

integrals over all other bilinear combinations are zero.

The ghost complex - BV fields (2)

$\mathrm{gh_m}(\cdot)$	2	1	0	-1
states		$g_{00}c$	g_{00}	
	$g_{01}c$	g_{01}	$g_{10}c$	g_{10}
		$g_{11}c$	g_{11}	
fields	$\mathcal{C}^{\#}$	$\Phi^{\#}$	Φ	$\mathcal{C}$

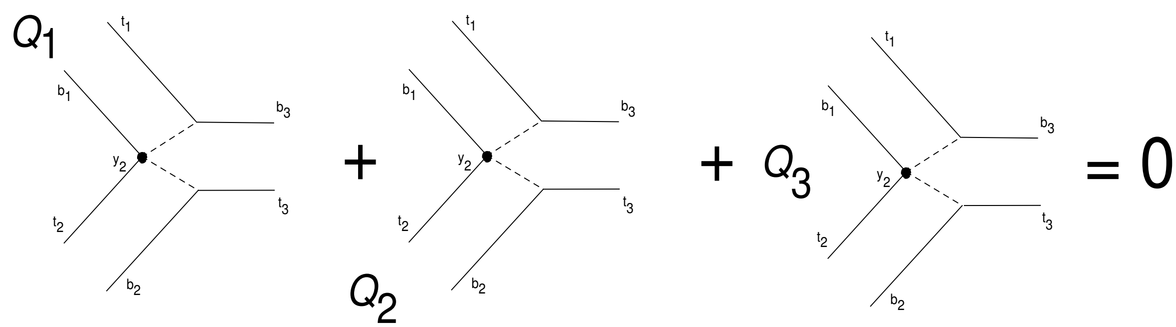
Ghost sector expansion:

$$\begin{aligned}\Psi(c,\gamma,\widetilde{\beta}) &= \int db(\Psi^c + b\Psi^b)e^{-bc} \\ &= \int db(\Psi^c_\alpha \mathcal{D}_\alpha + b\Psi^b_\alpha \mathcal{D}_\alpha) g e^{-bc}|_0 \\ &= \int db \Psi_\alpha(b)\mathcal{D}_\alpha\{e^{-bc} g(\gamma,\widetilde{\beta};\widetilde{\zeta},\eta)\}|_0 .\end{aligned}$$

Full expansion:

$$\begin{aligned}\Psi_\alpha(x,\xi;c,\gamma,\widetilde{\beta}) &= \frac{1}{(2\pi)^2}\int d^4p\, db\, \Psi^{(s)}_\alpha(b,p)\cdot D^{(s)}_J \mathcal{D}_\alpha \\ &\times \{e^{ip\cdot x} e^{-bc} g(\gamma,\widetilde{\beta};\widetilde{\zeta},\eta) f(\xi,J)\}|_0 .\end{aligned}$$

Gauge invariance



This is the semantic map of the first non-trivial product identity (cubic order)

$$\mathbf{pr}(\mathbf{pr}(\Phi_1,\Phi_2))+\mathbf{pr}(\Phi_1,\mathbf{pr}(\Phi_2))+\mathbf{pr}(\Phi_2,\mathbf{pr}(\Phi_1)) = 0$$

or

$$K\mathbf{pr}(\Phi_1,\Phi_2)+\mathbf{pr}(\Phi_1,K\Phi_2)+\mathbf{pr}(K\Phi_1,\Phi_2) = 0$$

What remains to be done

- Fix up the ghost sector

  so that cubic order gauge invariance can be checked

- Hone the formalism/calculational tecniques

  so that all orders gauge invariance can be checked

- Understand if it all make sense?

THE END

Thanks for your interest!

EXTRA: Strongly homotopy Lie algebras

$\mathbf{Z}_2$ graded vector space $V = V_0 \oplus V_1$, elements by x. Grading $\varrho(x) = 0$ if $x \in V_0$ and $\varrho(x) = 1$ if $x \in V_1$. A sequence of n-linear products denoted by brackets.

The graded n-linearity is expressed by

$$[x_1, \ldots, x_n, x_{n+1}, \ldots, x_m] =$$

$$(-)^{\varrho(x_n)\varrho(x_{n+1})}[x_1, \ldots, x_{n+1}, x_n, \ldots, x_m]$$

$$[x_1, \ldots, a_n x_n + b_n x'_n \ldots, x_m] =$$

$$a_n(-)^{\iota(a_n,n)}[x_1, \ldots, x_n, \ldots, x_m]+$$

$$b_n(-)^{\iota(b_n,n)}[x_1, \ldots, x'_n, \ldots, x_m]$$

where $\iota(a_n, n) = \varrho(a_n)(\varrho(x_1) + \ldots + \varrho(x_{n-1})$.

Strongly homotopy Lie algebras (2)

The defining identities for the algebra are, for all $n \in \mathsf{N}$

$$\sum_{\substack{k=0 \\ l=0}}^{k+l=n} \sum_{\pi(k,l)} \epsilon(\pi(k,l))$$

$$[[x_{\pi(1)}, \ldots, x_{\pi(k)}], x_{\pi(k+1)}, \ldots, x_{\pi(k+l)}] = 0.$$

where $\pi(k,l)$ stands for (k,l)-unshuffles.*

The low index, $n = 0$ and $n = 1$ brackets are treated separately, thus

$$[\cdot] = 0$$

$$[x] = \partial x,$$

with ∂ a derivation.

*A (k,l)-unshuffle is a permutation π of the indices $1, 2, \cdots, k+l$ such that $\pi(1) < \ldots < \pi(k)$ and $\pi(k+1) < \ldots < \pi(k+l)$. $\epsilon(\pi(k,l))$ is the sign picked up during the unshuffle as the points x_i with indices $0 \leq i \leq k$ are taken through the points x_j with indices $k+1 \leq j \leq l$. This is just the normal procedure in superalgebras.

Strongly homotopy Lie algebras (3)

A possible semantic target for abstract higher spin gauge field theory.

The image of a field Φ_n is a point x_n. The products $\mathbf{pr}(\cdot)$ map into the brackets $[\cdot]$. The sh-Lie algebra must be supplied with an inner product.

Which particular algebra to map to?

Map the corresponding categories.

A category of interacting fields $\mathbf{HSField}$.

A category for strongly homotopy Lie algebras $\mathbf{shLie}$.

The interpretation map $[|\cdot|]$ is then a functor

$$[|\cdot|] :: \mathbf{HSField} \to \mathbf{shLie}.$$

Categories, operads

A category is a class of objects. For every pair of objects X and Y, there is a set of arrows (morphisms). Arrows can be composed, and there are left and right identity arrows.

The objects can be sets with extra structure, but need not be.

The "paradigm" of category theory is to resist the temptation to peek into the objects.

What is the category **HSField**?

- Objects: Field configurations

  Morphisms: Field products

  Possibly multi-categories, or operads

- Objects: Field theories

  Morphisms: Structure preserving maps

EXTRA: Berends, Burgers and vanDam

Single higher spin interacting field theory attempt (late 80's), but can be generalized to an infinite tower of fields.

$$L = L_0(\phi,\phi) + gL_1(\phi,\phi,\phi) + g^2 L_2(\phi,\phi,\phi,\phi) + \cdots$$

$$\phi \to \phi' = \phi + \partial\xi + gT_1(\phi,\xi) + g^2 T_2(\phi,\phi,\xi) + \cdots$$

Assuming that the resulting gauge algebra is well-behaved (closed modulo field equations & Jacobi identities) it has been showed (Stasheff, Lada, Fulp) that an sh-Lie algebra results.

EXTRA: A way to look at $Q|\Phi\rangle$

In Fronsdal's equations

$$\partial^2\phi^{(s)} - \partial^{(1}\partial\cdot\phi^{s-2)} + \partial^{(2}\phi'^{\,s-2)} = 0$$

$$\delta\phi^{(s)} = \partial^{(1}\xi^{s-1)}$$

fields, derivatives and traces are all mixed up.

When we introduce mechanical ghosts (leading to extra fields) this can be written as

$$Q|\Phi\rangle = 0$$

$$\longleftarrow\longrightarrow$$

we're pulling the structure apart. Gauge invariance is encoded in

$$Q^2 = 0$$

Questions are

- In what ways can this pulling apart be done in the free field theory?

- How can these be extended to interactions?

EXTRA: on models

Case I. Reducible tower of higher spin gauge fields First class constraints G_0, G_-, G_+. Infinite tower of higher spin gauge fields.

Case II. Irreducible tower of higher spin gauge fields Plus second class trace constraints T and $T^\dagger$. Reproduces the Fronsdal equations.

Case III. Irreducible single higher spin gauge field $N - \lambda$ as a constraint turns T and $T^\dagger$ into first class (all are first class). Single higher spin gauge field.[Casalbuoni, Dominici and Longhi]

Case IV. Reducible single higher spin gauge field First class constraints G_0, G_-, G_+ and $N - \lambda$, but not $T, T^\dagger$. Fixes the spin to λ.

Case V. Regge trajectory of massive higher spin fields Combining G_0 and N into one constraint $G_0 + N$, all the rest of the constraints become second class. Regge trajectory of massive higher spin fields.[GershunPashnev1988].

EXTRA: Second quantisation

The fields can now be quantised by performing a normal mode expansion in terms of creators $\Lambda^+(\mathbf{p})_{\mu_1\dots\mu_n}$ and annihilators $\Lambda^-(\mathbf{p})_{\mu_1\dots\mu_n}$

$$\Phi^{\mu_1\cdots\mu_n}(x,\xi) =$$

$$\int \frac{d^3\mathbf{p}}{\sqrt{2p^0}}(\Lambda^+(\mathbf{p})_{\mu_1\dots\mu_n}\,\mathsf{exp}[-ip\cdot x]+$$

$$\Lambda^-(\mathbf{p})_{\mu_1\dots\mu_n}\,\mathsf{exp}[ip\cdot x])\,f^{\mu_1\cdots\mu_n}(\xi)$$

(This formula is somewhat qualitative - all details with polarisations et cetera can be fixed up.) [Casalbuoni, Dominici and Longhi, 1976]

EXTRA: Implementing the product (II)

Fock space vertex operator

Semantic map

$$\Phi_k \hookrightarrow |\Phi_k\rangle \qquad \text{for } k \in N$$

$$\mathbf{pr}(\Phi_1, \ldots, \Phi_n) \hookrightarrow \langle\Phi_1| \ldots \langle\Phi_n|\mathcal{V}_{n+1}\rangle \qquad \text{for } n \geq 1.$$

evaluates to a field

$$|\Phi_{n+1}\rangle$$

The vertex $|\mathcal{V}_{n+1}\rangle$ is very complex - computable but impractical.

EXTRA on Fock space vertices (I)

Recursive equations for the vertices

Base:

$$\sum_{r=1}^{3} Q_r|\mathcal{V}_3\rangle = 0$$

Step:

$$\sum_{r=1}^{n+1} Q_r|\mathcal{V}_{n+1}\rangle = -\sum_{p=0}^{\lfloor (n-3)/2 \rfloor} |\mathcal{V}_{p+3}\rangle \diamond |\mathcal{V}_{n-p}\rangle .$$

EXTRA on vertices (II)

$$\mathcal{V}_n = \kappa^{(\frac{nD}{2}-D-n)} \exp\big(\sum_m^{\infty} \Delta^n_{2m}\big)$$

where

$$\Delta^n_{2m} = {}^nY^{r_1 s_1 \cdots r_m s_m}_{a_1 \cdots a_m} \eta^{a_1}_{r_1 s_1} \cdots \eta^{a_m}_{r_m s_m}$$

and

- all r_i and s_i, $i \in [1..m]$ are summed over the list $[1..n]$
- all a_i, $i \in [1..m]$ are summed over the list [1..5]
- ${}^nY^{r_1 s_1 \cdots r_m s_m}_{a_1 \cdots a_m}$ are algebraic numbers to be determined
- $$\begin{cases} \eta^1_{rs} = \alpha^\dagger_r \cdot \alpha^\dagger_s \\ \eta^2_{rs} = \kappa \alpha^\dagger_r \cdot p_s \\ \eta^3_{rs} = c^+_r b_{-s} \\ \eta^4_{rs} = \kappa c^+_r b_{0s} \\ \eta^5_{rs} = \kappa^2 p_r \cdot p_s \end{cases}$$