On Einstein - Weyl unified model of gravity, dark energy and dark matter:

brief history, new interpretation, simplest dimensional reductions, static and cosmological solutions

A.T. Filippov (Alexandre.Filippov@jinr.ru)

H.Weyl, **Raum, Zeit, Materie (Transl. from 1921 edition.)** !

$$\mathbf{V} = (\mathbf{G} + \alpha \mathbf{l}) + \frac{\epsilon^3}{4}\sqrt{g}\{1 - 3(\phi_i\phi^i)\},$$

$$\Gamma^r_{ik} = \left\{ {ik \atop r} \right\} + \tfrac{1}{2}\epsilon^2(\delta^r_i\phi_k + \delta^r_k\phi_i - g_{ik}\phi^r).$$

Thus, by neglecting the exceedingly small cosmological terms, we arrive exactly at the classical Maxwell-Einstein theory of electricity and gravitation. To make the expression correspond exactly with that of § 34 we must set $\frac{\epsilon^2}{2} = \lambda$. Hence our theory necessarily gives us Einstein's cosmological term $\frac{1}{2}\lambda\sqrt{g}$. The uniform distribution of electrically neutral matter at rest over the whole of (spherical) space is thus a state of equilibrium which is compatible with our law. But, whereas in Einstein's Theory (cf. § 34) there must be a pre-established harmony between the universal physical constant λ that occurs in it, and the total mass of the earth (because each of these quantities in themselves already determine the curvature of the world), here (where λ **denotes** merely the curvature), we have that the mass present in the world **determines** the curvature. It seems to the author that just this is what makes Einstein's cosmology physically possible. In the case in which a physical field is present, Einstein's cosmological term must be supplemented by the further term $-\frac{3}{2}\lambda\sqrt{g}(\phi_i\phi^i)$; and in the com-

Eddington's ideas

In 1919 Eddington proposed a more radical modification of general relativity [7], [8]. His idea was to start with the pure affine formulation of the gravitation, i.e. using first the general symmetric affine connection and only at some later stage introducing a metric tensor. Indeed, the curvature tensor can be defined without metric:

$$r^i_{klm} = -\Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} + \Gamma^i_{km,l} - \Gamma^i_{nm}\Gamma^n_{kl}. \quad (1)$$

Then the Ricci-like (but non-symmetric) curvature tensor can be defined by contracting the indices i, m (or, i, l):

$$r_{kl} = -\Gamma^m_{kl,m} + \Gamma^m_{nl}\Gamma^n_{km} + \Gamma^m_{km,l} - \Gamma^m_{nm}\Gamma^n_{kl} \quad (2)$$

(let us stress once more that $\Gamma^m_{nl} = \Gamma^m_{ln}$ but $r_{kl} \neq r_{lk}$).

Invariants and interpretation

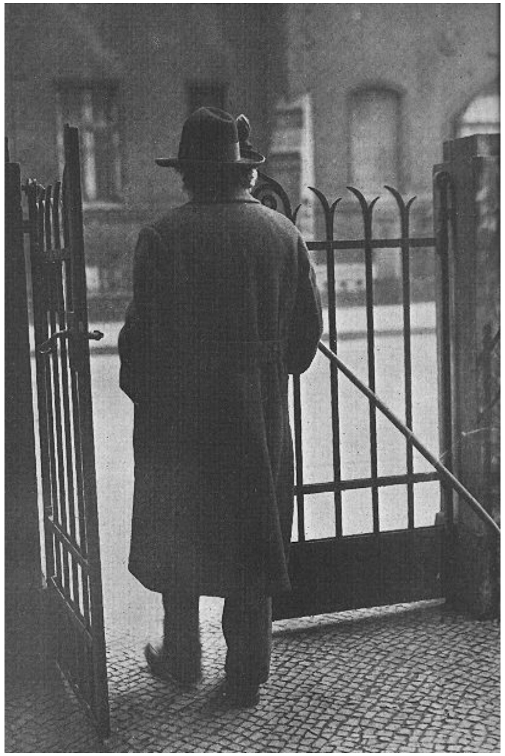
In particular, Eddington discussed different sorts of tensor densities, e.g.,

$$\hat{\mathcal{L}} \equiv \sqrt{-\det(r_{kl})} \equiv \sqrt{-r}\,, \tag{3}$$

which resembles the fundamental scalar density of the Riemannian geometry, $\sqrt{-\det(g_{kl})} \equiv \sqrt{-g}$. Eddington suggested to identify the symmetric part of r_{kl} with the metric tensor. The anti-symmetric part,

$$\phi_{kl} \equiv \frac{1}{2}(\Gamma^m_{km,l} - \Gamma^m_{lm,k})\,, \qquad \phi_{kl,m} + \phi_{lm,k} + \phi_{mk,l} \equiv 0\,, \tag{4}$$

strongly resembles the electro-magnetic field tensor and it seems natural to identify it with this tensor. Eddington tried to write consistent equations of the generalized theory but this problem was solved only by Einstein.



Einstein in Berlin

The first paper

К ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ *

§ 1. Общая часть. Вывод уравнения поля

Математическое построение общей теории относительности первоначально было полностью основано на метрике, т. е. на инварианте

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu. \qquad (1)$$

Величины $g_{\mu\nu}$ и их производные изображали метрическое, а также гравитационное поле. Напротив, компоненты электрического поля оставались по отношению к ним совершенно чужеродными. Желание свести гравитационное и электромагнитное поля в одно единое по своей сущности поле в последние годы владеет умами теоретиков.

Навстречу этим стремлениям идет математическое открытие, сделанное Леви-Чивитой и Вейлем: тензор кривизны Римана, имеющий фундаментальное значение для общей теории относительности, наиболее естественным путем можно получить с помощью закона «параллельного переноса» векторов («аффинная связь»):

$$\delta A^\mu = -\Gamma^\mu_{\alpha\beta} A^\alpha\, dx_\beta. \qquad (2)$$

Этот закон сводится к формуле (1), если постулировать, что длина вектора при параллельном переносе не меняется; однако этот шаг не является логически необходимым. Впервые это обстоятельство обнаружил Г. Вейль, построивший на нем обобщение римановой геометрии, которое, по его мнению, содержало теорию электромагнитного поля. Вейль придает инвариантный смысл не длине линейного элемента или вектора, а только отношению длин двух линейных элементов или векторов, исходящих из одной точки. Параллельный перенос (2) должен быть таким, чтобы это от-

---

* *Zur allgemeinen Relativitätstheorie*. Sitzungsber. preuss. Akad. Wiss., phys.-math. Kl., 1923, 32—38.

Существуют такие тензорные плотности, которые являются рационыльными функциями второй степени по величинам $R^i_{k,lm}$; они могут быть получены с помощью тензорной плотности δ^{iklm}, компоненты которой равны 1 или —1, в зависимости от того, образуют индексы $iklm$ четную или нечетную перестановку цифр 1, 2, 3, 4. Такой тензорной плотностью является, например,

$$R^i_{k,\,lm}R^k_{i,\,\sigma\tau}\delta^{lm\sigma\tau}.$$

Однако я считаю правильным ограничиться такими тензорными плотностями, которые образуются из свернутого тензора R_{kl} или из величин S_{kl} и ϕ_{kl}, поскольку мы можем придавать физический смысл только этим величинам. Тогда мы должны допускать и иррациональные функции, к чему мы уже привыкли в общей теории относительности (например, $\sqrt{-g}$). Но в этом случае существуют еще различные возможности, из которых наиболее интересной нам представляется следующая:

$$\mathfrak{H} = 2\sqrt{-|R_{kl}|}. \tag{10}$$

Это выражение, являющееся аналогом тензорной плотности элемента объема, образовано из величин R_{kl} без расщепления на симметричную и антисимметричную части. Если эта функция Гамильтона окажется хорошей, то теория придет идеальным способом к объединению гравитации и электричества в одно-

First paper – exposition of ideas

The starting point of Einstein in his first paper (72 in [1]) was to write the action principle and to suppose (3) to be the Lagrangian density depending on 40 connection functions Γ^m_{kl}. Varying the action w.r.t. these functions he derived 40 equations that allowed him to find the general expression for Γ^m_{kl} (the derivation is similar to that of the standard general relativity):

$$\Gamma^m_{kl} = \frac{1}{2}[s^{mn}(s_{kn,l} + s_{ln,k} - s_{kl,n}) - s_{kl}\, i^n + \frac{1}{3}(\delta^m_k\, i_l + \delta^m_l\, i_k)]. \qquad (5)$$

Here s_{kl} is a symmetric tensor (s^{mn} is the inverse matrix to s_{kl}), which Einstein interpreted as the metric tensor (then the first term is the Christoffel symbol for this metric), and i_n is a vector which he tried to connect with the electro-magnetic field.

This identification apparently follows from the equations

$$r_{kl} = R_{kl} + \frac{1}{6}[(i_{k,l} - i_{l,k}) + i_k i_l]\,, \tag{6}$$

$$\phi_{kl} = \frac{1}{6}(i_{k,l} - i_{l,k})\,, \tag{7}$$

which can be obtained by inserting the expression (5) into (2), (4); R_{kl} is the standard Ricci curvature tensor for the metric s_{kl} . Einstein's interpretation of ϕ_{kl} as the Maxwell field is not so natural because of the term $i_k i_l$ in the r.h.s. of Eq.(6) which in fact makes this interpretation impossible.

The Weyl - Einstein connection

NB: The connection (5) is a special case of the general expression. **The most general symmetric affine connection** has the form:

$$\Gamma^m_{kl} = \frac{1}{2}[s^{mn}(s_{nk,l} + s_{ln,k} - s_{kl,n}) + s^{mn}(s_{nkl} + s_{lnk} - s_{kln})], \qquad (8)$$

where s_{kl} is an arbitrary symmetric tensor, s^{mn} is inverse to s_{kl}, and s_{kln} is symmetric in k and l. Both Weyl and Einstein connections correspond to:

$$s_{kln} = \alpha\, s_{kl}\, i_n + \beta(s_{nk}\, i_l + s_{ln}\, i_k)\,. \qquad (9)$$

The **Weyl-Einstein connection** (defining the Weyl-Einstein spaces):

$$\Gamma^m_{kl} = \frac{1}{2}[s^{mn}(s_{nk,l} + s_{ln,k} - s_{kl,n}) + \alpha(\delta^m_k\, i_l + \delta^m_l\, i_k) - (\alpha - 2\beta)s_{kl}\, i^m]\,. \qquad (10)$$

Einstein's connection: $\alpha = -\beta = \frac{1}{3}$. Weyl's connection: $\alpha = 1$, $\beta = 0$.

The second paper

ЗАМЕЧАНИЕ К МОЕЙ РАБОТЕ „К ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ“ *

Подставляя выражение (4) для Г в формулу (2), получаем уравнения поля:

$$r_{kl} = R_{kl} + \frac{1}{6}\left[\left(\frac{\partial i_k}{\partial x_l} - \frac{\partial i_l}{\partial x_k}\right) + i_k i_l\right], \quad (13)$$

которые после разложения на симметричные и антисимметричные части дают уравнения гравитационного и электромагнитного полей. Здесь R_{kl} представляет собой римановский тензор кривизны, образованный из s_{kl} как метрического фундаментального тензора.

Уравнения (13) могут быть записаны в форме принципа Гамильтона

$$\delta\left\{\int\left[-2\sqrt{-r} + \mathfrak{R} - \frac{1}{6}\mathfrak{s}^{\alpha\beta} i_\alpha i_\beta\right] d\tau\right\} = 0, \quad (14)$$

где $\mathfrak{R}$ — скалярная плотность римановской кривизны, относящейся к фун-

The second paper (`no details for clarity')

In the second paper Einstein proposes the effective Lagrangian density:

$$\hat{\mathcal{L}} = -2\sqrt{-\det(r_{mn})} + \hat{R} - \frac{1}{6}\hat{s}^{mn}\hat{i}_m\hat{i}_n \,. \tag{11}$$

This should be varied w.r.t. $\hat{s}_{kl}$ and $\hat{f}_{kl}$, which are **the tensor densities defined with the aid of the scalar density** $\sqrt{-\det(s_{kl})}$ and corresponding to the tensors in the decomposition,

$$r_{kl} = s_{kl} + \phi_{kl} \,; \tag{12}$$

$\hat{R}$ is the scalar curvature density for the metric s_{kl}. The Lagrangian (11) contains a very complex term $\sqrt{-\det r_{mn}}$ which is more general than the so called Born-Infeld Lagrangian proposed ten years later M. Born (the first attempts to construct nonlinear electro-dynamics were undertaken by G. Mie).

The third (final) original paper

The main idea of the very beautiful third paper (‘*Zur affinen Feldtheorie*’) is to take for the Lagrangian $\hat{\mathcal{L}}$ an arbitrary function of s_{kl} and ϕ_{kl} . Then he introduces the Legendre transformation and the transformed (effective) Lagrangian density $\hat{\mathcal{L}}^*$:

$$\frac{\partial \hat{\mathcal{L}}}{\partial s_{kl}} \equiv \hat{g}^{kl}, \qquad \frac{\partial \hat{\mathcal{L}}}{\partial \phi_{kl}} \equiv \hat{f}^{kl}; \qquad s_{kl} = \frac{\partial \hat{\mathcal{L}}^*}{\partial \hat{g}^{kl}}, \qquad \phi_{kl} = \frac{\partial \hat{\mathcal{L}}^*}{\partial \hat{f}^{kl}} \tag{13}$$

Introducing the Riemann metric tensor g_{kl} and the i^k-vector,

$$g^{kl}\sqrt{-g} = \hat{g}^{kl}, \quad g_{kl}\, g^{lm} = \delta_k^m; \qquad \hat{i}^k = \partial_l \hat{f}^{kl}, \tag{14}$$

he claims (without proof) that Eq.(5) is valid with s_{kl} replaced by g_{kl} and thus the affine geometry is the same for any $\hat{\mathcal{L}}(s_{kl}, \phi_{kl})$.

Finally, he uses the freedom in choosing $\hat{\mathcal{L}}^*(\hat{g}^{kl}, \hat{f}^{kl})$ and proposes:

$$\hat{\mathcal{L}}^* = 2\alpha\sqrt{-g} - \frac{1}{2}\beta f_{kl}\,\hat{f}^{kl}\,, \qquad (15)$$

where α and β are some constants not defined by the theory.

Then Einstein rewrites the Lagrangian so that the equations of motion can be obtained by varying it in the metric and the vector field tensors, g_{kl} and f_{kl}. Neglecting dimensions ($\hbar = c = \kappa = 1$) and changing Einstein's notation we write it as follows:

$$\hat{\mathcal{L}} = \sqrt{-g}\,[R - 2\Lambda - F_{kl}F^{kl} - m^2 A_k A^k], \qquad F_{kl} \equiv A_{k,l} - A_{k,l}\,. \qquad (16)$$

Obviously, A_k is a neutral massive vector field with coupling to gravity only. We call it **vecton**, that is an old fashioned but proper term for this 'geometric' particle. This particle has not been directly observed but it can be considered as one of the possible candidates for **dark matter**. In view of the fact that the affine theory also predicted the cosmological constant term which is one of the best candidates for explaining **dark energy**, Einstein's theory may be considered as the first **unified model of dark energy and dark matter**.

2 Spherical reduction - static and cosmological solutions

2.1 Vecton-dilaton gravity

At first sight, the theory (16) is very close to the well-understood Einstein-Maxwell theory which can be obtained when $m = 0$. However, we will show that the two theories are qualitatively different and it is hardly possible to construct a reasonable perturbation theory in the parameter m^2.

The general spherically symmetric metric is $(i, j = 0, 1;\ x^0 = t, x^1 = r)$:

$$ds^2 = g_{ij}(t,r)\, dx^i dx^j + \varphi(t,r)(\sin^2\theta\, d\theta + d\phi^2)\,. \tag{17}$$

Supposing that all other functions also depend on t, r, inserting the metric (17) into the action with the Lagrangian (16), and integrating out the angle variables θ, ϕ one can derive the following effective two-dimensional Lagrangian

$$\mathcal{L}^{(2)} = \sqrt{-g}\,[\varphi\, R^{(2)} + 2 - 2\Lambda\varphi + (\partial\varphi)^2/2\varphi - \varphi\, F_{ij}F^{ij} - \varphi\, m^2 A_i A^i]. \tag{18}$$

E-W model in D=4

$$\hat{\mathcal{L}} = \sqrt{-g}\,[R - 2\Lambda - F_{kl}F^{kl} - m^2 A_k A^k]$$

Spherical reduction:

$$ds^2 = g_{ij}(t,r)\ dx^i dx^j + \varphi(t,r)(\sin^2\theta\, d\theta + d\phi^2)$$

$$\mathcal{L}^{(2)} = \sqrt{-g}\,[\varphi\, R^{(2)} + 2 - 2\Lambda\varphi + (\partial\varphi)^2/2\varphi - \varphi\, F_{ij}F^{ij} - \varphi\, m^2 A_i A^i]$$

Weyl - transformed Lagrangian

$$\mathcal{L}^{(2)}_W = \sqrt{-g}\,[\varphi\, R^{(2)} + 2\varphi^{-1/2} - 2\Lambda\varphi^{1/2} - \varphi^{-3/2}\, F^2 - \varphi\, m^2 A^2]$$

Below we use the Weyl rescaled Lagrangian ($g_{ij} = \varphi^{-\frac{1}{2}} g^{W}_{ij}$)

$$\mathcal{L}_W^{(2)} = \sqrt{-g}\,[\varphi\, R^{(2)} + 2\varphi^{-1/2} - 2\Lambda\varphi^{1/2} - \varphi^{-3/2} F^2 - \varphi\, m^2 A^2]$$

The equations of motion in a generic metric g_{ij} are equivalent to the Einstein equations for the spherically symmetric solutions. By varying w.r.t. the diagonal metric functions g_{ii} we fist write the energy and momentum constraints. In the light cone (LC) metric, $ds^2 = -4f(u,v)\,du\,dv$, these constraints are:

The most important eqs. are the CONSTRAINTS

$$f\partial_i\,(\partial_i\,\varphi/\,f)] + \varphi\, m^2 A_i^2 = 0, \quad i = u, v$$

that can be solved if the second term vanishes

Other EOM in the LC metric

$$\partial_u \partial_v \varphi + f(2\varphi^{-1/2} - 2\Lambda\varphi^{1/2} - \frac{1}{2}\varphi^{3/2} f^{-2} F_{uv}^2) = 0,$$

$$\partial_j \left(\varphi^{3/2} f^{-1} F_{ij}\right) = \varphi\, m^2 A_j \qquad i, j = u, v$$

$$F_{uv} \equiv A_{u,v} - A_{v,u}$$

$$\partial_v(\varphi A_u) + \partial_u(\varphi A_v) = 0$$

\`No spin zero’ condition in D=4

(Ludwig Lorenz condition)

2.2 Static states and horizons

Let us write the **static** equations corresponding to the naive reduction to one spatial dimension, $r = u + v$, etc. To derive them one can reduce either the equations or the Lagrangian. Define two additional functions, χ and B

$$\varphi'(r) = \chi(r), \qquad A'(r) = f(r)\varphi^{-3/2}(r)B(r)\,, \tag{23}$$

where $A_v(r) = -A_u(r) \equiv -A(r)$.

Then the other equations are

$$\chi' = -fU\,, \qquad B' = -\frac{1}{2}\varphi m^2 A\,, \qquad f' = (f/\chi)[-fU + \varphi m^2 A^2]\,, \tag{24}$$

$$U \equiv 2(\varphi^{-1/2} - \Lambda\varphi^{1/2} - \varphi^{-3/2}B^2)$$

Consider **solutions near possible horizons** that are defined as zeroes of the metric, $f \to 0$ for $\varphi \to \varphi_0$ and expand them in powers of $\tilde{\varphi} \equiv \varphi - \varphi_0$.

Further analysis shows that A, $\tilde{F} \equiv f/\chi$ and $\tilde{A} = A/\chi$ must be finite and thus:

$$\tilde{F}'(\varphi) = \varphi \, \tilde{F}(\varphi) \, m^2 \, \tilde{A}^2(\varphi) \,,$$

$$\chi'(\varphi) = -\tilde{F}(\varphi) \, U(\varphi) \,, \quad B'(\varphi) = -\frac{1}{2} \varphi \, m^2 \, \tilde{A}(\varphi) \,,$$

$$\tilde{A}'(\varphi)\chi(\varphi) = \tilde{F}(\varphi) \left[\varphi^{-3/2} B(\varphi) + U(\varphi) \tilde{A}(\varphi) \right],$$

the prime denotes differentiation in φ.

φ_0, $\tilde{A}_0$, B_0, $\tilde{F}_0$ can be taken arbitrary up to one relation

$$\tilde{A}_0 U_0 + \varphi_0^{-3/2} B_0 = 0\,, \qquad U_0 \equiv U(\varphi_0, B_0).$$

This equation can be solved w.r.t any parameter. It has two solutions for φ_0 which means that **there may exist two horizons.**

The solutions are the power series expansions convergent near the horizons

Naïve cosmological reduction (to be corrected below!)

$$ds^2 = e^{2\alpha} dr^2 - e^{2\gamma} dt^2$$

$$\mathcal{L}_c = 6ke^{\alpha+\gamma} - 6\dot{\alpha}^2 e^{3\alpha-\gamma} - 2\Lambda e^{3\alpha+\gamma} + \dot{A}^2 e^{\alpha-\gamma} - m^2 A^2 e^{\alpha+\gamma}$$

the gauge fixing condition $\gamma = 0$

$$\mathcal{H}_0^c \equiv f[-6\dot{f}^2 - 6k + 2\Lambda f^2 + \dot{A}^2 + m^2 A^2] = 0$$

the LC gauge $\alpha = \gamma$

$$\mathcal{H}_1^c \equiv -6\dot{f}^2 - 6kf^2 + 2\Lambda f^4 + \dot{A}^2 + f^2 m^2 A^2 = 0$$

Consistency of the reduction for the vecton

To have a consistent reduction we should check all the equations similar to Eqs.(17) - (19) but written in the Einstein frame and in the (r, t) coordinates. The main restrictions on the separated functions are given by the constraints and the equations for the vector field (analogous to (17) and (19)). Omitting details we give here the main points only.

Let us make the above statements more precise

(Corrections to the standard approach)

To get all spherically symmetric cosmological models we take the general spherically symmetric metric

$$ds^2 = e^{2\alpha} dr^2 - e^{2\gamma} dt^2 + 2e^{2\delta} dr dt + e^{2\beta} d\Omega_2^2 , \qquad (55)$$

where α, β, γ, δ depend on t and r. Then we separate the variables by the additive *Ansatz* for the metric functions, $\alpha = \alpha_0(t) + \alpha_1(r)$, etc., and take A_i depending only on t.

All homogeneous isotropic cosmologies should satisfy the following necessary conditions:

$$\dot{\alpha} = \dot{\beta}\,, \quad \gamma' = 0\,, \quad {\beta_1}'' + ke^{-2\beta_1} = 0\,, \quad ke^{-2\beta_1} - 3{\beta_1'}^2 - 2{\beta_1}'' = C\,, \qquad (56)$$

where C is a constant proportional to the 3-curvature (its time dependence is given by the factor $e^{-2\alpha_0}$) and the third equation is the isotropy condition. Neglecting inessential constant factors, we also have chosen $\alpha_1 = \gamma_1 = 0$.

For the SCALAR matter

$$\mathcal{L}_c = 6ke^{\alpha+\gamma} - 6\dot{\alpha}^2 e^{3\alpha-\gamma} - 2\Lambda e^{3\alpha+\gamma} + \dot{\psi}^2 e^{3\alpha-\gamma} - \mu^2\psi^2 e^{3\alpha+\gamma}\,.$$

Reduction of the vecton equations - additional conditions

In the vecton model the equations for $A_i(t,r)$ **do give additional constraints** on $\beta_1(r)$ and the effective Lagrangian (25) must be corrected to account for these constraints and thus to be consistent with its higher-dimensional origin. These equations are:

$$\partial_0[e^{\alpha_0-\gamma_0+2\beta_1}\dot{A}_1] = -\mu^2 e^{\alpha_0+\gamma_0+2\beta_1} A_1 \,,$$

$$\partial_1[e^{\alpha_0-\gamma_0+2\beta_1}\dot{A}_1] = -\mu^2 e^{3\alpha_0-\gamma_0+2\beta_1} A_0 \,.$$

Temporal notation

$$\rho \equiv \frac{1}{3}(\alpha + 2\beta)\,, \qquad \sigma \equiv \frac{1}{3}(\beta - \alpha)\,,$$

$$A_{\pm} = e^{-2\rho+4\sigma}(\dot{A}^2 \pm \mu^2 e^{2\gamma} A^2)\,, \qquad \bar{V} \equiv V(\psi) + 2\Lambda\,.$$

General E-W plus scalar Lagrangian

$$\mathcal{L}^{(1)} = e^{2\rho-\gamma}(\dot{\psi}^2 - 6\dot{\rho}^2 + 6\dot{\sigma}^2) + e^{3\rho-\gamma} A_- - e^{3\rho+\gamma}\,\bar{V}(\psi)$$

Constraint

$$\dot{\psi}^2 - 6\dot{\rho}^2 + 6\dot{\sigma}^2 + A_- + e^{2\gamma}\,\bar{V} = 0$$

Equations of motion

$$\ddot{A} + (\dot{\rho} + 4\dot{\sigma} - \dot{\gamma})\dot{A} + e^{2\gamma}\mu^2 A = 0\,,$$

$$4\ddot{\rho} + 6\dot{\rho}^2 - 4\dot{\rho}\dot{\gamma} - 6\dot{\sigma}^2 + \frac{1}{3}A_- + \dot{\psi}^2 - e^{2\gamma}\,\bar{V} = 0\,,$$

$$\ddot{\sigma} + 3\dot{\sigma}\dot{\rho} - \dot{\sigma}\dot{\gamma} - \frac{1}{3}A_- = 0\,.$$

$$\ddot{\psi} + (3\dot{\rho} - \dot{\gamma})\dot{\psi} + \frac{1}{2}e^{2\gamma}\,\bar{V}_\psi = 0\,,$$

Approximate (model) Lagrangian
for the E-W cosmology

$$\mathcal{L}_c = -6\dot{\alpha}^2 e^{3\alpha-\gamma} - 2\Lambda e^{3\alpha+\gamma} + \dot{A}^2 e^{\alpha-\gamma} - \mu^2 A^2 e^{\alpha+\gamma}$$

ONLY MODEL!!
Not giving exact solution for D=4

equations of motion in the LC gauge $\alpha = \gamma$

$$\dot{f} = fF \,, \qquad \dot{F} + F^2 + k = \frac{2}{3}\Lambda f^2 + \frac{1}{6}m^2 A^2$$

$$\dot{A} = B, \qquad \dot{B} = -m^2 f^2 A$$

$$-6f^2F^2 - 6kf^2 + B^2 + 2\Lambda f^4 + m^2 f^2 A^2 = 0$$

Similarly to our consideration of the static equations, we change the independent variable to $\alpha \equiv \ln f$.

Then we have two equations

$$2F^2A'' + (F^2)'A' + 2m^2e^{2\alpha}A = 0$$

$$(F^2)' = -2F^2 - 2k + \frac{4}{3}\Lambda e^{2\alpha} + \frac{1}{3}m^2A^2$$

and the constraint

$$[F^2(6 - e^{-2\alpha}A'^2) + 6k - 2\Lambda e^{2\alpha} - m^2A^2] = 0$$

that are easy to solve in the asymptotic region $\alpha \longrightarrow -\infty$

$$A = \sum_{n=0}^{\infty} A_n e^{\alpha n}\,, \qquad F^2 = e^{-2\alpha}\left[C_\infty + \sum_{n=2}^{\infty} F_n^{(2)} e^{\alpha n}\right]$$

C_∞, A_0, A_1 are arbitrary constants

$$A_1 = \pm\sqrt{6}, \quad A_2 = 0, \qquad A_4 = \frac{m^2}{4C_\infty} A_0\,;$$

$$A_3 = -\frac{1}{A_1 C_\infty}\left[\frac{1}{6} m^2 A_0^2 - k\right],$$

etc. etc. etc.

$$F_2^{(2)} = \frac{1}{6} m^2 A_0^2 - k\,, \qquad F_3^{(2)} = \frac{2}{9} m^2 A_0 A_1\,;$$

Thus we find the differential equation for $f(t)$:

$$\frac{d}{dt}(e^{\alpha}) \equiv \dot{f} = \sqrt{C_{\infty}}\,[1 + 2\psi_2 f^2 + 2\psi_3 f^3 + ...]^{\frac{1}{2}}$$

Neglecting the third term in the r.h.s.

$$f(t) \cong \sqrt{C_{\infty}}\left(\frac{1}{6}m^2 A_0^2 - k\right)^{-\frac{1}{2}} \sinh\left[\left(\frac{1}{6}m^2 A_0^2 - k\right)^{\frac{1}{2}}(t - t_0)\right]$$

a possibility of an **inflation**

The discussed **solution is not unique**

A SPECIAL SOLUTION

$$A = \sum_{n=0}^{\infty} A_n e^{2n\alpha}, \qquad F = \sum_{n=0}^{\infty} F_n e^{2n\alpha}$$

$$\dot{f} \equiv \frac{de^{\alpha}}{dt} = [F_0^2 + 2F_1 F_0^{-1} f^2(t) + \ldots]^{\frac{1}{2}}$$

In this approximation

$$f \equiv e^{\alpha} = 2e^{F_0 t}(1 - 2F_1 F_0^{-1} e^{2F_0 t})^{-1}$$

$F_1 F_0^{-1}$ strongly depends on A_0, Λ; if $k = 0$

$$F_1 F_0^{-1} = \left(\Lambda - \frac{3}{4}m^2\right)\left(m^2 A_0^2\right)^{-1}$$

3. The expansions for both cosmological solutions of Section 2.3 can be generally written as follows. Let us introduce a somewhat more convenient notation:

$$\mathcal{F}(\alpha) \equiv F^2(\alpha) \equiv C_\infty e^{-2\alpha} + \sum_{n=0}^{\infty} \mathcal{F}_n e^{-n\alpha},$$

$$\mathcal{F}_n = \mu^2 \mathcal{A}_n / [3(n+2)] - k\,\delta_{n,0} + \bar{\Lambda}\,\delta_{n,2},$$

$$\mathcal{A}_n \equiv \sum_{l=0}^{n} A_l\, A_{n-l}, \qquad \bar{\Lambda} \equiv \Lambda/3.$$

$$2\mathcal{F}A'' + \mathcal{F}'A' + 2\mu^2 e^{2\alpha} A = 0$$

$$A(\alpha) \equiv \sum_n A_n e^{n\alpha}$$

$$2C_\infty (n+1)(n+2) A_{n+2}$$

$$= -- \sum_{m=1}^{n} m(m+n)\mathcal{F}_{n-m} A_m + 2\mu^2 A_{n-2}$$

The system of recurrence relations

$$\frac{d\bar{\alpha}}{d\tau} = \left[1 + \sum_{k=1}^{\infty} f_{2k}\, e^{2k\bar{\alpha}(\tau)}\right]^{\frac{1}{2}}, \qquad \tau \equiv \mu A_0/\sqrt{6}\, t\,.$$

$$A(\alpha)/A_0 = 1 + \sum_{k=1}^{\infty} a_{2k}\, e^{2k\bar{\alpha}}$$

$$\mathcal{F}/\mathcal{F}_0 = 1 + \sum_{k=1}^{\infty} f_{2k}\, e^{2k\bar{\alpha}}$$

$$\bar{\alpha} \equiv \alpha - \ln A_0$$

Problems:

A detailed description of inflation requires adding matter and solving our equations beyond the asymptotic regions (in the positive asymptotic region there are problems definitely requiring matter).

It is easy to add some scalar matter but a realistic theory with matter is a problem.

In principle the E-W model contains some inflation mechanism (at least, is compatible with inflation: non-isotropic?).

Vecton dark matter can be produced in **strong** gravitational fields.

Quantum gravity is necessary!

Possibly, so produced dark matter can influence inflation?

Anyway, inflation and dark matter are crucial things to study and test

References: ATF: 0812.2616 v1 (E-W-E)

V. de Alfaro, ATF: 0902.4451, hep-th/0612258 (T and L)

ATF: hep-th/0605276 (SC – duality)

D. Maison, ATF: gr-qc/0210081 (expansions near horizons)

D. Garfinkle e.a. gr-qc/0305014 (numer. BH with mass. vect.)

L.Ford, Ph.R.D. 40 (1989) 967;
V.Mukhanov e.a 0802.2068v3..
M.Banados, 0801.4103v4

[1] A.Einstein, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse, 1923, pp. 32-38, 76-77, 137-140. ('Sobranie nauchnykh trudov', vol.2, papers 72-74, Moscow, 1966 (in Russian)).

[2] A. Einstein, Nature, 112, pp. 448-449; Appendix to the book: A.Eddington,'Relativitäts theorie in mathematischer Behandlung', Springer, Berlin, 1925. ('Sobranie nauchnykh trudov', vol.2, papers 75, 77, Moscow, 1966 (in Russian)).

[3] E. Schrödinger, Space - time structure, NY, 1950.

[4] W. Pauli, Relativitätstheorie (Enzykl. d. Math. Wiss., Bd.5), 1921 (Russian transl. 1947);
Theory of Relativity, NY, 1958 (English transl. with author's comments and addenda).

[5] H. Weyl, Raum-Zeit-Materie, 5-th edition, 1923 (1-st ed. 1918; English translation 1950).

[7] A.S. Eddington, Proc. Roy. Soc. A99 (1919) 742.

[8] A.S. Eddington, The mathematical theory of relativity, Camb., 1923 (German translation of the 2-nd ed. 1925).