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# Confinement from dyons

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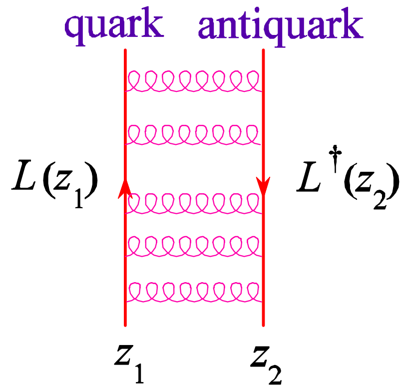
Confinement criteria in a pure glue theory (no dynamical quarks):

1) Average Polyakov line

$$< \mathrm{Tr}\ L(\mathbf{z}) > \; = \; \left\langle \mathrm{Tr}\,\mathcal{P}\,\exp\left(i\int_0^{1/T} A_4\,dt\right)\right\rangle$$

$$= \; e^{-M_{\text{quark}}\,/\,T} \begin{cases} = 0 & \text{below } T_c \\ \neq 0 & \text{above } T_c \end{cases}$$

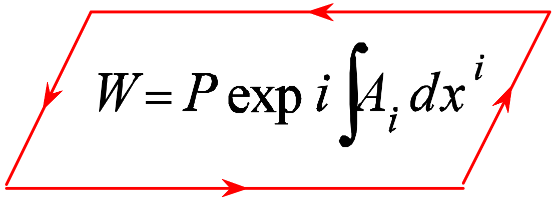
2) Linear rising potential energy of static quark and antiquark




$$< \mathrm{Tr}\ L(\mathbf{z_1})\ \mathrm{Tr}\ L^{\dagger}(\mathbf{z_2}) >= e^{-V(z_1-z_2)/T}$$

$$V(z_1 - z_2) = |\mathbf{z_1} - \mathbf{z_2}|\,\sigma$$

3) Area law for the average Wilson loop



$$W = P\exp i\int A_i\,dx^i \sim \exp(-\sigma\ \text{Area})$$

4) Mass gap: no massless states, only massive glueballs

We shall consider quantum Yang-Mills theory at nonzero T, as we shall be interested not only in confinement at small T but also in the deconfinement phase transition at T>Tc. Quarks are switched off.

According to Feynman, the partition function is given by a path integral over all connections periodic in imaginary time, with period 1/T :

$$\mathcal{Z} \quad = \quad \int DA_\mu^a(t,\mathbf{x}) \, \exp\left(-\frac{1}{4g^2}\int_0^{\frac{1}{T}} dt \int d^3x \, F_{\mu\nu}^a F_{\mu\nu}^a\right),$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c \,,$$

$$A_\mu^a\left(t + \frac{1}{T}, \mathbf{x}\right) = A_\mu^a\,(t, \mathbf{x})\,.$$

A helpful way to estimate integrals is by the saddle point method.
*Dyons are saddle points*, i.e. field configurations satisfying the non-linear Maxwell equation:

$$D_\mu^{ab} F_{\mu\nu}^b = 0, \qquad D_\mu^{ab} = \delta^{ab}\partial_\mu + f^{acb} A_\mu^c\,.$$

Bogomol'nyi-Prasad-Sommerfield **monopoles** or **dyons** are self-dual configurations of the Yang-Mills field, whose asymptotic electric and magnetic fields are Coulomb-like, and the eigenvalues of the Polyakov line are *non-trivial*.

For the *SU(N)* gauge group there are *N* kinds of elementary dyons:

$$\mathbf{E} = \mathbf{B} \stackrel{|\mathbf{x}|\to\infty}{=} \frac{\mathbf{x}}{|\mathbf{x}|^3} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

hence, "dyons"

$$L(\mathbf{x}) = \mathcal{P} \exp\left( i \int_0^{1/T} dx^4\, A_4(\mathbf{x}, x^4) \right) \xrightarrow{|\mathbf{x}|\to\infty} \begin{pmatrix} e^{2\pi i \mu_1} & 0 & 0 \\ 0 & e^{2\pi i \mu_2} & 0 \\ 0 & 0 & e^{2\pi i \mu_3} \end{pmatrix}$$ "holonomy"

$$\mu_1 \le \mu_2 \le \mu_3 \le \mu_1 + 1, \qquad \mu_1 + \mu_2 + \mu_3 = 0.$$

Inside the dyons' cores, whose size is $\frac{1}{2\pi T(\mu_m - \mu_n)}$, the field is large and, generally, time-dependent, the non-linearity is essential. Far away the field is weak and static.

In the saddle point method, one has to compute small-oscillation determinants about classical solutions.

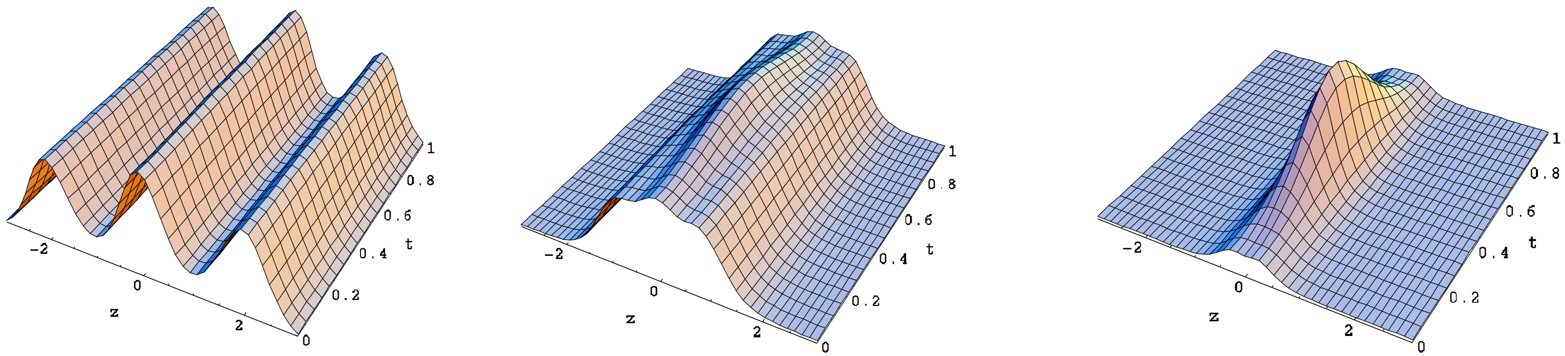
The small-oscillation determinant about a single dyon is infrared-divergent (because of the Coulomb asymptotics at infinity)

$\Rightarrow$ isolated dyons are unacceptable, they have zero weight

One has to take *neutral* clusters of *N* kinds of dyons. The corresponding exact solutions are known as Kraan-van Baal-Lee-Lu (KvBLL) calorons or **instantons with non-trivial holonomy** (1998).

The KvBLL instantons generalize standard instantons to the case when the Polyakov loop (the holonomy) is nontrivial, $\mu_1,\ \mu_2,\ \ldots \mu_N \neq \frac{k}{N}, \qquad k = 0, 1, \ldots, (N-1)$

The analytical solution shows what happens when dyons come close to each other:




Action density as function of time of three dyons of the SU(3) group.

At large dyon separations, we have three *static* dyons.

When dyons merge, they become a standard *time-dependent* instanton.

In all cases the full action is the same.

The small-oscillation determinant about KvBLL instantons is finite;
computed *exactly* by Diakonov, Gromov, Petrov, Slizovskiy (2004)
as function of

- separations between *N* dyons
- the phases of the Polyakov line $\mu_1,\ \mu_2, \ldots, \mu_N$
- temperature T
- $\Lambda$, the renormalized scale parameter

The 1-loop statistical weight (or probablity) of an instanton with non-trivial holonomy:

$$W = \int d\mathbf{x}_1 \ldots d\mathbf{x}_N \, \det G \, f^N .$$

$$f = \frac{4\pi}{g^4} \frac{\Lambda^4}{T} c,$$

"fugacity"

$$c = (\mathrm{Det}(-\triangle))^{-1}_{\text{reg, norm}} \approx \exp\left(-VT^3 P^{\text{pert}}(\mu_m)\right)$$

$$G_{mn}^{N\times N} = \delta_{mn}\left(4\pi\nu_m + \frac{1}{T|\mathbf{x}_{m,m-1}|} + \frac{1}{T|\mathbf{x}_{m,m+1}|}\right) - \frac{\delta_{m,n-1}}{T|\mathbf{x}_{m,m+1}|} - \frac{\delta_{m,n+1}}{T|\mathbf{x}_{m,m-1}|}$$

$$\nu_m = \mu_{m+1} - \mu_m, \qquad \sum \nu_m = 1.$$

Gibbons and Manton (1995); Lee, Weinberg and Yi (1996); Kraan (2000); DD and Gromov

The expression for the metric of the moduli space G is exact, *valid for all separations* between dyons.

If holonomy is trivial, or T -> 0, the measure reduces to that of the standard instanton, written in terms of center, size and orientations [Diakonov and Gromov (2005)].

The perturbative potential energy (it is present even in the absence of dyons ) as function of the Polyakov loop phases $\mu_m$ :

$$P^{\rm pert}(\mu_m) = \frac{(2\pi)^2\,T^3}{3}\sum_{m>n}^{N}(\mu_m-\mu_n)^2[1-(\mu_m-\mu_n)]^2\Big|_{\rm mod\ 1} \quad \sim \; T^3$$

It has *N* degenerate minima when all $\mu_m$ are equal (*mod 1*) i.e. when the Polyakov loop belongs to one of the *N* elements of the group center:

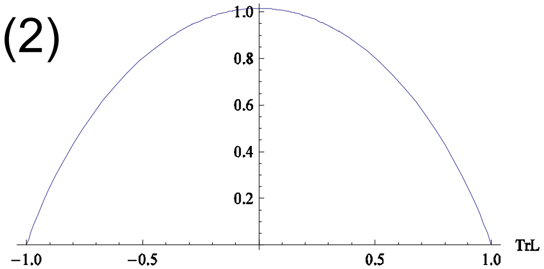
$$L = e^{\frac{2\pi i k}{N}}\,{\rm diag}(1,1\ldots 1) \in Z_N, \qquad k = 1,2\ldots N\,.$$

In perturbation theory, deviation from these values are forbidden as *exp*(- const.*V*).

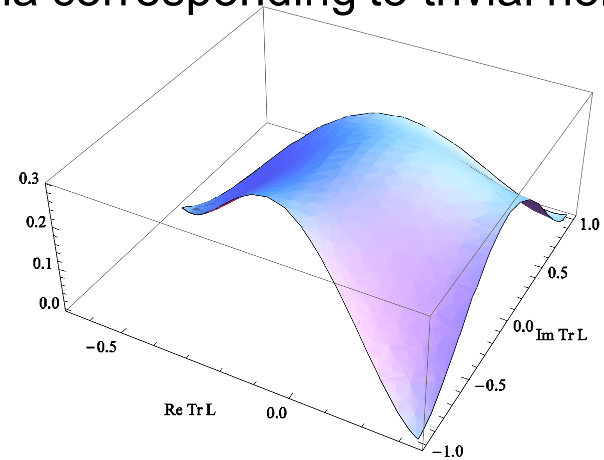
For confinement, one needs *Tr L = 0* , which is achieved at the **maximum** of the perturbative energy!

Perturbative potential energy has *N* minima corresponding to trivial holonomy:

SU(2)

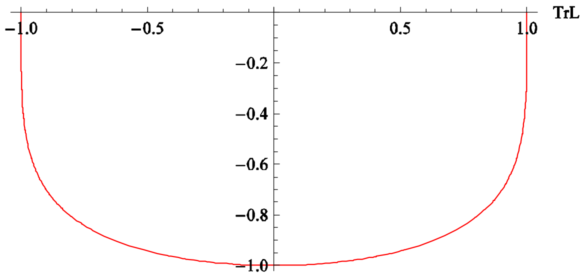


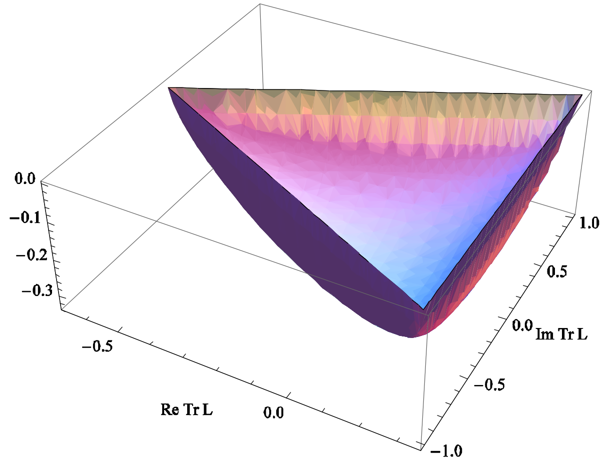

scale as T^4




SU(3)

However, the non-perturbative free energy of the ensemble of $\mathcal{O}(V)$ dyons has the **minimum** at *Tr L = 0* ! At low *T* it wins → confinement !
[ DD (2003) ]




At T<Tc the dyon-induced free energy prevails and forces the system to pick the “confining” holonomy

$$L = \begin{pmatrix} e^{-\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{\frac{0\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \end{pmatrix} \quad \text{for } SU(3), \qquad \mathrm{Tr}\, L = 0$$

To see it, one has to calculate the partition function of the grand canonical ensemble of an arbitrary number of dyons of *N* kinds and arbitrary $\mu_m$'s, and then minimize the free energy in $\mu_m$'s

(and also compute the essential correlation functions).

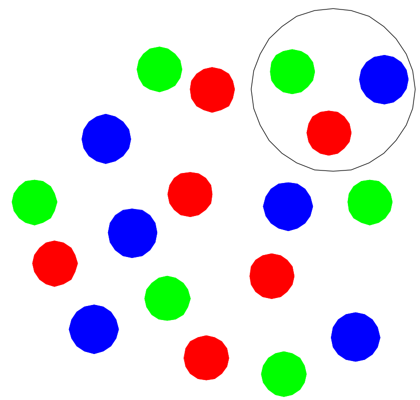
fugacity, function of T, $\Lambda$

$$\mathcal{Z} = \sum_{K_1\ldots K_N} \frac{1}{K_1!\ldots K_N!} \prod_{m=1}^{N} \prod_{i=1}^{K_m} \int (d^3\mathbf{x}_{mi}\, f)\ \det\ G(\mathbf{x})\ .$$

moduli space metric,
function of dyon separations

$K_m$ number of dyons of kind *m*

$\mathbf{x}_{mi}$ 3d coordinate of the *i*-th dyon of kind *m*



G is the “moduli space metric tensor” whose dimension is the total # of dyons:

$$G_{mi,nj} = \delta_{mn}\delta_{ij}\left(4\pi\nu_m + \sum_k \frac{1}{|\mathbf{x}_{mi}-\mathbf{x}_{m-1,k}|} + \sum_k \frac{1}{|\mathbf{x}_{mi}-\mathbf{x}_{m+1,k}|} - 2\sum_{k\neq i}\frac{1}{|\mathbf{x}_{mi}-\mathbf{x}_{mk}|}\right) - \frac{\delta_{m,n-1}}{|\mathbf{x}_{mi}-\mathbf{x}_{m+1,j}|} - \frac{\delta_{m,n+1}}{|\mathbf{x}_{mi}-\mathbf{x}_{m-1,j}|} + 2\,\frac{\delta_{mn}}{|\mathbf{x}_{mi}-\mathbf{x}_{mj}|}\Bigg|_{i\neq j} .$$

Properties:

1) the metric is hyper-Kaehler (a very non-trivial requirement)
2) same-kind dyons repulse each other, whereas different-kind attract e.o.
3) if dyons happen to organize into well separated neutral clusters with N dyons in each (= instantons), then *det G* is factorized into exact measures!
4) identical dyons are symmetric under permutations: they should not “know” what instanton they belong to!

This is an unusual statistical physics based not on the Boltzmann *exp(-U/T)* but on the measure *det G*; it can be written as *exp(Tr Log G)*, but then there will be many-body forces!

It turns out that this statistical ensemble is equivalent to an **exactly solvable** 3d Quantum Field Theory!

Use two tricks to present the ensemble as a QFT:

1) «fermionization» [Berezin]

$$\det\, G = \int \prod_A d\psi_A^\dagger\, d\psi_A\, \exp\left(\psi_A^\dagger\, G_{AB}\, \psi_B\right)$$

anticommuting Grassmann variables

2) «bosonization» [Polyakov]

auxiliary boson field

$$\exp\left(\sum_{m,n} \frac{Q_m Q_n}{|\mathbf{x}_m - \mathbf{x}_n|}\right) \quad = \quad \int D\phi\, \exp\left(-\int d\mathbf{x}(\partial_i\phi\partial_i\phi + \rho\phi)\right)$$

$$= \quad \exp\left(\int \rho \frac{1}{\triangle} \rho\right), \qquad \rho = \sum Q_m\, \delta(\mathbf{x} - \mathbf{x}_m)$$

Here the «charges» $Q$ are Grassmann variables but they can be easily integrated out [ Diakonov and Petrov (2007) ]

The partition function of the dyon ensemble can be presented identically as a QFT with *2N* boson fields *v_m, w_m*, and *2N* anticommuting (ghost) fields:

$$\mathcal{Z} = \int D\chi^\dagger D\chi D v D w \, \exp \int d^3x \left\{ \frac{T}{4\pi} \left( \partial_i \chi_m^\dagger \partial_i \chi_m + \partial_i v_m \partial_i w_m \right) + f \left[ (-4\pi\mu_m + v_m) \frac{\partial \mathcal{F}}{\partial w_m} + \chi_m^\dagger \frac{\partial^2 \mathcal{F}}{\partial w_m \partial w_n} \chi_n \right] \right\}$$

$$\mathcal{F} = \sum_{m=1}^{N} e^{w_m - w_{m+1}} .$$

⟸ periodic *N*-particleToda potential

$$\int D v_m \longrightarrow \delta \left( -\frac{T}{4\pi} \partial^2 w_m + f \frac{\partial \mathcal{F}}{\partial w_m} \right)$$

$$\int D w_m \, \delta \left( -\frac{T}{4\pi} \partial^2 w_m + f \frac{\partial \mathcal{F}}{\partial w_m} \right)$$

$$\longrightarrow \det{}^{-1} \left( -\frac{T}{4\pi} \partial^2 \delta_{mn} + f \left. \frac{\partial^2 \mathcal{F}}{\partial w_m \partial w_n} \right|_{w=\bar{w}} \right)$$

$$\int D\chi_m^\dagger D\chi_m \longrightarrow \det \left( -\frac{T}{4\pi} \partial^2 \delta_{mn} + f \left. \frac{\partial^2 \mathcal{F}}{\partial w_m \partial w_n} \right|_{w=\bar{w}} \right)$$

**boson and ghost determinants cancel. Classical calculation is exact!**

1st result , 1st criterion of confinement:

The minimum of the free energy is at equidistant values of $\mu_m$ corresponding to the zero average value of the Polyakov line!

Indeed, the dyon-induced potential energy as function of $\mu_m$ ,

$$\mathcal{P} = -4\pi f \, N \, (\nu_1 \nu_2 ... \nu_N)^{\frac{1}{N}}, \qquad \nu_1 + \nu_2 + ... + \nu_N = 1,$$

$$\nu_m = \mu_{m+1} - \mu_m$$

has the minimum at

$$\nu_1 = \nu_2 = ... = \nu_N = \frac{1}{N}, \qquad \mathcal{P}^{\min} = -4\pi f.$$

i.e. at equidistant $\mu_m$ , which implies *Tr L = 0* !

# Confinement-deconfinement in the exceptional group G2 ?

rank=2, trivial center (contrary to SU(N)!), lowest dimensional representation dim=7.
Question: is there a confinement-deconfinement phase transition in G2 ?
Lattice answer [Pepe and Weise (2007), Greensite et al. (2007), Di Giacomo et al. (2007)]: **Yes!**

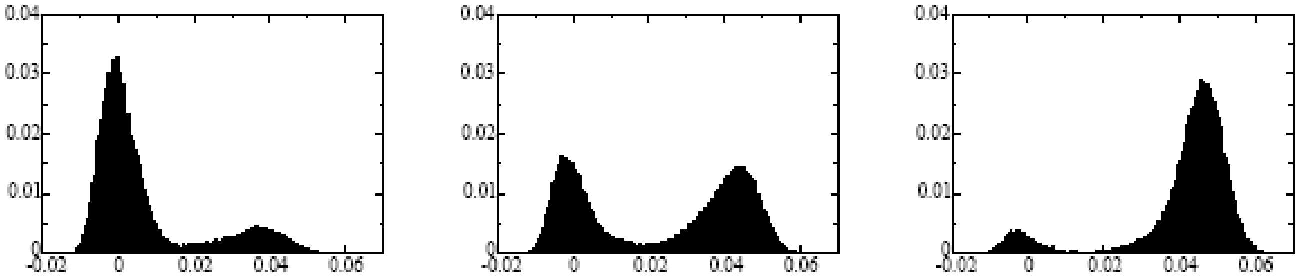



Figure 4: Polyakov loop probability distributions in the region of the deconfinement phase transition in (3+1)-d $G(2)$ Yang-Mills theory. The temperature increases from left to right. The simulations have been performed on a $20^3 \times 6$ lattice at the three

Since G2 is centerless, the transition cannot be attributed to the spontaneous breaking of center symmetry.

Dyons explain < Tr L > = 0 at low T, and a first order phase transition at a critical Tc !!

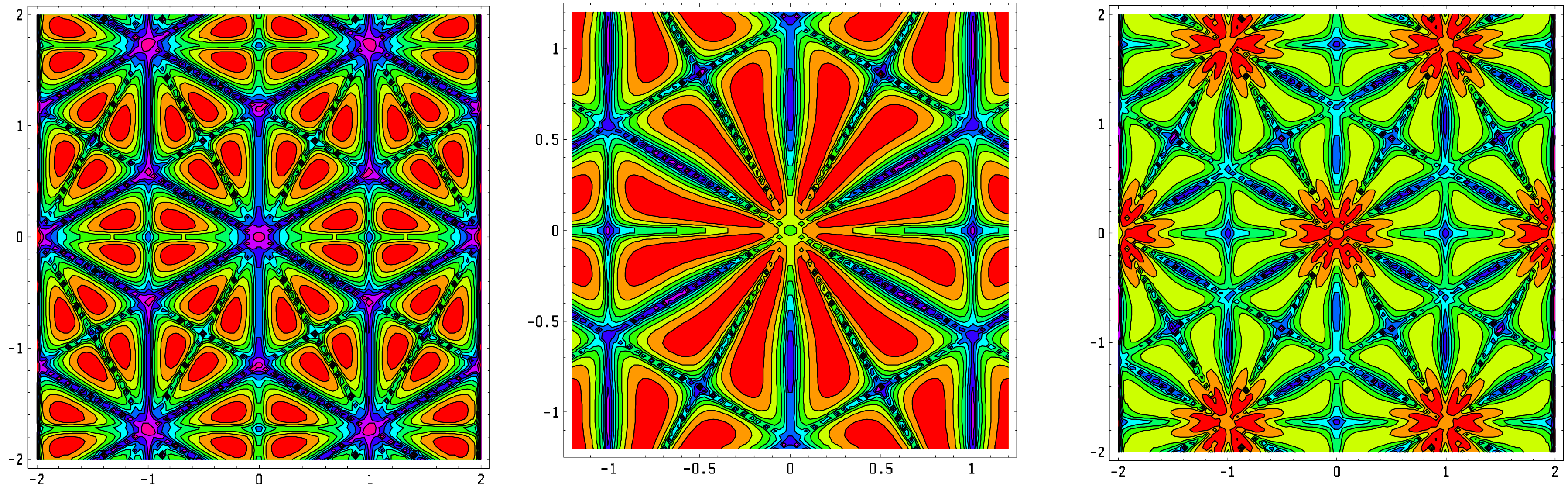
At low T<Tc, the free energy induced by dyons, has the minimum at

L = *diag (exp(2 pi i (-5/12, -4/12,-1/12, 0, 1/12, 4/12, 5/12)),*     *Tr L = 0 !!*

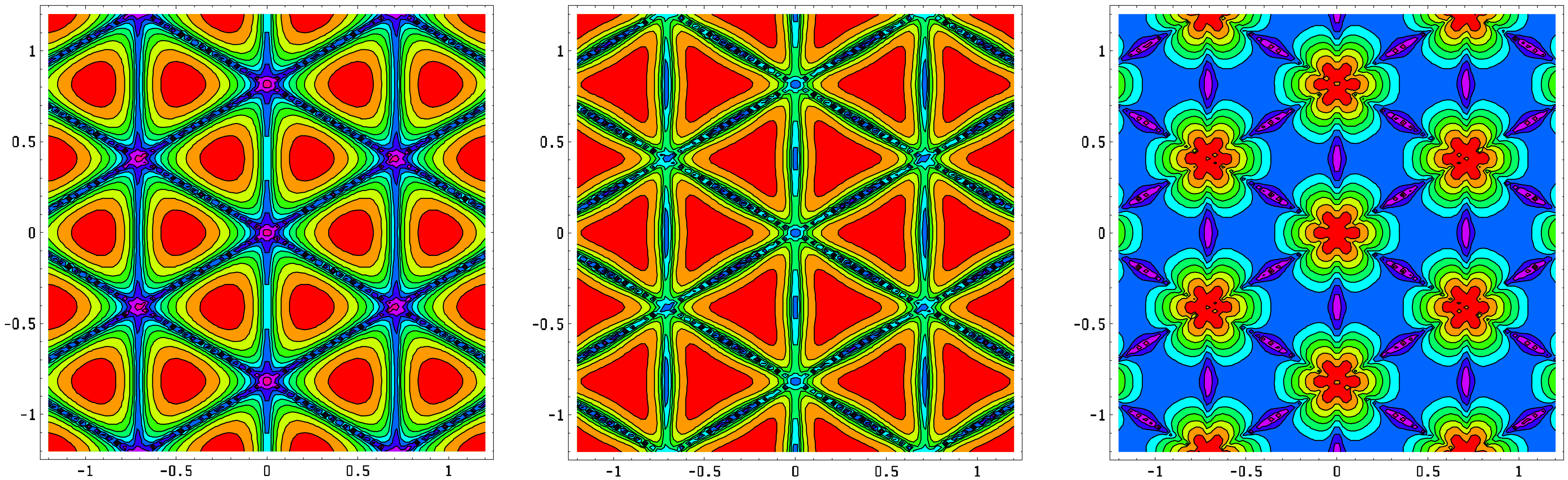
G2 instanton is made of 4 dyons of 3 kinds:

Contour plots of the **effective potential** as function of two eigenvalues of A4 :

G2:




SU(3): T=0 T=Tc T=1.5 Tc

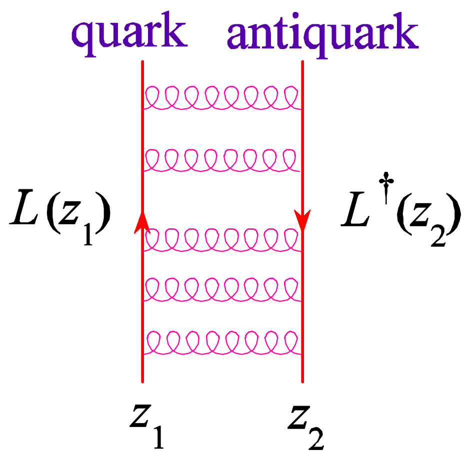


The correlation function of two Polyakov lines defines the potential energy between two static quarks:

$$\left\langle \mathrm{Tr}\, L(\mathbf{x}) \mathrm{Tr}\, L^\dagger(\mathbf{y}) \right\rangle = C \exp\left( -\frac{V(\mathbf{x}-\mathbf{y})}{T} \right)$$

$$V(\mathbf{x}-\mathbf{y}) = \sigma\, |\mathbf{x}-\mathbf{y}|, \qquad \sigma \approx 1\, \frac{\mathrm{GeV}}{\mathrm{fm}}$$

$$\sigma = \frac{\Lambda^2}{\lambda} \frac{N_c}{\pi} \sin\frac{\pi}{N_c}, \qquad N_c = 3$$




2nd result , 2nd criterion of confinement:

The potential energy of static quark and antiquark is linearly rising with separation, with a calculable slope, or string tension.

The string tension has a finite limit at small T.
It is stable in the number of colours Nc, as it should be.

## 3d result, 3d criterion

$$W = P \exp i \oint A_i \, dx^i \sim \exp(-\sigma \, \text{Area})$$

Along the surface spanning the loop there is a large (dual) field, “the string”, leading to the area behaviour of the average Wilson loop !

At low T the “magnetic” string tension coincides with the “electric” one, as it should be: $\sigma_{\text{electr}} = \sigma_{\text{magn}}$ , $T^{o} \to 0$

The Lorentz symmetry is restored, despite the 3d formulation.

Moreover, in *SU(N)* there are *N* different string tensions, classified by the “*N*-ality” of the representation, in which the Wilson loop is considered. We find

$$\sigma_{\text{electr}}(k) = \sigma_{\text{magn}}(k) = \frac{\Lambda^2}{\lambda} \frac{N_c}{\pi} \sin \frac{\pi k}{N_c}$$

the results for the two string tensions are the same although they are computed in two very different ways

for the rank-*k* antisymmetric tensor representation.

The string tension in the adjoint representation (*k=0*) is asymptotically zero.

## 4th result, thermodynamics of the deconfinement phase transition:

In the confinement phase, the free energy is

$$\frac{F}{V} = -N_c^2 \frac{\Lambda^4}{2\pi^2\lambda^2} + T^4 \frac{\pi^2}{45}\left(N_c^2 - \frac{1}{N_c^2}\right) - T^4 \frac{\pi^2}{45}\left(N_c^2 - 1\right)$$

dyon-induced — perturbative energy at maximum — Stefan-Boltzmann

$\mathcal{O}(N_c^2)$ gluons are cancelled from the free energy, as it should be in the confining phase!

The 1st order confinement-deconfinement phase transition is expected at

$$T_c^4 = \frac{45}{2\pi^4} \frac{N_c^4}{N_c^4 - 1} \frac{\Lambda^4}{\lambda^2}$$

(At Nc = 2 the free energy depends only on one variable, and the phase transition is explicitly 2nd order, in agreement with the lattice data.)

Critical temperature T_c in units of the string tension for various numbers N_c :

| | $N_c = 3$ | 4 | 6 | 8 |
|---|---|---|---|---|
| $T_c/\sqrt{\sigma}$, theory | 0.6430 | 0.6150 | 0.5967 | 0.5906 |
| $T_c/\sqrt{\sigma}$, lattice | 0.6462(30) | 0.6344(81) | 0.6101(51) | 0.5928(107) |

[lattice data: Lucini, Teper and Wenger (2003)]

Another important quantity characterizing the non-perturbative vacuum –
the “topological susceptibility” :

$$\frac{(< Q_T^2 >)^{\frac{1}{4}}}{\sqrt{\sigma}} = \begin{cases} 0.439, & \text{theory} \\ 0.434(10), & \text{lattice} \end{cases} \qquad \text{for } N_c = 3.$$

# Summary

1) The statistical weight of gluon field configurations in the form of $N$ kinds of dyons has been computed exactly to 1-loop

2) Statistical physics of the ensemble of interacting dyons is governed by an exactly solvable 3d QFT

3) The ensemble of dyons self-organizes in such a way that all criteria of confinement are fulfilled

Non-trivial holonomy allows the existence of dyons,
dyons request the holonomy to be maximally non-trivial !

4) All quantities computed are in good agreement with lattice data

5) A simple picture of a semi-classical vacuum based on dyons works surprisingly well!