$\mathcal{N}=6$ CHERN-SIMONS THEORIES IN HARMONIC SUPERSPACE

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D=3 Chern-Simons theory

$$S_{CS}=\frac{k}{4\pi}\int d^3x\varepsilon^{mnr}\mathbf{Tr}\,\{A_m(\partial_nA_r+\tfrac{i}{3}[A_n,A_r])\}$$

where A_m is the three-dimensional gauge field. The classical CS solutions are pure gauge fields $F_{nr}(A)=0$.

D=3 BF-theory

$$S_{BF}=\int d^3x\varepsilon^{mnr}\mathbf{Tr}\,\{B_mF_{nr}(A)\}$$

This model has the additional noncompact Abelian b-transformations

$$\delta B_m=\partial_mb+i[A_m,b]+i[B_m,a],\qquad \delta A_m=\partial_ma+i[A_m,a]$$

$D=3,\mathcal{N}=1$ **supersymmetric CS theory in the superspace $z=(x^m,\theta^\alpha)$ is described by the spinor gauge superfield $A_\alpha(z)$ and the superfield strength W_α [W. Siegel (1979), J. Schonfeld (1981)].**

$D=3,\mathcal{N}=2$ **supersymmetric CS theory in the superspace $z=(x^m,\theta^\alpha,\bar\theta^\alpha)$ is described by the prepotential V and the pseudoscalar superfield strength W [B. Zupnik, D. Pak (1988)].**

$$S_{CS}=\int d^3xd^4\theta VW=\int d^3xd^4\theta VD^\alpha\bar D_\alpha V$$

$D=3,\mathcal{N}=3$ general superspace: $x^m, \theta^\alpha_{(ik)}, \quad i,k=1,2$
$D=3,\mathcal{N}=3$ harmonic superspace uses the $SU(2)/U(1)$ harmonics $u^\pm_i$

$$\theta^{++}_\alpha = \theta^{ik}_\alpha u^+_i u^+_k, \quad \theta^0_\alpha = \theta^{ik}_\alpha u^+_i u^-_k$$

$\mathcal{N}=3$ gauge prepotential V^{++} and superfield strength W^{++} live in the same analytic superspace [D. Khetselius, B. Zupnik (1988)]

$$S_{CS} = \int d^3x_A d\theta^{-4} du V^{++} W^{++} = \int d^3x d^6\theta du V^{++} V^{--}$$

$$D^{++}V^{--} = D^{--}V^{++}, \quad W^{++} = -\tfrac{1}{4} D^{++\alpha} D^{++}_\alpha V^{--}$$

The $\mathcal{N}=6$ ABJM model was reformulated in this $\mathcal{N}=3$ superfield formalism [J. Buchbinder, E. Ivanov, O. Lechtenfeld, N. Pletnev, I. Samsonov, B. Zupnik (2008)]. This formalism was presented in the talk of I. Samsonov on this conference.

The action of the $\mathcal{N}=3$ abelian BF theory can be constructed as the difference of two Chern-Simons actions

$$\tfrac{1}{4}\int d\zeta^{-4} du [(V^{++}+A^{++})(W^{++}_V+W^{++}_A)-(V^{++}+A^{++})(W^{++}_V+W^{++}_A)]$$

$$= \tfrac{1}{2}\int d\zeta^{-4}[V^{++}W^{++}_A + A^{++}W^{++}_V]$$

where the prepotentials V^{++} and A^{++} have opposite parities. The fourth supersymmetry transformations of these prepotentials are

$$\delta_4 V^{++} = \epsilon^\alpha D^0_\alpha V^{++}, \quad \delta_4 A^{++} = -\epsilon^\alpha D^0_\alpha A^{++}$$

This model can be also constructed in the $\mathcal{N}=4$ superspace.

$D=3, \mathcal{N}=4$ **general superspace** $x^m, \theta^\alpha_{ia}, \quad i,k=1,2, \quad a=1,2$ **is covariant with respect to the group** $SU_L(2)\times SU_R(2)$**. We introduce left harmonics** $u^\pm_k$ **and right harmonics** $v^{(\pm)}_a$**. The mirror map interchanges indices of two groups**

$$\mathcal{M}: \quad SU_L(2)\leftrightarrow SU_R(2), \quad u^\pm_k \leftrightarrow v^{(\pm)}_a, \quad \theta^\alpha_{12}\leftrightarrow\theta^\alpha_{21}$$

The $\mathcal{N}=4$ **left harmonic superspace uses the harmonics** $u^\pm_k$ **and the analytic projection of the Grassmann coordinates** $\theta^{+\alpha}_a = -u^{+k}\theta^\alpha_{ka}$**. It is analogous to the** $D=4, \mathcal{N}=2$ **harmonic superspace [A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, E. Sokatchev (1984)]. The left gauge prepotential** V^{++} **and its superfield strength**

$$W^{ab} = -\tfrac{1}{4}D^{+a\alpha}D^{+b}_\alpha V^{--}$$

are defined in different superspaces. The superfield $W=\sqrt{W^{ab}W_{ab}}$ **plays the role of the superconformal dilaton. The constants** φ **and** C^ab **break down spontaneously the superconformal symmetry in the representation** $W^{ab}=\varphi(C^{ab}+w^{ab})$**. We can construct the superconformal generalization of the** $\mathcal{N}=4$ **superfield abelian gauge action [B. Zupnik (1999)]**

$$S^0_W = \int d^3x d^8\theta du W^{-1}V^{++}V^{--}$$

The mirror $\mathcal{N}=4$ **right harmonic superspace uses the harmonics** $v^{(\pm)}_a$ **and the corresponding projection of the Grassmann coordinates** $\theta^{(+)\alpha}_k = -v^{(+)a}\theta^\alpha_{ka}$**. The mirror abelian prepotential** $A^{(++)}$ **has the left analytic superfield strength**

$$L^{++}(A) = -\tfrac{1}{4}u^+_k u^+_l D^{(+)k\alpha}D^{(+)l}_\alpha A^{(--)}, \quad D^{++}L^{++}=0,$$

$$D^{(++)}A^{(--)} = D^{(--)}A^{(++)}, \quad D^{(+)l}_{\alpha}A^{(++)} = 0$$

The action of the $\mathcal{N}=4$ superfield abelian BF theory

$$S_{BF} = \beta \int d\zeta_L^{-4} du V^{++} L^{++}(A)$$

In components, it connects fields of the mirror gauge multiplets [R. Brooks, S.J. Gates (1994)]

$$S_{BF} = \beta \int d^3x (2\varepsilon^{mnp} A_m \partial_n B_p - \tfrac{1}{2}\phi^{ab} Y_{ab} - \tfrac{1}{2}\Lambda^{ik} X_{ik} + 2\rho^{ka}_{\alpha} \lambda^{\alpha}_{ka})$$

This term yields the non-trivial interaction of the left and right abelian gauge multiplets in the $\mathcal{M}$-invariant action

$$S^0_W + \mathcal{M} S^0_W + S_{BF}$$

which describes the interactions of the topologically massive gauge fields with the scalar and fermion fields. The mixed scalar potential term is

$$P = -\frac{\sqrt{2}\beta^2}{4} \int d^3x [\phi^{ab}\phi_{ab}\sqrt{\Lambda^{kl}\Lambda_{kl}} + \Lambda^{kl}\Lambda_{kl}\sqrt{\phi^{ab}\phi_{ab}}]$$

We can add interactions of the left hypermultiplet q^+_a and right hypermultiplet $Q^{(+)}_k = \mathcal{M} q^+_a$ with the corresponding gauge multiplets

$$S(q,V) = \int d\zeta_L^{-4} du q^+_a [D^{++}\delta^a_b + (\tau_3)^a_b V^{++}] q^{+b},$$

$$S(Q,A) = \mathcal{M}S(q,V) = \int d\zeta_R^{(-4)} dv Q^{(+)}_k [D^{(++)}\delta^k_l + (\tau_3)^k_l A^{(++)}] Q^{(+)l}$$

$\mathcal{N}=6$ **general superspace** $z=(x^m,\theta^\alpha_a)$ **where** $a=1,\dots 6$ **is the 6-vector index of the group** $SO(6)$**. We analyze the CS$^6_3$ theory using the** $SO(6)/U(3)$ **harmonics [P.S. Howe, M.I. Leeming (1994)]**

$$U=(U^{+k}_a,U^-_{ka})$$

where a **is the** $SO(6)$ **vector index,** $+,-$ **are the** $U(1)$ **charges, and** $k=1,2,3$ **are indices of the spinor representations of** $SU(3)$**. The basic relations for these harmonics are**

$$\begin{aligned}&U^{+k}_aU^{+l}_a=0,\quad U^{+k}_aU^-_{la}=\delta^k_l,\quad U^-_{ka}U^-_{la}=0,\\&U^{+k}_aU^-_{kb}+U^-_{ka}U^{+k}_b=\delta_{ab}.\end{aligned}$$

We treat the $SO(6)/U(3)$ **harmonics as real with respect to the special** $\sim$ **conjugation.**

The $SO(6)$ **invariant harmonic derivatives are**

$$\begin{aligned}&\partial^{++}_k=\varepsilon_{kjl}U^{+j}_a\frac{\partial}{\partial U^-_{la}},\quad \partial^{--k}=\varepsilon^{kjl}U^-_{ja}\frac{\partial}{\partial U^{+l}_a},\\&\partial^k_l=U^{+k}_a\frac{\partial}{\partial U^{+l}_a}-U^-_{la}\frac{\partial}{\partial U^-_{ka}}-\frac{1}{3}\delta^k_l\left(U^{+j}_a\frac{\partial}{\partial U^{+j}_a}-U^-_{ja}\frac{\partial}{\partial U^-_{ja}}\right),\\&\partial^0=U^{+j}_a\frac{\partial}{\partial U^{+j}_a}-U^-_{ja}\frac{\partial}{\partial U^-_{ja}}.\end{aligned}$$

In the analytic basis (AB) **of the** $SO(6)/U(3)$ **harmonic superspace, we introduce harmonic projections of the** N=6 **spinor coordinates**

$$\theta^{+k\alpha}=\theta^\alpha_aU^{+k}_a,\quad \theta^{-\alpha}_k=\theta^\alpha_aU^-_{ka}$$

and the corresponding analytic vector coordinates

$$y^m=x^m+i(\theta_a\gamma^m\theta_b)U^{+k}_aU^-_{kb}$$

where γ_m are the 3D gamma-matrices. The analytic integral measure is pure imaginary and dimensionless

$$d\mu^{(-6)} = dUd^3yd\theta^{(-6)}, \quad \int d\theta^{(-6)}(\theta^{+1})^2(\theta^{+2})^2(\theta^{+3})^2 = 1.$$

This integral measure is odd with respect to the P parity transformation

$$(y^0, y^1, y^2) \to (y^0, -y^1, y^2), \quad \theta^{+k\alpha} \to (\gamma_1)^\alpha_\beta \theta^{+k\beta}$$

The analytic-superspace representation of the harmonic derivatives have the following form:

$$\mathcal{D}^{++}_k = \partial^{++}_k - i\varepsilon_{kjl}(\theta^{+j}\gamma^m\theta^{+l})\partial_m - \varepsilon_{kjl}\theta^{+j\alpha}\partial^{+l}_\alpha.$$

where

$$\partial^-_{k\alpha} = \frac{\partial}{\partial\theta^{+k\alpha}}, \quad \partial^{+k}_\alpha = \frac{\partial}{\partial\theta^{-\alpha}_k}.$$

The AB-representation of the spinor derivatives is

$$D^-_{k\alpha} = \partial^-_{k\alpha} - 2i\theta^{-\beta}_k(\gamma^m)_{\alpha\beta}\partial_m, \quad D^{+k}_\alpha = \partial^{+k}_\alpha.$$

The analytic superfield gauge parameters satisfy the conditions

$$D^{+k}_\alpha \Lambda(\zeta) = 0, \quad \mathcal{D}^k_l\Lambda = \mathcal{D}^0\Lambda = 0$$

In the gauge group $SU(n)$, we use the following covariant derivatives and the analytic prepotentials:

$$\nabla^{++}_k = \mathcal{D}^{++}_k + V^{++}_k(\zeta), \quad (V^{++}_k)^\dagger = -V^{++}_k,$$
$$\mathcal{D}^k_l V^{++}_j = \tfrac{1}{3}\delta^k_l V^{++}_j - \delta^k_j V^{++}_l, \quad \mathcal{D}^0 V^{++}_k = 2V^{++}_k$$

The superfield action of the N=6 CS theory has the following form in our notation:

$$S_{CS} = \tfrac{i}{12}\int d\mu^{(-6)}\varepsilon^{klj}\mathbf{Tr}\,\{V^{++}_k\mathcal{D}^{++}_l V^{++}_j + \tfrac{1}{3}V^{++}_k[V^{++}_l, V^{++}_j]\}$$

The corresponding classical equations of motion $F^{(+4)}_{kl} = 0$ have the pure gauge solutions only.

The $N=6$ BF theory contains interactions of V^{++}_k with the second analytic gauge superfield B^{++}_k

$$\delta_\Lambda B^{++}_k = [B^{++}_k, \Lambda], \quad \delta_\Sigma B^{++}_k = \mathcal{D}^{++}_k \Sigma + [V^{++}_k, \Sigma]$$

where Σ is the independent superfield matrix parameter describing Abelian translations in the group space.

The corresponding superfield action has the following form:

$$S_{BF} = i\int d\mu^{(-6)} \varepsilon^{klj} \mathbf{Tr}\,\{\tfrac{1}{2} B^{++}_k F^{(+4)}_{lj}(V)\}. \qquad (0.1)$$

The BF-action preserves the P parity if B^{++}_k is the P-odd superfield.

The classical equations of the BF theory are

$$F^{(+4)}_{lj}(V) = 0, \quad \varepsilon^{klj}\nabla^{++}_l B^{++}_j = 0$$

They have the pure gauge solutions for both superfields.

The superconformal transformations $\mathbf{SC}^6_3$ of the N=6 analytic coordinates

$$\delta_{sc} y^m = c^m + 2l\, y^m + \varepsilon^{mnr} L_n y_r + (y^n k_n) y^m - \tfrac{1}{2} y^2 k^m - 2i(\epsilon^-_k \gamma^m \theta^{+k})$$
$$-i\omega_{ab} U^-_{ka} U^-_{lb} (\theta^{+k}\gamma^m\theta^{+l}) + i y^m \theta^{+k\alpha}\eta^-_{k\alpha} + i\varepsilon^{mnr} y_n (\theta^{+k}\gamma_r \eta^-_k),$$

$$\delta_{cs}\theta^{+k\alpha} = \epsilon^{+k\alpha} + L^\alpha_\beta \theta^{+k\beta} + l\,\theta^{+k\alpha} - \omega_{bc} U^{+k}_b U^-_{lc}\theta^{+l\alpha}$$
$$+\tfrac{1}{2} y^{\alpha\beta}\theta^{+k\gamma} k_{\beta\gamma} + \tfrac{1}{2} y^{\alpha\beta}\eta^{+k}_\beta - i\theta^{+l\alpha}\theta^{+k\beta}\eta^-_{l\beta},$$

$$\delta_{sc} U^{+k}_a = \varepsilon^{klj}\lambda^{++}_l U^-_{ja}, \quad \delta_{sc} U^-_{ka} = 0,$$
$$\lambda^{++}_l = \varepsilon_{ljn}[\tfrac{i}{2} k_{\alpha\beta}\theta^{+j\alpha}\theta^{+n\beta} + i\theta^{+j\alpha}\eta^{+n}_\alpha + \tfrac{1}{2}\omega_{cb} U^{+j}_c U^{+n}_b],$$

where c^m, L^m, l and k^m are parameters of the 3D conformal group, ω_{ab} are the SO(6) parameters, and the harmonic projections of the odd parameters ϵ^α_a and η^α_a are used

$$\epsilon^{+k\alpha} = \epsilon^\alpha_a U^{+k}_a, \quad \epsilon^{-\alpha}_k = \epsilon^\alpha_a U^-_{ka}, \quad \eta^{+k\alpha} = \eta^\alpha_a U^{+k}_a, \quad \eta^{-\alpha}_k = \eta^\alpha_a U^-_{ka}.$$

The analytic integral measure is invariant with respect to these transformations.

The SC^6_3 transformations of the harmonic derivatives are

$$\delta_{sc}\mathcal{D}^{++}_k = \lambda^{++}_l \mathcal{D}^l_k - \tfrac{2}{3}\lambda^{++}_k \mathcal{D}^0, \quad \delta_{sc}\mathcal{D}^k_l = \delta_{sc}\mathcal{D}^0 = 0$$

The superfield CS and BF actions are invariant with respect to the superconformal transformations

$$\delta_{sc}V^{++}_k = 0, \quad \delta_{sc}F^{(+4)}_{kl} = 0, \quad \delta_{sc}B^{++}_k = 0.$$

The Grassmann decompositions of the gauge BF-superfield V^{++}_k contain the vector gauge field plus an infinite number of auxiliary fields

$$V^{++}_k = \varepsilon_{kjl}(\theta^{+j}\gamma^m\theta^{+l})A_m + i\Theta^{(+3)\alpha\beta\gamma}U^-_{ka}\Psi^a_{\alpha\beta\gamma} + i\Theta^{(+3)l\alpha}_k U^-_{la}\Psi^a_\alpha + \ldots$$

where $\Psi^a_{\alpha\beta\gamma}(x)$ is the unusual auxiliary field.

We do not know how to describe interaction of this CS prepotential with the $\mathcal{N} = 6$ matter superfields.

The equivalent formalism of the $\mathcal{N}=6$ CS theory can be formulated in the $\mathcal{N}=5$ harmonic superspace [Zupnik (2007)]. We used the $SO(5)/U(2)$ harmonics

$$U^K_a=(U^{+i}_a,U^0_a,U^-_{ia})=(U^{+1}_a,U^{+2}_a,U^0_a,U^-_{1a},U^-_{2a})$$

where $a=1,\dots 5$ is the vector index of the group $SO(5)$, $i=1,2$ is the spinor index of the group $SU(2)$, and $U(1)$-charges are denoted by symbols $+,-,0$. The basic relations for these harmonics are

$$U^{+i}_aU^{+k}_a=U^{+i}_aU^0_a=0,\quad U^-_{ia}U^-_{ka}=U^-_{ia}U^0_a=0,$$
$$U^{+i}_aU^-_{ka}=\delta^i_k,\quad U^0_aU^0_a=1$$

We consider the $SO(5)$ invariant harmonic derivatives with nonzero $U(1)$ charges

$$\partial^{+i}=U^{+i}_a\frac{\partial}{\partial U^0_a}-U^0_a\frac{\partial}{\partial U^-_{ia}},\quad \partial^{+i}U^0_a=U^{+i}_a,\quad \partial^{+i}U^-_{ka}=-\delta^i_kU^0_a,$$

$$\partial^{++}=U^+_{ia}\frac{\partial}{\partial U^-_{ia}},\quad [\partial^{+i},\partial^{+k}]=\varepsilon^{ki}\partial^{++},\quad \partial^{+i}\partial^+_i=\partial^{++}$$

The $SO(5)/U(2)$ harmonics allow constructing projections of the spinor coordinates and the partial spinor derivatives

$$\theta^{+i\alpha}=U^{+i}_a\theta^\alpha_a,\quad \theta^{0\alpha}=U^0_a\theta^\alpha_a,\quad \theta^{-\alpha}_i=U^-_{ia}\theta^\alpha_a$$

We use the following representation of the vector coordinate:

$$x^m_A=x^m+i(\theta^{+k}\gamma^m\theta^-_k)$$

and the harmonic derivatives in the AB representation:

$$\mathcal{D}^{+k}=\partial^{+k}-i(\theta^{+k}\gamma^m\theta^0)\partial_m+\theta^{+k\alpha}\partial^0_\alpha-\theta^{0\alpha}\partial^{+k}_\alpha,$$

$$\mathcal{D}^{++} = \partial^{++} + i(\theta^{+k}\gamma^m\theta^+_k)\partial_m + \theta^{+\alpha}_k\partial^{+k}_\alpha$$

The integration measure in the analytic superspace $d\mu^{(-4)}$ has the dimension zero

$$d\mu^{(-4)} = dUd^3x_A(\partial^0_\alpha)^2(\partial^-_{i\alpha})^4 = dUd^3x_A d\theta^{(-4)}$$

The gauge superfields (prepotentials) $V^{+k}(\zeta)$ and $V^{++}(\zeta)$ in the harmonic $SO(5)/U(2)$ superspace satisfy the Grassmann analyticity and $U(2)$-covariance conditions

$$D^{+k}_\alpha V^{+k} = D^{+k}_\alpha V^{++} = 0, \quad \mathcal{D}^i_j V^{+k} = \delta^k_j V^{+i}, \quad \mathcal{D}^i_j V^{++} = \delta^i_j V^{++}$$

We treat these prepotentials as connections in the covariant gauge derivatives

$$\nabla^{+i} = \mathcal{D}^{+i} + V^{+i}, \quad \nabla^{++} = \mathcal{D}^{++} + V^{++}$$

The superfield action in the analytic $SO(5)/U(2)$ superspace is defined on three prepotentials V^{+k} and V^{++}

$$S_{CS} = \frac{ik}{12\pi}\int d\mu^{(-4)}\mathbf{Tr}\,\{V^{+j}\mathcal{D}^{++}V^+_j + 2V^{++}\mathcal{D}^+_j V^{+j}$$

$$+(V^{++})^2 + V^{++}[V^+_j, V^{+j}]\}$$

The transformation of the sixth supersymmetry can be defined on the analytic $\mathcal{N}$=5 superfields

$$\delta_6 V^{++} = \epsilon^\alpha_6 D^0_\alpha V^{++}, \quad \delta_6 V^{+k} = \epsilon^\alpha_6 D^0_\alpha V^{+k}$$