

# Infrared-finite Observables in N=4 Super Yang-Mills Theory

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# Outline

- 1 Introduction
  - N=4 Syper Yang Mills Theory
  - Gluon scattering amplitudes
- 2 Infrared Divergences
  - Weak Coupling Case
  - Strong Coupling Case
- 3 Cancellation of IR Divergences
  - Toy model: electron-quark scattering
  - Gluon scattering in N=4 Super Yang-Mills Theory
- 4 Summary and Outlook

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# N=4 Super Yang-Mills Theory

- $\mathcal{N} = 4$  Super Yang-Mills theory is the most supersymmetric theory possible without gravity
- Field content: 1 massless gauge boson, 4 massless (Majorana) spin 1/2 fermions, 6 real (or 3 complex) massless spin 0 bosons  
All fields are in adjoint representation of the gauge group (Take  $SU(N_c)$ )
- The theory is exactly scale invariant, conformal field theory at quantum level, i.e. the  $\beta$  function identically vanishes at all orders of PT

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# AdS/CFT Correspondence

- $N_c \rightarrow \infty$  (planar limit) is expected to be integrable and solvable
- Maldacena's conjecture: Planar Limit of N=4 SYM at strong coupling is dual to weakly coupled type II b supergravity in 10 dimensional  $AdS_5 * S_5$  space.
- How might PT series be organized to produce simple strong coupling result?
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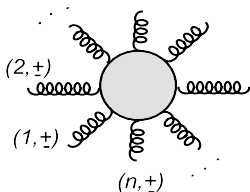
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# Gluon scattering amplitudes



All outgoing gluons with helicity + or -  
on mass shell

In the leading  $N_c$  order (planar limit)

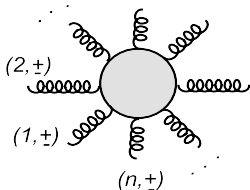
- Colour decomposition of amplitudes in N=4 SYM theory for  $N_c \rightarrow \infty$

$$\mathcal{A}_n^{(l)} = g^{n-2} \left( \frac{g^2 N_c}{16\pi^2} \right)^l \sum_{perm} \text{Tr}(T^{a_{\sigma(1)}}, \dots, T^{a_{\sigma(n)}}) A_n^{(l)}(\mathbf{a}_{\sigma(1)}, \dots, \mathbf{a}_{\sigma(n)}),$$

where  $\mathcal{A}_n$  - physical amplitude,  $A_n$  - partial amplitude,  $a_i$  - is color index of  $i$ -th external "gluon"

- **Maximal helicity violating (MHV)** amplitudes (two negative helicities and the rest positive) have observed a simple structure on tree level (and even in loops) and one can **speculate** that this is the consequence of SUSY

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# Perturbation theory

- Bern, Dixon & Smirnov's conjecture:  $M_n^{(L)}(\varepsilon) \equiv A_n^{(L)}/A_n^{(0)}$

$$\mathcal{M}_n \equiv 1 + \sum_{L=1}^{\infty} \left( \frac{g^2 N_c}{16\pi^2} \right)^L M_n^{(L)}(\varepsilon) = \exp \left[ \sum_{l=1}^{\infty} \left( \frac{g^2 N_c}{16\pi^2} \right)^l \left( f^{(l)}(\varepsilon) M_n^{(1)}(l\varepsilon) + C^{(l)} + E_n^{(l)}(\varepsilon) \right) \right]$$

$$f^{(l)}(\varepsilon) = f_0^{(l)}(\varepsilon) + \varepsilon f_1^{(l)}(\varepsilon) + \varepsilon^2 f_2^{(l)}(\varepsilon)$$

$$\begin{aligned} \mathcal{M}_n(\varepsilon) = & \exp \left[ -\frac{1}{8} \sum_{l=1}^{\infty} \left( \frac{g^2 N_c}{16\pi^2} \right)^l \left( \frac{\gamma_K^{(l)}}{(l\varepsilon)^2} + \frac{2G_0^{(l)}}{l\varepsilon} \right) \sum_{i=1}^n \left( \frac{\mu^2}{-s_{i,i+1}} \right)^{l\varepsilon} \right. \\ & \left. + \frac{1}{4} \sum_{l=1}^{\infty} \left( \frac{g^2 N_c}{16\pi^2} \right)^l \gamma_K^{(l)} F_n^{(1)}(0) \right] \end{aligned}$$

$$F_4^{(1)}(0) = \frac{1}{2} \log^2 \left( \frac{-t}{-s} \right) + 4\zeta_2$$

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# Cusp anomalous dimension

- **Cusp anomalous dimension** appears in RG eq. for the expectation value of a Wilson line with a cusp

Loop expansion 
$$\gamma_K = \sum_{l=1}^{\infty} \left( \frac{g^2 N_c}{16\pi^2} \right)^l \gamma_K^{(l)}$$

$$\gamma_K^{(1)} = 8, \quad \gamma_K^{(2)} = -16\zeta_2, \quad \gamma_K^{(3)} = 176\zeta_4, \dots$$

- It also controls the large spin limit of anomalous dimension of leading-twist operators

$$O_j \equiv \bar{q}(\gamma_+ \mathcal{D}^+)^j q$$

$$\gamma_j = \frac{1}{2} \gamma_K(\alpha) \log j + \mathcal{O}(j^0), \quad j \rightarrow \infty$$

- and large  $x$  limit of the DGLAP kernel for p.d.f.

$$P_{gg} = \frac{1}{2} \frac{\gamma_K(\alpha)}{(1-x)_+} + \dots, \quad x \rightarrow 1, \quad \gamma(j) = - \int_0^1 dx x^{j-1} P(x)$$

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# Strong coupling expansion/AdS

- Classical solution (Alday & Maldacena) for the scattering amplitude

$$\mathcal{M}_4(\varepsilon) = \exp[-S_{cl}^E]$$

$$\begin{aligned} \bullet \quad S_{cl}^E &= \frac{1}{\varepsilon^2} \frac{\sqrt{g^2 N_c}}{\pi} \left[ \left( \frac{\mu_{IR}^2}{-s} \right)^{\varepsilon/2} + \left( \frac{\mu_{IR}^2}{-t} \right)^{\varepsilon/2} \right] \\ &+ \frac{1}{\varepsilon} \frac{\sqrt{g^2 N_c}}{2\pi} (1-\log 2) \left[ \left( \frac{\mu_{IR}^2}{-s} \right)^{\varepsilon/2} + \left( \frac{\mu_{IR}^2}{-t} \right)^{\varepsilon/2} \right] - \frac{\sqrt{g^2 N_c}}{8\pi} \left[ \log^2\left(\frac{s}{t}\right) + c \right] + \mathcal{O}(\varepsilon) \end{aligned}$$

- $\gamma_K(g^2) \sim \frac{\sqrt{g^2 N_c}}{\pi}$ ,  $G_0(g^2) \sim \sqrt{g^2 N_c} \frac{1-\log 2}{2\pi}$ , for  $g^2 N_c \rightarrow \infty$
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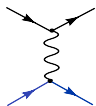
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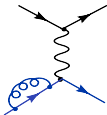
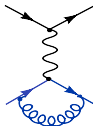
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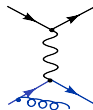
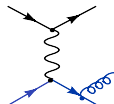
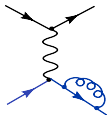
# Electron-quark scattering



Born



Virtual



Real Emission

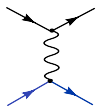
- Virtual Correction

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{2E^2} \left(\frac{s^2 + u^2 - \epsilon t^2}{t^2}\right) \left(\frac{\mu^2}{s}\right)^\epsilon$$

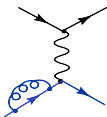
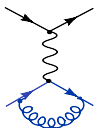
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$$\left(\frac{d\sigma}{d\Omega}\right)_{virt} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[ 1 - 2C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{-t}\right)^\epsilon \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8\right) \right]$$

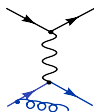
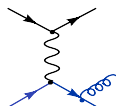
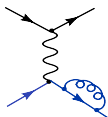
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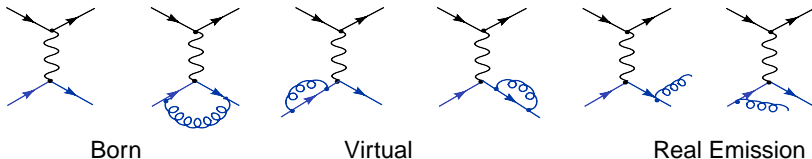
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## Real Emission

$$\left(\frac{d\sigma}{d\Omega}\right)_{real} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[ 2C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{-t}\right)^\epsilon \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8\right) \right] \\ + C_F \frac{\alpha^2}{E^2} \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{-t}\right)^\epsilon \left(\frac{f_1}{\epsilon} + f_2\right),$$

- where the functions  $f_1$  and  $f_2$  in the c.m. frame are ( $c = \cos \theta$ )

$$f_1 = -2 \frac{(c^3 + 5c^2 - 3c + 5) \log\left(\frac{1-c}{2}\right) + (1-c^2)(c-11)/4}{(1-c)(1+c)^2}$$

$$f_2 = -\frac{1}{(1-c^2)^2} \left[ (1-c)(c^3 + 5c^2 - 3c + 5) \log^2\left(\frac{1-c}{2}\right) \right. \\ \left. + \frac{1}{2}(1-c)(3c^3 + 15c^2 + 77c - 31) \log\left(\frac{1-c}{2}\right) + \frac{1}{2}(1-c^2)(5c^2 - 42c - 23) \right. \\ \left. + (1+c)^2(c^2 + 5c + 3)\pi^2 - 12(9c^2 + 2c + 5) Li_2\left(\frac{1+c}{2}\right) \right].$$

## Real Emission



$$\left(\frac{d\sigma}{d\Omega}\right)_{real} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[ 2C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{-t}\right)^\epsilon \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8\right) \right] \\ + C_F \frac{\alpha^2}{E^2} \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{-t}\right)^\epsilon \left(\frac{f_1}{\epsilon} + f_2\right),$$

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$$f_1 = -2 \frac{(c^3 + 5c^2 - 3c + 5) \log\left(\frac{1-c}{2}\right) + (1-c^2)(c-11)/4}{(1-c)(1+c)^2}$$

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## Real Emission



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## Initial state splitting



$$\left(\frac{d\sigma}{d\Omega}\right)_{split} = \frac{1}{\epsilon} \frac{\alpha_s}{2\pi} \int_0^1 dz \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon P_{qq}(z) \frac{d\sigma_0}{d\Omega}(pz)$$



$$\left(\frac{d\sigma}{d\Omega}\right)_{split} = C_F \frac{\alpha^2}{2E^2} \frac{\alpha_s}{2\pi} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{-t}\right)^\epsilon \left(-\frac{f_1}{\epsilon} + f_3\right),$$

- where for  $Q_f^2 = \hat{t}$

$$f_3 = -\frac{1}{(1-c)^2(1+c)^2} \left[ 2(1-c)(c^3 + c^2 - 33c + 7) \log\left(\frac{1-c}{2}\right) + 12(9c^2 + 2c + 5)L_2\left(\frac{1+c}{2}\right) - (1+c)^2(c^2 + 5c + 3)\pi^2 - \frac{1}{2}(1-c)(1+c)(11c^2 - 19) \right].$$

## Initial state splitting



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## Initial state splitting



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$$\left(\frac{d\sigma}{d\Omega}\right)_{split} = C_F \frac{\alpha^2}{2E^2} \frac{\alpha_s}{2\pi} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{-t}\right)^\epsilon \left(-\frac{f_1}{\epsilon} + f_3\right),$$

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## Initial state splitting



$$\left(\frac{d\sigma}{d\Omega}\right)_{split} = \frac{1}{\epsilon} \frac{\alpha_s}{2\pi} \int_0^1 dz \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon P_{qq}(z) \frac{d\sigma_0}{d\Omega}(pz)$$



$$\left(\frac{d\sigma}{d\Omega}\right)_{split} = C_F \frac{\alpha^2}{2E^2} \frac{\alpha_s}{2\pi} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{-t}\right)^\epsilon \left(-\frac{f_1}{\epsilon} + f_3\right),$$

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$$\begin{aligned} f_3 = & -\frac{1}{(1-c)^2(1+c)^2} \left[ 2(1-c)(c^3 + c^2 - 33c + 7) \log\left(\frac{1-c}{2}\right) \right. \\ & + 12(9c^2 + 2c + 5)Li_2\left(\frac{1+c}{2}\right) - (1+c)^2(c^2 + 5c + 3)\pi^2 \\ & \left. - \frac{1}{2}(1-c)(1+c)(11c^2 - 19) \right]. \end{aligned}$$



# Infrared-free observable = inclusive cross-section

$$\begin{aligned}
 \left(\frac{d\sigma}{d\Omega}\right)_{\text{observ}} &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{virt}}^{2\rightarrow 2} + \left(\frac{d\sigma}{d\Omega}\right)_{\text{real}}^{2\rightarrow 3} + \left(\frac{d\sigma}{d\Omega}\right)_{\text{split}}^{2\rightarrow 2} \\
 &= \frac{\alpha^2}{2E^2} \left\{ \frac{c^2 + 2c + 5}{(1-c)^2} \right. \\
 &\quad - \frac{\alpha_s}{2\pi} \frac{C_F}{(1-c)(1+c)^2} \left[ (c^3 + 5c^2 - 3c + 5) \log^2 \frac{1-c}{2} \right. \\
 &\quad \left. \left. + \frac{1}{2}(7c^3 + 19c^2 - 55c - 3) \log \frac{1-c}{2} - (1+c)(3c^2 + 21c + 2) \right] \right\}
 \end{aligned}$$

# Outline

- 1 Introduction
  - N=4 Syper Yang Mills Theory
  - Gluon scattering amplitudes
- 2 Infrared Divergences
  - Weak Coupling Case
  - Strong Coupling Case
- 3 Cancellation of IR Divergences
  - Toy model: electron-quark scattering
  - Gluon scattering in N=4 Super Yang-Mills Theory
- 4 Summary and Outlook

## From partial amplitudes to cross-sections

To obtain the cross sections from partial amplitudes one have to compute the square of them. In the the planar limit it is just:

$$\Phi_n(p_1^\pm, \dots, p_n^\pm) = g^{2n-4} \left( \frac{g^2 N_c}{16\pi^2} \right)^{2l} \sum_{\text{colors}} \mathcal{A}_n^{(l)} \mathcal{A}_n^{(l)*} =$$

$$2g^{2n-4} N_c^{n-2} (N_c^2 - 1) \left( \frac{g^2 N_c}{16\pi^2} \right)^{2l} \sum_{\text{perm}} |\mathcal{A}_n^{(l)}(\mathbf{a}_{\sigma(1)}, \dots, \mathbf{a}_{\sigma(n-1)}, \mathbf{a}_n)|^2$$

Then the cross-section is

$$d\sigma_n(p_{in}) = \Phi_n(p_1^\pm, \dots, p_n^\pm) d\phi_k,$$

where  $d\phi_k$  is the phase space of the outgoing particles:

$$d\phi_k \sim \delta^D(p_{in} - p_{fin}) \mathcal{S}_n \prod_k \delta^+(p_k^2) d^D p_k,$$

where  $\mathcal{S}_n$  - is so called measurement function and integration goes over  $D = 4 - 2\epsilon$  dimensions.

## Virtual Correction (MHV)



$$\left(\frac{d\sigma}{d\Omega}\right)_0^{-\rightarrow\rightarrow\rightarrow} = \frac{\alpha^2 N_c^2}{8E^2} \frac{s^4 (s^2 + t^2 + u^2)}{s^2 t^2 u^2} \left(\frac{\mu^2}{s}\right)^\epsilon = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \frac{3 + c^2}{(1 - c^2)^2}$$



$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{virt}^{-\rightarrow\rightarrow\rightarrow} &= \frac{\alpha^2 N_c^2}{8E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left\{ \frac{\alpha N_c}{2\pi} \frac{s^4}{s^2 t^2 u^2} \left[ -\frac{8}{\epsilon^2} \left( \left(\frac{\mu^2}{-t}\right)^\epsilon + \left(\frac{\mu^2}{-u}\right)^\epsilon \right) s^2 \right. \right. \\ &\quad \left. \left. + \left( \left(\frac{\mu^2}{s}\right)^\epsilon + \left(\frac{\mu^2}{-t}\right)^\epsilon \right) u^2 + \left( \left(\frac{\mu^2}{s}\right)^\epsilon + \left(\frac{\mu^2}{-u}\right)^\epsilon \right) t^2 \right] \right. \\ &\quad \left. + \frac{16}{3} \pi^2 (s^2 + t^2 + u^2) + 4(u^2 \log^2\left(\frac{-s}{t}\right) + t^2 \log^2\left(\frac{-s}{u}\right) + s^2 \log^2\left(\frac{t}{u}\right)) \right\} \end{aligned}$$



$$\begin{aligned} &= \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left\{ \frac{\alpha N_c}{2\pi} \left[ -\frac{16}{\epsilon^2} \frac{3 + c^2}{(1 - c^2)^2} + \frac{4}{\epsilon} \left( \frac{5 + 2c + c^2}{(1 - c^2)^2} \log\left(\frac{1 - c}{2}\right) \right. \right. \right. \\ &\quad \left. \left. + (c \leftrightarrow -c) \right) + \frac{16(3 + c^2)\pi^2}{3(1 - c^2)^2} - \frac{16}{(1 - c^2)^2} \log\left(\frac{1 - c}{2}\right) \log\left(\frac{1 + c}{2}\right) \right\} \end{aligned}$$

## Virtual Correction (MHV)

$$\left(\frac{d\sigma}{d\Omega}\right)_0^{-+} = \frac{\alpha^2 N_c^2}{8E^2} \frac{s^4 (s^2 + t^2 + u^2)}{s^2 t^2 u^2} \left(\frac{\mu^2}{s}\right)^\epsilon = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \frac{3 + c^2}{(1 - c^2)^2}$$

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{virt}^{-+} &= \frac{\alpha^2 N_c^2}{8E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left\{ \frac{\alpha N_c}{2\pi} \frac{s^4}{s^2 t^2 u^2} \left[ -\frac{8}{\epsilon^2} \left( \left(\frac{\mu^2}{-t}\right)^\epsilon + \left(\frac{\mu^2}{-u}\right)^\epsilon \right) s^2 \right. \right. \\ &\quad \left. \left. + \left( \left(\frac{\mu^2}{s}\right)^\epsilon + \left(\frac{\mu^2}{-t}\right)^\epsilon \right) u^2 + \left( \left(\frac{\mu^2}{s}\right)^\epsilon + \left(\frac{\mu^2}{-u}\right)^\epsilon \right) t^2 \right] \right. \\ &\quad \left. + \frac{16}{3} \pi^2 (s^2 + t^2 + u^2) + 4(u^2 \log^2\left(\frac{-s}{t}\right) + t^2 \log^2\left(\frac{-s}{u}\right) + s^2 \log^2\left(\frac{t}{u}\right)) \right\} \end{aligned}$$

$$\begin{aligned} &= \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left\{ \frac{\alpha N_c}{2\pi} \left[ -\frac{16}{\epsilon^2} \frac{3 + c^2}{(1 - c^2)^2} + \frac{4}{\epsilon} \left( \frac{5 + 2c + c^2}{(1 - c^2)^2} \log\left(\frac{1 - c}{2}\right) \right. \right. \right. \\ &\quad \left. \left. + (c \leftrightarrow -c) \right) + \frac{16(3 + c^2)\pi^2}{3(1 - c^2)^2} - \frac{16}{(1 - c^2)^2} \log\left(\frac{1 - c}{2}\right) \log\left(\frac{1 + c}{2}\right) \right\} \end{aligned}$$

## Virtual Correction (MHV)



$$\left(\frac{d\sigma}{d\Omega}\right)_0^{-++} = \frac{\alpha^2 N_c^2}{8E^2} \frac{s^4 (s^2 + t^2 + u^2)}{s^2 t^2 u^2} \left(\frac{\mu^2}{s}\right)^\epsilon = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \frac{3 + c^2}{(1 - c^2)^2}$$



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$$\begin{aligned} = & \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left\{ \frac{\alpha N_c}{2\pi} \left[ -\frac{16}{\epsilon^2} \frac{3 + c^2}{(1 - c^2)^2} + \frac{4}{\epsilon} \left( \frac{5 + 2c + c^2}{(1 - c^2)^2} \log\left(\frac{1 - c}{2}\right) \right. \right. \right. \\ & \left. \left. + (c \leftrightarrow -c) \right) + \frac{16(3 + c^2)\pi^2}{3(1 - c^2)^2} - \frac{16}{(1 - c^2)^2} \log\left(\frac{1 - c}{2}\right) \log\left(\frac{1 + c}{2}\right) \right\} \end{aligned}$$

## Virtual Correction (MHV)



$$\left(\frac{d\sigma}{d\Omega}\right)_0^{-++} = \frac{\alpha^2 N_c^2}{8E^2} \frac{s^4 (s^2 + t^2 + u^2)}{s^2 t^2 u^2} \left(\frac{\mu^2}{s}\right)^\epsilon = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \frac{3 + c^2}{(1 - c^2)^2}$$



$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{virt}^{-++} &= \frac{\alpha^2 N_c^2}{8E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left\{ \frac{\alpha N_c}{2\pi} \frac{s^4}{s^2 t^2 u^2} \left[ -\frac{8}{\epsilon^2} \left( \left(\frac{\mu^2}{-t}\right)^\epsilon + \left(\frac{\mu^2}{-u}\right)^\epsilon \right) s^2 \right. \right. \\ &\quad \left. \left. + \left( \left(\frac{\mu^2}{s}\right)^\epsilon + \left(\frac{\mu^2}{-t}\right)^\epsilon \right) u^2 + \left( \left(\frac{\mu^2}{s}\right)^\epsilon + \left(\frac{\mu^2}{-u}\right)^\epsilon \right) t^2 \right) \right. \\ &\quad \left. \left. + \frac{16}{3} \pi^2 (s^2 + t^2 + u^2) + 4(u^2 \log^2(\frac{-s}{t}) + t^2 \log^2(\frac{-s}{u}) + s^2 \log^2(\frac{t}{u})) \right] \right\} \end{aligned}$$



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## Real Emission (MHV)

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{Born}^{(--++++)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon^2} \frac{8(3+c^2)}{(1-c^2)^2} \right. \\
 + \frac{1}{\epsilon} \left[ \frac{2}{(1+c)^2} \log\left(\frac{1-c}{2}\right) + \frac{2}{(1-c)^2} \log\left(\frac{1+c}{2}\right) + \frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2} \right. \\
 \left. \left. + \frac{12(3+c^2)(\log(1-\delta) - \log(\delta))}{(1-c^2)^2} \right] + \text{Finite part} \right\}$$

- ▶ What is  $\delta$ ? One has a singularity as  $\delta \rightarrow 1$ .
- ▶ This is the cut-off in external momenta of the scattered gluon:  
 $|\vec{p}| \leq \frac{E}{2}(1-\delta)$ .
- ▶ This allows one to distinguish identical final gluons, so that the gluon scattered at angle  $\theta$  has non-zero momentum



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$$\begin{aligned} \left(\frac{d\sigma}{d\Omega_{14}}\right)_{Born}^{(--++++)} &= \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon^2} \frac{8(3+c^2)}{(1-c^2)^2} \right. \\ &+ \frac{1}{\epsilon} \left[ \frac{2}{(1+c)^2} \log\left(\frac{1-c}{2}\right) + \frac{2}{(1-c)^2} \log\left(\frac{1+c}{2}\right) + \frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2} \right. \\ &\left. \left. + \frac{12(3+c^2)(\log(1-\delta) - \log(\delta))}{(1-c^2)^2} \right] + \text{Finite part} \right\} \end{aligned}$$

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 &+ \frac{1}{\epsilon} \left[ \frac{2}{(1+c)^2} \log\left(\frac{1-c}{2}\right) + \frac{2}{(1-c)^2} \log\left(\frac{1+c}{2}\right) + \frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2} \right. \\
 &\left. \left. + \frac{12(3+c^2)(\log(1-\delta) - \log(\delta))}{(1-c^2)^2} \right] + \text{Finite part} \right\}
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- ▶ This allows one to distinguish identical final gluons, so that the gluon scattered at angle  $\theta$  has non-zero momentum

## Real Emission (MHV)



$$\begin{aligned} \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Born}^{(---+++)} &= \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^{2\varepsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\varepsilon^2} \frac{8(3+c^2)}{(1-c^2)^2} \right. \\ &+ \frac{1}{\varepsilon} \left[ \frac{2}{(1+c)^2} \log\left(\frac{1-c}{2}\right) + \frac{2}{(1-c)^2} \log\left(\frac{1+c}{2}\right) + \frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2} \right. \\ &\left. \left. + \frac{12(3+c^2)(\log(1-\delta) - \log(\delta))}{(1-c^2)^2} \right] + \text{Finite part} \right\} \end{aligned}$$

- ▶ What is  $\delta$ ? One has a singularity as  $\delta \rightarrow 1$ .
- ▶ This is the cut-off in external momenta of the scattered gluon:  
 $|\vec{p}| \leq \frac{E}{2}(1-\delta)$ .
- ▶ This allows one to distinguish identical final gluons, so that the gluon scattered at angle  $\theta$  has non-zero momentum

## Real Emission (Anti MHV)

$$\begin{aligned}
 & \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Born}^{(---+-)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon^2} \frac{8(3+c^2)}{(1-c^2)^2} \right. \\
 & + \frac{2}{\epsilon} \left[ \frac{64}{3(1-c^2)^3} + \delta \frac{(6\delta^2 - 3\delta + 30)c^2 + (10\delta^2 - 57\delta + 66)}{3(1-c^2)^2} - \frac{6(c^2+3)\log\delta}{(1-c^2)^2} + \right. \\
 & \left. \left( \frac{3c^2 - 24c + 85}{(1-c)(1+c)^3} \log \frac{1-c}{2} - \frac{4(c^2 - 6c + 21)}{(1-c)(1+c)^3} \log \frac{1+\delta - (1-\delta)c}{2} \right. \right. \\
 & \left. \left. + \frac{16\delta(5-c)}{(1-c^2)^2(1+\delta - (1-\delta)c)} - \frac{32(1-c)}{3(1+c)^3(1+\delta - (1-\delta)c)^3} + (c \leftrightarrow -c) \right) \right] \\
 & \left. + \text{Finite part} \right\}
 \end{aligned}$$

## Real Emission (Anti MHV)



$$\begin{aligned}
 \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Born}^{(---+-)} &= \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon^2} \frac{8(3+c^2)}{(1-c^2)^2} \right. \\
 + \frac{2}{\epsilon} &\left[ \frac{64}{3(1-c^2)^3} + \delta \frac{(6\delta^2 - 3\delta + 30)c^2 + (10\delta^2 - 57\delta + 66)}{3(1-c^2)^2} - \frac{6(c^2+3)\log\delta}{(1-c^2)^2} + \right. \\
 &\left( \frac{3c^2 - 24c + 85}{(1-c)(1+c)^3} \log \frac{1-c}{2} - \frac{4(c^2 - 6c + 21)}{(1-c)(1+c)^3} \log \frac{1+\delta - (1-\delta)c}{2} \right. \\
 &\left. + \frac{16\delta(5-c)}{(1-c^2)^2(1+\delta - (1-\delta)c)} - \frac{32(1-c)}{3(1+c)^3(1+\delta - (1-\delta)c)^3} + (c \leftrightarrow -c) \right] \\
 &\left. + \text{Finite part} \right\}
 \end{aligned}$$

## Real Emission (Matter)( $\delta = 1$ )

- Fermions

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{Real}^{(--+\bar{q}q)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{4}{\epsilon} \left[ \frac{32(79 - 25c^2)}{3(1 - c^2)^2} + \frac{64(3 - c)^2}{(1 - c)(1 + c)^3} \log\left(\frac{1 - c}{2}\right) + \frac{64(3 + c)^2}{(1 - c)^3(1 + c)} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

- Sfermions

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{Real}^{(--+\bar{q}\bar{q})} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{128(10 + 7c^2)}{(1 - c^2)^2} - \frac{192(5 - c)}{(1 + c)^3} \log\left(\frac{1 - c}{2}\right) - \frac{192(5 + c)}{(1 - c)^3} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

## Real Emission (Matter)( $\delta = 1$ )

### ● Fermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(--+\bar{q}q)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{4}{\epsilon} \left[ \frac{32(79 - 25c^2)}{3(1 - c^2)^2} + \frac{64(3 - c)^2}{(1 - c)(1 + c)^3} \log\left(\frac{1 - c}{2}\right) + \frac{64(3 + c)^2}{(1 - c)^3(1 + c)} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

### ● Sfermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(--+\bar{q}\bar{q})} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{128(10 + 7c^2)}{(1 - c^2)^2} - \frac{192(5 - c)}{(1 + c)^3} \log\left(\frac{1 - c}{2}\right) - \frac{192(5 + c)}{(1 - c)^3} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$



## Real Emission (Matter)( $\delta = 1$ )

- Fermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(--+\bar{q}q)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{4}{\epsilon} \left[ \frac{32(79 - 25c^2)}{3(1 - c^2)^2} + \frac{64(3 - c)^2}{(1 - c)(1 + c)^3} \log\left(\frac{1 - c}{2}\right) + \frac{64(3 + c)^2}{(1 - c)^3(1 + c)} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

- Sfermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(--+\bar{q}\bar{q})} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^{2\epsilon} \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{128(10 + 7c^2)}{(1 - c^2)^2} - \frac{192(5 - c)}{(1 + c)^3} \log\left(\frac{1 - c}{2}\right) - \frac{192(5 + c)}{(1 - c)^3} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

## Initial and final state splitting (MHV)

### ● Initial

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{InSplit}^{(--++++)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon$$

$$\frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{4(c^2+3)}{(1-c^2)^2} \left( \log \frac{1-c}{2} + \log \frac{1+c}{2} \right) - \frac{8(c^2+3)}{(1-c^2)^2} \log \frac{1-\delta}{\delta} \right. \right.$$

$$\left. \left. - \frac{16\delta(2\delta-3)}{(1-\delta^2)(1-c^2)^2} \right] + \text{Finite part} \right\}$$

### ● Final

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{FnSplit}^{(--++++)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon$$

$$\frac{\alpha N_c}{2\pi} \left\{ -\frac{1}{\epsilon} \frac{4(c^2+3)}{(1-c^2)^2} \log \frac{1-\delta}{\delta} + \text{Finite part} \right\}$$

## Initial and final state splitting (MHV)

### • Initial

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{InSplit}^{(--++++)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon$$

$$\frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{4(c^2+3)}{(1-c^2)^2} \left( \log \frac{1-c}{2} + \log \frac{1+c}{2} \right) - \frac{8(c^2+3)}{(1-c^2)^2} \log \frac{1-\delta}{\delta} \right. \right.$$

$$\left. \left. - \frac{16\delta(2\delta-3)}{(1-\delta^2)(1-c^2)^2} \right] + \text{Finite part} \right\}$$

### • Final

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{FnSplit}^{(--++++)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon$$

$$\frac{\alpha N_c}{2\pi} \left\{ -\frac{1}{\epsilon} \frac{4(c^2+3)}{(1-c^2)^2} \log \frac{1-\delta}{\delta} + \text{Finite part} \right\}$$

## Initial and final state splitting (MHV)

- Initial

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{InSplit}^{(--++++)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon$$

$$\frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{4(c^2+3)}{(1-c^2)^2} \left( \log \frac{1-c}{2} + \log \frac{1+c}{2} \right) - \frac{8(c^2+3)}{(1-c^2)^2} \log \frac{1-\delta}{\delta} \right. \right.$$

$$\left. \left. - \frac{16\delta(2\delta-3)}{(1-\delta^2)(1-c^2)^2} \right] + \text{Finite part} \right\}$$

- Final

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{FnSplit}^{(--++++)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon$$

$$\frac{\alpha N_c}{2\pi} \left\{ -\frac{1}{\epsilon} \frac{4(c^2+3)}{(1-c^2)^2} \log \frac{1-\delta}{\delta} + \text{Finite part} \right\}$$

## Initial and final state splitting (Anti MHV)

### ● Initial

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{InSplit}^{(--++--)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ \left( \frac{4(c^3 - 15c^2 + 51c - 45)}{(1-c)^2(1+c)^3} \log \frac{1-c}{2} \right. \right. \right. \\ \left. \left. - \frac{16(c^2 - 3c + 3)}{(1-c)^2(1+c)^3} \log \frac{1+\delta - c(1-\delta)}{2} + \frac{8(c^2 + 3)}{(1-c^2)^2} \log \delta + (c \leftrightarrow -c) \right) \right. \\ \left. - \frac{4\delta}{3(1-c^2)^2((1+\delta)^2 - c^2(1-\delta)^2)^3} \left( c^8(1-\delta)^6(2\delta^2 + 3\delta + 6) - 4c^6(1-\delta)^4(\delta^4 + 10\delta^3 \right. \right. \\ \left. \left. - 23\delta^2 + 114\delta - 33) + 2c^4(1-\delta)^2(39\delta^5 - 102\delta^4 + 86\delta^3 + 658\delta^2 + 183\delta - 312) \right. \right. \\ \left. \left. + 4c^2(\delta^8 - 12\delta^7 + 39\delta^6 + 216\delta^5 - 42\delta^4 - 720\delta^3 - 421\delta^2 + 300\delta + 208) \right. \right. \\ \left. \left. - (1+\delta)^3(2\delta^5 - 9\delta^4 + 63\delta^3 + 455\delta^2 + 579\delta + 198) \right) \right] + \text{Finite part} \left. \right\}$$

### ● Final

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{FnSplit}^{(--++--)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \frac{4(c^2 + 3)}{(1-c^2)^2} \left[ \log \delta - \frac{\delta}{3}(2\delta^2 - 9\delta + 18) \right] + \text{F.p.} \right\}$$

## Initial and final state splitting (Anti MHV)

### ● Initial

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{InSplit}^{(--+--+)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ \left( \frac{4(c^3 - 15c^2 + 51c - 45)}{(1-c)^2(1+c)^3} \log \frac{1-c}{2} \right. \right. \right. \\ \left. \left. - \frac{16(c^2 - 3c + 3)}{(1-c)^2(1+c)^3} \log \frac{1+\delta - c(1-\delta)}{2} + \frac{8(c^2 + 3)}{(1-c^2)^2} \log \delta + (c \leftrightarrow -c) \right) \right. \\ \left. - \frac{4\delta}{3(1-c^2)^2((1+\delta)^2 - c^2(1-\delta)^2)^3} \left( c^8(1-\delta)^6(2\delta^2 + 3\delta + 6) - 4c^6(1-\delta)^4(\delta^4 + 10\delta^3 \right. \right. \\ \left. \left. - 23\delta^2 + 114\delta - 33) + 2c^4(1-\delta)^2(39\delta^5 - 102\delta^4 + 86\delta^3 + 658\delta^2 + 183\delta - 312) \right. \right. \\ \left. \left. + 4c^2(\delta^8 - 12\delta^7 + 39\delta^6 + 216\delta^5 - 42\delta^4 - 720\delta^3 - 421\delta^2 + 300\delta + 208) \right. \right. \\ \left. \left. - (1+\delta)^3(2\delta^5 - 9\delta^4 + 63\delta^3 + 455\delta^2 + 579\delta + 198) \right) \right] + \text{Finite part} \left. \right\}$$

### ● Final

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{FnSplit}^{(--+--+)} = \frac{\alpha^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \frac{4(c^2 + 3)}{(1-c^2)^2} \left[ \log \delta - \frac{\delta}{3}(2\delta^2 - 9\delta + 18) \right] + \text{F.p.} \right\}$$

## Initial and final state splitting (Anti MHV)

### • Initial

$$\begin{aligned}
 \left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(- - + + -)} &= \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ \left( \frac{4(c^3 - 15c^2 + 51c - 45)}{(1-c)^2(1+c)^3} \log \frac{1-c}{2} \right. \right. \right. \\
 &- \frac{16(c^2 - 3c + 3)}{(1-c)^2(1+c)^3} \log \frac{1+\delta - c(1-\delta)}{2} + \frac{8(c^2 + 3)}{(1-c^2)^2} \log \delta + (c \leftrightarrow -c) \\
 &- \frac{4\delta}{3(1-c^2)^2((1+\delta)^2 - c^2(1-\delta)^2)^3} \left( c^8(1-\delta)^6(2\delta^2 + 3\delta + 6) - 4c^6(1-\delta)^4(\delta^4 + 10\delta^3 \right. \\
 &- 23\delta^2 + 114\delta - 33) + 2c^4(1-\delta)^2(39\delta^5 - 102\delta^4 + 86\delta^3 + 658\delta^2 + 183\delta - 312) \\
 &+ 4c^2(\delta^8 - 12\delta^7 + 39\delta^6 + 216\delta^5 - 42\delta^4 - 720\delta^3 - 421\delta^2 + 300\delta + 208) \\
 &\left. \left. \left. - (1+\delta)^3(2\delta^5 - 9\delta^4 + 63\delta^3 + 455\delta^2 + 579\delta + 198) \right) \right] + \text{Finite part} \right\}
 \end{aligned}$$

### • Final

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{FnSplit}^{(- - + + -)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \frac{4(c^2 + 3)}{(1-c^2)^2} \left[ \log \delta - \frac{\delta}{3} (2\delta^2 - 9\delta + 18) \right] + \text{F.p.} \right\}$$

## Initial state splitting (Matter) ( $\delta = 1$ )

### ● Fermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(---\bar{q}q)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{4}{\epsilon} \left[ \frac{32(79 - 25c^2)}{3(1 - c^2)^2} + \frac{64(3 - c)^2}{(1 - c)(1 + c)^3} \log\left(\frac{1 - c}{2}\right) + \frac{64(3 + c)^2}{(1 - c)^3(1 + c)} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

### ● Sfermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(---\bar{q}\bar{q})} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{128(10 + 7c^2)}{(1 - c^2)^2} - \frac{192(5 - c)}{(1 + c)^3} \log\left(\frac{1 - c}{2}\right) - \frac{192(5 + c)}{(1 - c)^3} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$



## Initial state splitting (Matter) ( $\delta = 1$ )

### Fermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(--+\bar{q}q)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{4}{\epsilon} \left[ \frac{32(79 - 25c^2)}{3(1 - c^2)^2} + \frac{64(3 - c)^2}{(1 - c)(1 + c)^3} \log\left(\frac{1 - c}{2}\right) + \frac{64(3 + c)^2}{(1 - c)^3(1 + c)} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

### Sfermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(--+\bar{q}\bar{q})} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{128(10 + 7c^2)}{(1 - c^2)^2} - \frac{192(5 - c)}{(1 + c)^3} \log\left(\frac{1 - c}{2}\right) - \frac{192(5 + c)}{(1 - c)^3} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

## Initial state splitting (Matter) ( $\delta = 1$ )

### • Fermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(--+\bar{q}q)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{4}{\epsilon} \left[ \frac{32(79 - 25c^2)}{3(1 - c^2)^2} + \frac{64(3 - c)^2}{(1 - c)(1 + c)^3} \log\left(\frac{1 - c}{2}\right) + \frac{64(3 + c)^2}{(1 - c)^3(1 + c)} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

### • Sfermions

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(--+\bar{q}\bar{q})} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha N_c}{2\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{128(10 + 7c^2)}{(1 - c^2)^2} - \frac{192(5 - c)}{(1 + c)^3} \log\left(\frac{1 - c}{2}\right) - \frac{192(5 + c)}{(1 - c)^3} \log\left(\frac{1 + c}{2}\right) \right] + \text{Finite part} \right\}$$

## Infrared-free sets (for any arbitrary $\delta$ )

$$A^{MHV} = \frac{1}{2} \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Virt}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{FnSplit}^{(----)}$$

$$B^{AntiMHV} = \frac{1}{2} \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Virt}^{(---+)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(---+)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(---+)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{FnSplit}^{(---+)}$$

$$C^{Matter} = \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(---+\bar{q}q,\bar{q}\bar{q})} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(---+\bar{q}q,\bar{q}\bar{q})} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{FnSplit}^{(---+\bar{q}q,\bar{q}\bar{q})}$$

## Infrared-free sets (for any arbitrary $\delta$ )

$$A^{MHV} = \frac{1}{2} \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Virt}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{FnSplit}^{(----)}$$

$$B^{AntiMHV} = \frac{1}{2} \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Virt}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{FnSplit}^{(----)}$$

$$C^{Matter} = \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(---+\bar{q}q,\bar{q}\bar{q})} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(---+\bar{q}q,\bar{q}\bar{q})} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{FnSplit}^{(---+\bar{q}q,\bar{q}\bar{q})}$$

## Infrared-free sets (for any arbitrary $\delta$ )

$$A^{MHV} = \frac{1}{2} \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Virt}^{(----)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{Real}^{(---+)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{InSplit}^{(---+)} + \left( \frac{d\sigma}{d\Omega_{14}} \right)_{FnSplit}^{(---+)}$$

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## Infrared-free observables

- Registration of two fastest gluons of positive chirality

$$A^{MHV} \Big|_{\delta=2/3} + B^{AntiMHV} \Big|_{\delta=1}$$

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The simplest IR finite answer so far ( $Q_f = E$ ): **N=4 SYM Anti MHV**

$$\left( \frac{d\sigma}{d\Omega_{14}} \right)_{AntiMHV} = \frac{\alpha^2 N_c^2}{E^2} \left\{ \frac{3 + c^2}{(1 - c^2)^2} - \frac{\alpha N_c}{2\pi} \left[ 2 \frac{(c^4 + 2c^3 + 4c^2 + 6c + 19) \log^2\left(\frac{1-c}{2}\right)}{(1-c)^2(1+c)^4} + 2 \frac{(c^4 - 2c^3 + 4c^2 - 6c + 19) \log^2\left(\frac{1+c}{2}\right)}{(1-c)^4(1+c)^2} - 8 \frac{(c^2 + 1) \log\left(\frac{1+c}{2}\right) \log\left(\frac{1-c}{2}\right)}{(1-c^2)^2} - \frac{6\pi^2(c^2 - 1) + 5(61c^2 + 99)}{9(1-c^2)^2} + 2 \frac{(11c^3 + 31c^2 - 47c + 133) \log\left(\frac{1+c}{2}\right)}{3(1-c)^3(1+c)^2} - 2 \frac{(11c^3 - 31c^2 - 47c - 133) \log\left(\frac{1-c}{2}\right)}{3(1+c)^3(1-c)^2} \right] \right\}$$

# Summary

- In observable cross-sections the IR divergences do cancel in accordance with Kinoshita-Lee-Nauenberg theorem

$$d\sigma_{obs}^{incl} = \sum_{n=2}^{\infty} \int_0^1 dz_1 q_1(z_1, \frac{Q_f^2}{\mu^2}) \int_0^1 dz_2 q_2(z_2, \frac{Q_f^2}{\mu^2}) \prod_{i=1}^n \int_0^1 dx_i q_i(x_i, \frac{Q_f^2}{\mu^2}) \times \\ \times d\sigma^{2 \rightarrow n}(z_1 p_1, z_2 p_2) S_n(\{z\}, \{x\}) = g^4 \sum_{L=0}^{\infty} \left( \frac{g^2}{16\pi^2} \right)^L d\sigma_L^{Finite}(s, t, u, Q_f^2)$$

- The simple structure of the MHV amplitude DOES NOT reveal at the level of IR finite cross-sections;
- In some cases the cancellation of complicated functions occurs, though not always;
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