Theoretical foundations of stellar dynamics: potentials, orbits, actions

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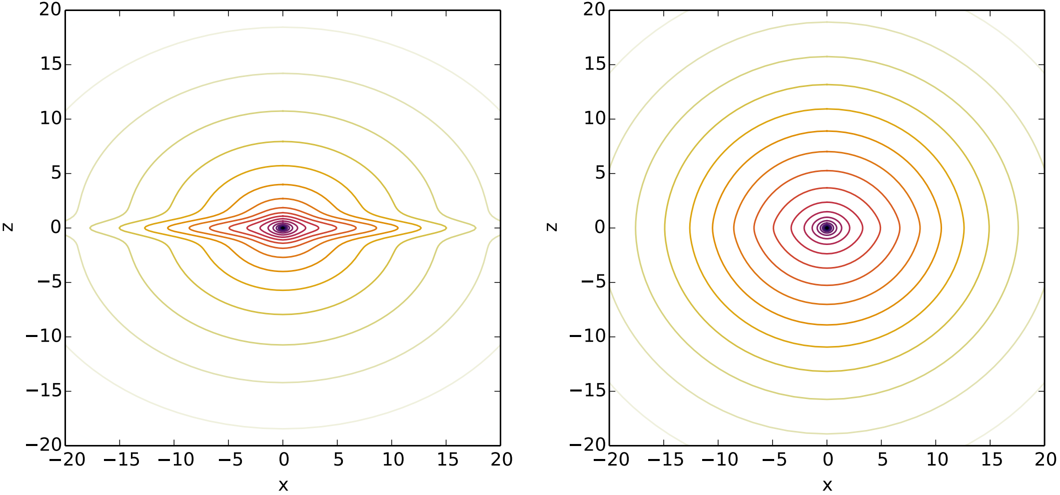
Gravitational potential

$$\nabla^2 \Phi = 4\pi \, G \, \rho \qquad \textbf{(in Newtonian gravity)}$$

In galactic dynamics:

- neglect relativity;
- neglect cosmological expansion;
- Φ is negative and tends to zero at infinity.

Potential and density



isodensity surfaces are more flattened than equipotential surfaces

Stellar orbits

Fundamental concepts of Hamiltonian mechanics:

Hamiltonian $H(\mathbf{q}, \mathbf{p})$,
where $\mathbf{q}$ are generalized coordinates, $\mathbf{p}$ are generalized momenta.

Equations of motion:

$$\begin{aligned} \dot{\mathbf{q}} &= \partial H / \partial \mathbf{p}, \\ \dot{\mathbf{p}} &= -\partial H / \partial \mathbf{q}. \end{aligned}$$

In the simplest case, $\mathbf{q}$ would be Cartesian coordinates $\mathbf{x} \equiv \{x, y, z\}$,
and $\mathbf{p}$ – corresponding velocity components $\mathbf{v} \equiv \{v_x, v_y, v_z\}$.

$$H(\mathbf{x}, \mathbf{v}) = \Phi(\mathbf{x}) + \tfrac{1}{2}|\mathbf{v}|^2.$$

Stellar orbits – 1d

Simplest possible Hamiltonian system with bound motion:

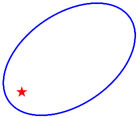
$$\text{harmonic oscillator:} \quad H(x, v) = \tfrac{1}{2}\Omega^2 x^2 + \tfrac{1}{2} v^2.$$

Not a totally idealized case:

$$\Phi(x) \propto x^2 \implies \rho(x) \propto \frac{d^2\Phi}{dx^2} \propto \text{const.}$$

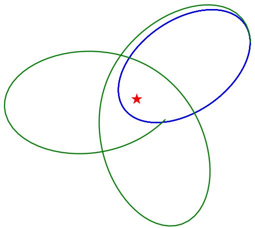
Constant-density cores are commonly encountered in many stellar systems.

Stellar orbits – 2d (planar motion)



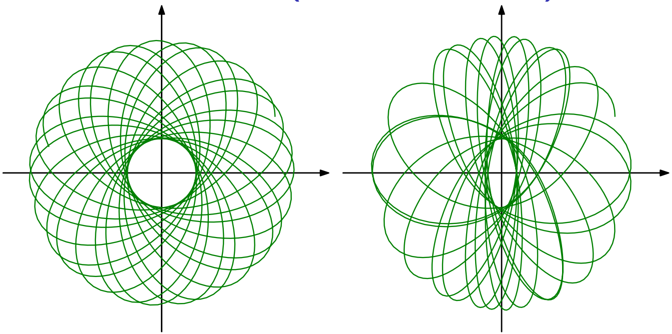
Next-simplest case: Keplerian motion

Stellar orbits – 2d (planar motion)



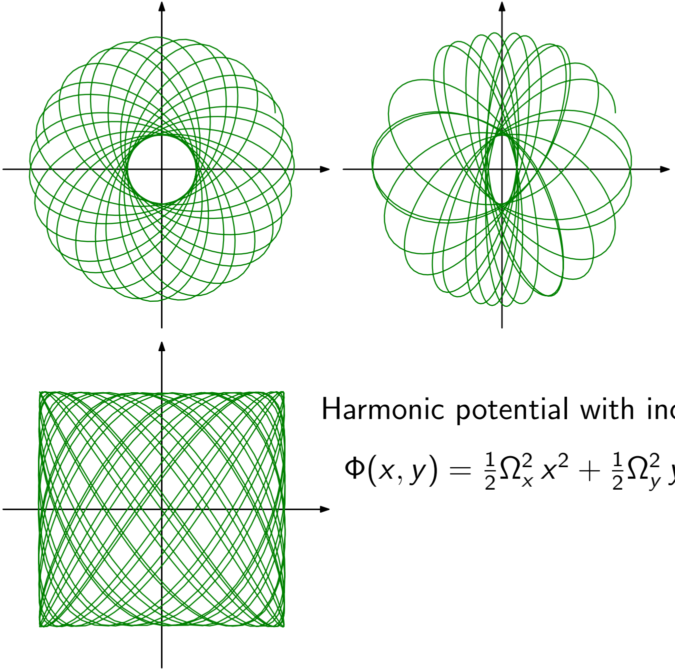
In a general spherically-symmetric potential, orbits are not closed and form rosette figures

Stellar orbits – 2d (planar motion)



In a non-axisymmetric potential,
these rosettes (tube orbits) are squashed

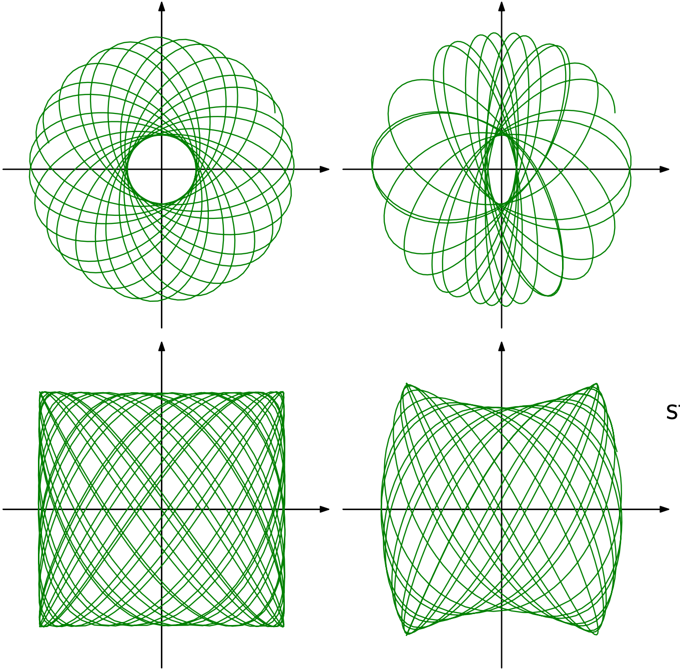
Stellar orbits – 2d (planar motion)



Harmonic potential with incommensurable frequencies:

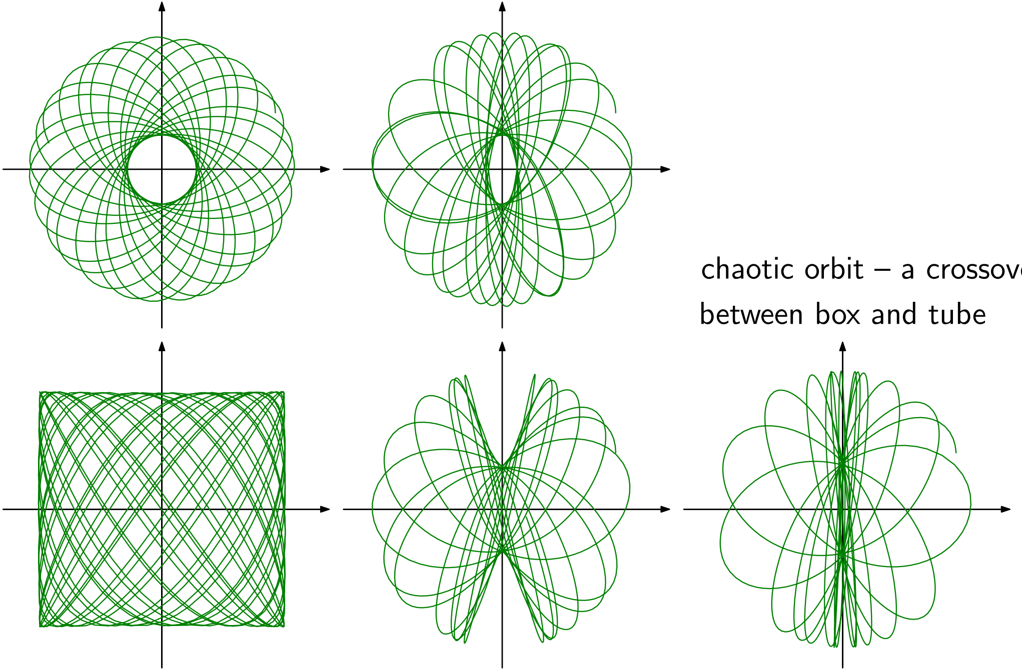
$$\Phi(x, y) = \frac{1}{2}\Omega_x^2\, x^2 + \frac{1}{2}\Omega_y^2\, y^2 \quad \Longrightarrow \quad \text{box orbit}$$

Stellar orbits – 2d (planar motion)



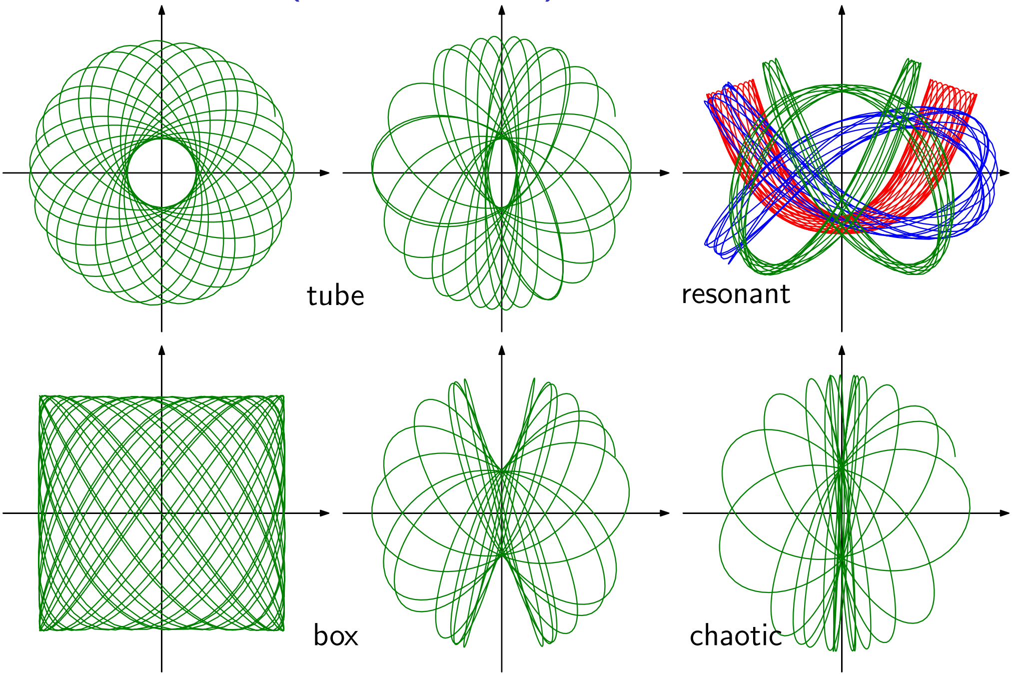
still a (deformed) box orbit

Stellar orbits – 2d (planar motion)

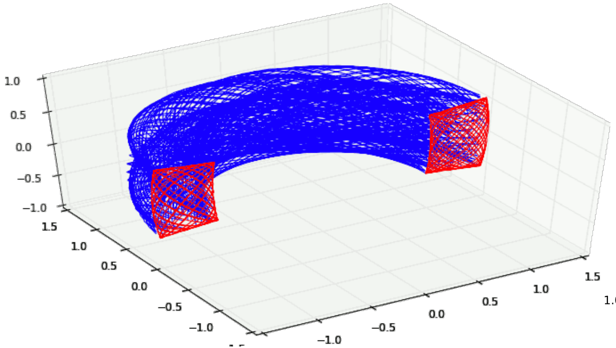


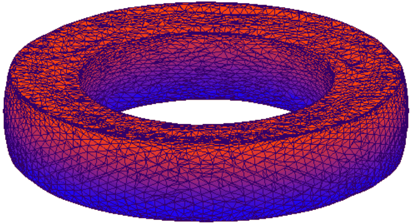
chaotic orbit – a crossover between box and tube

Stellar orbits – 2d (planar motion)



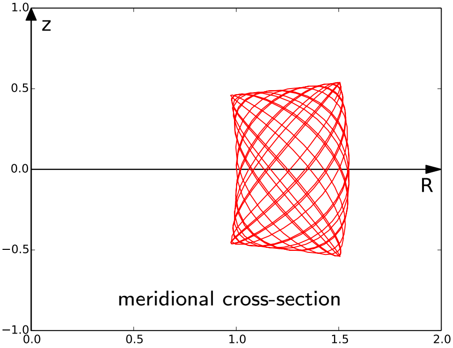
Stellar orbits – 3d





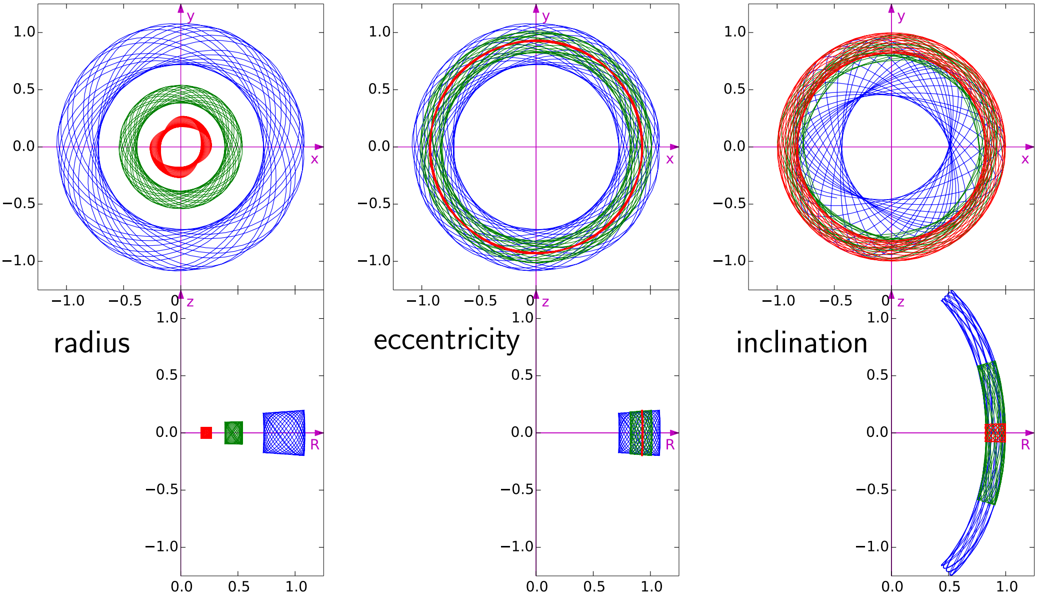
tube orbit

(note that the variety of orbits in 3d is much larger than in 2d, but the main classes are the same – box, tube, various resonant families and chaotic orbits)



Stellar orbits and integrals of motion

How large is the variety of orbits?



Stellar orbits and integrals of motion

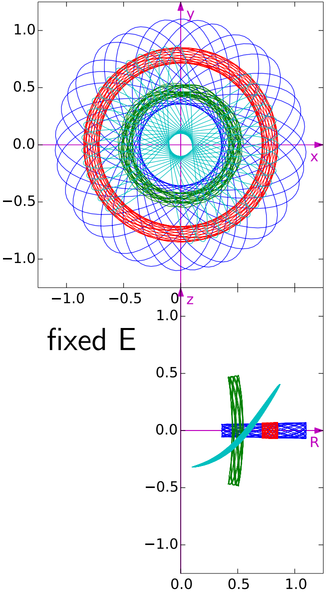
How large is the variety of orbits?

In general, *regular* orbits in *typical* stationary galactic potentials conserve 3 integrals of motion: one is always the energy E, and at a fixed energy, there are two more degrees of freedom (roughly corresponding to eccentricity and inclination).

The spherical case is degenerate in that it supports 4 integrals of motion (E and three components of the angular momentum $\mathbf{L}$).

In the axisymmetric case, the second integral is L_z, but the third integral I_3 (if exists), does not have an explicit expression (except in a special class of fully integrable potentials known as Stäckel potentials).

Not all orbits have the same number of integrals, and the physical meaning of these integrals is different between orbit families.



The holy grail of Hamiltonian mechanics

What is the simplest possible Hamiltonian system?

A free particle!

$$H(\mathbf{q},\mathbf{p}) = \tfrac{1}{2}|\mathbf{p}|^2 \quad \Longrightarrow \quad p_i(t) = \text{const}, \quad q_i(t) = p_i\, t + \text{const}$$

Unfortunately, it corresponds to an unbound motion,
unlike [most] stars in galaxies.

The next simplest (and more realistic) thing? Periodic motion:

$$H(\mathbf{q},\mathbf{p}) = H(\mathbf{p}) \quad \Longrightarrow \quad p_i(t) = \text{const}, \quad q_i(t) = \frac{\partial H}{\partial p_i}\, t + \text{const} \equiv \Omega\, t + \text{const},$$

where $\mathbf{q}$ are treated as angle-like (periodic) variables, $q + 2\pi \cong q$,
and $\mathbf{p}$ are integrals of motion.

These are action–angle variables

Action–angle variables for a 1d simple harmonic oscillator

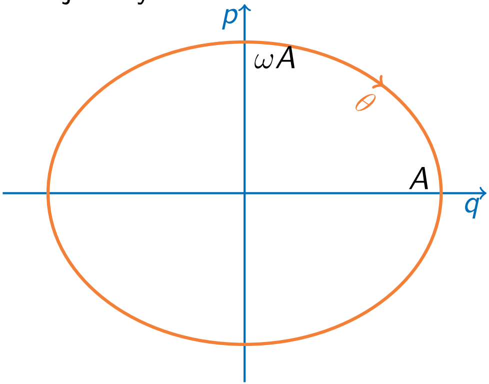
Hamiltonian: $H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$.

The trajectory is $q(t) = A\sin(\omega t + \phi_0)$, $p(t) = A\omega\cos(\omega t + \phi_0)$, and the energy is $E = \frac{1}{2}\omega^2 A^2$.

The motion is periodic with frequency ω ($\Leftrightarrow$ period $2\pi/\omega$), so we define the angle $\theta = \omega t + \phi_0$.

The action J is $\frac{1}{2\pi}\times$area enclosed by the trajectory:

$$
\begin{aligned}
J &= \frac{1}{2\pi}\oint p\, dq \\
&= \frac{1}{2\pi}\int_0^{2\pi} p(\theta)\,\frac{dq}{d\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} A^2\omega\cos^2\theta\,d\theta \\
&= \frac{A^2\omega}{2} = \frac{E}{\omega}
\end{aligned}
$$

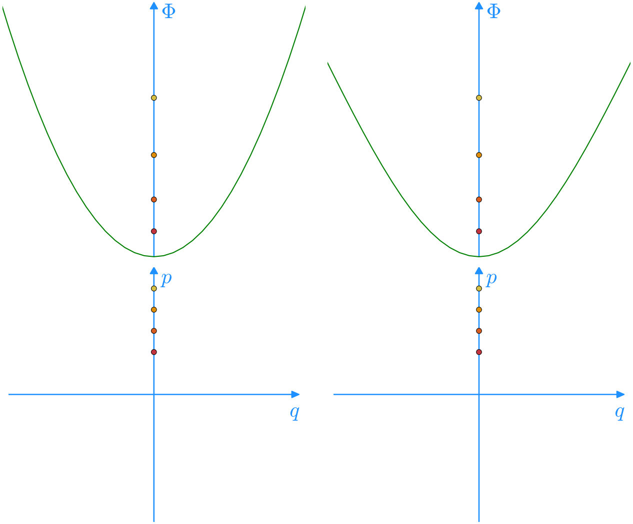


Action–angle variables for a generic 1d potential

For a generic 1d Hamiltonian $H(p, q) = \frac{1}{2}p^2 + \Phi(q)$, the action is still defined as

$$J = \frac{1}{2\pi} \oint p\, dq = \frac{2}{\pi} \int_0^{\Phi^{-1}(E)} \sqrt{2\left[E - \Phi(q)\right]}\, dq,$$

and the frequency $\Omega \equiv \dfrac{dH}{dJ}$ usually varies with energy.



harmonic anharmonic

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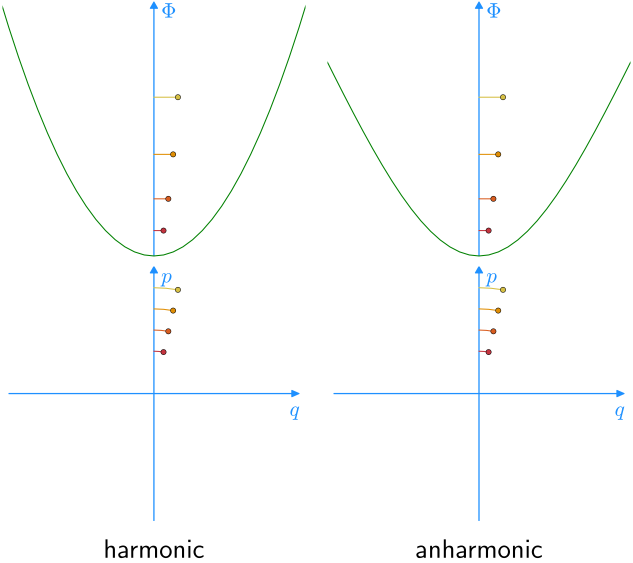

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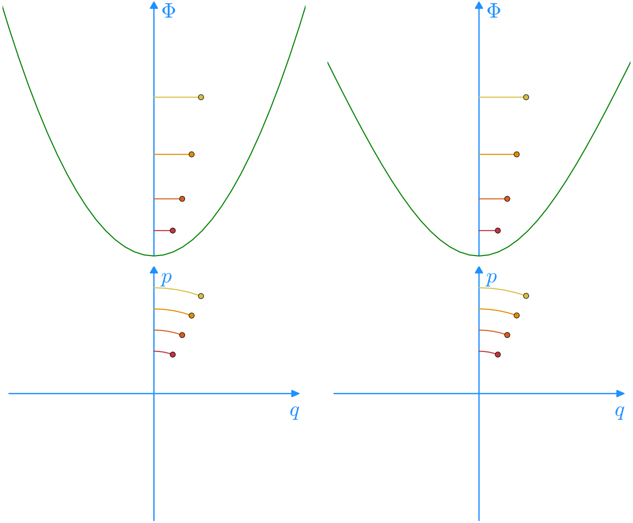

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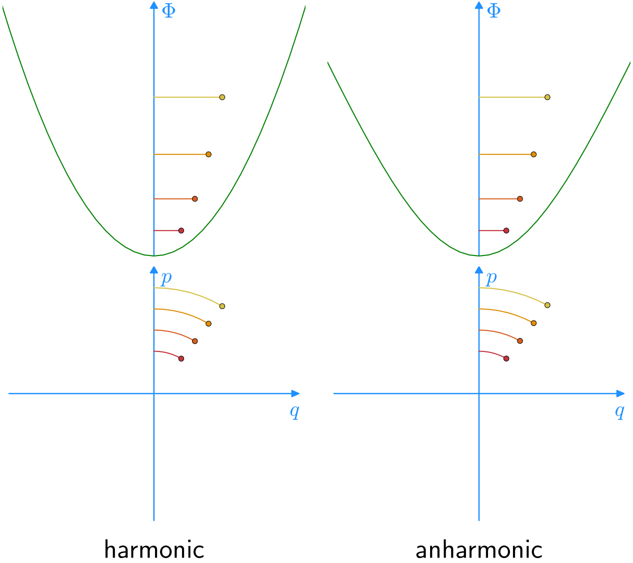

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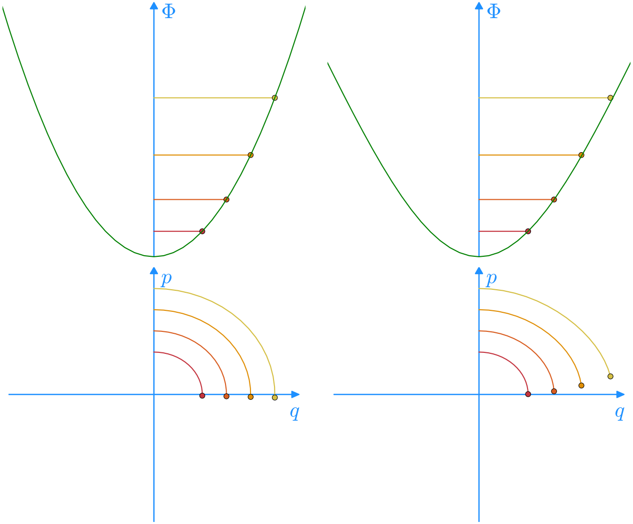

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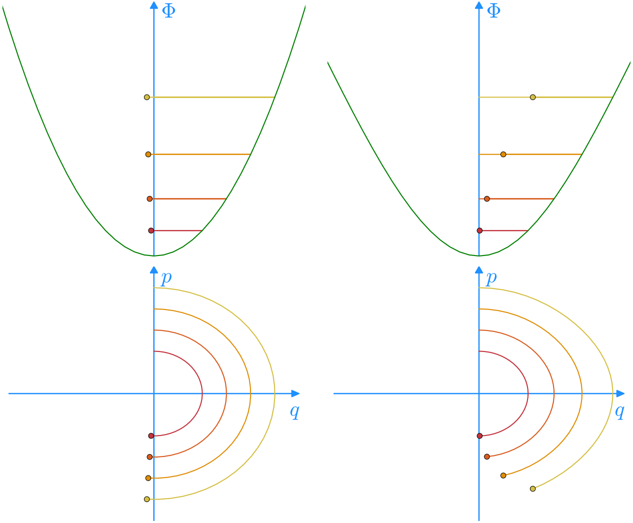

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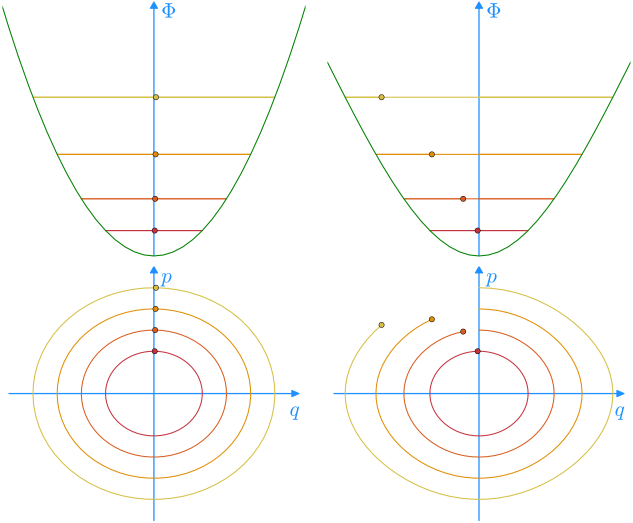


harmonic

anharmonic

Action–angle variables for a 2d simple harmonic oscillator

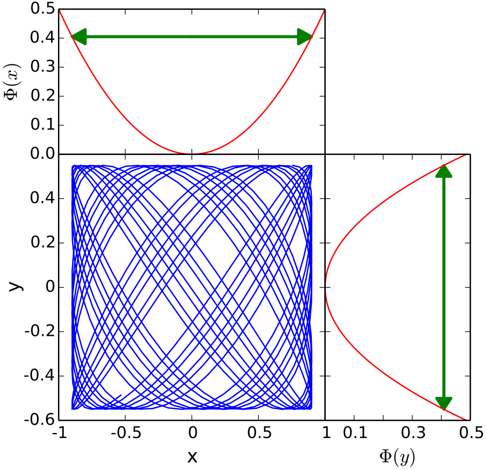
The same thing but in two dimensions: $\mathbf{q} = \{x, y\}, \ \ \mathbf{p} = \{p_x, p_y\}$;

Hamiltonian:

$$H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\left(p_x^2 + \omega_x^2 x^2\right) + \frac{1}{2}\left(p_y^2 + \omega_y^2 y^2\right)$$

$$\equiv \quad H_x(x, p_x) \quad + \quad H_y(y, p_y)$$

Motion is separable in x, y –
two uncoupled simple harmonic oscillators,
two integrals of motion E_x, E_y,
actions are $J_x = E_x/\omega_x, J_y = E_y/\omega_y$.



Action–angle variables for a 2d planar axisymmetric potential

A slightly more complicated system: two degrees of freedom, motion in an axisymmetric potential $\Phi(x, y) = \Phi(R)$, where $R \equiv \sqrt{x^2 + y^2}$.

Canonical coordinates: $\mathbf{q} = \{R, \phi\}, \mathbf{p} = \{p_R, p_\phi\}$

$$\text{Hamiltonian: } H = \Phi(R) + \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2}\right) \equiv \Phi_{\text{eff}}(R) + \frac{1}{2}p_R^2$$

$$\text{equations of motion: } \dot{R} = p_R, \quad \dot{\phi} = \frac{p_\phi}{R^2}, \quad \dot{p}_R = -\frac{d\Phi_{\text{eff}}}{dR}, \quad \dot{p}_\phi = 0$$

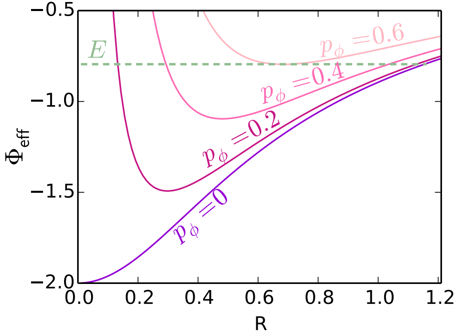
integrals of motion: E and p_ϕ

Motion in R is described by a 1d effective potential $\Phi_{\text{eff}}(R) \equiv \Phi(R) + p_\phi^2/R^2$

The radial action is

$$J_R = \frac{1}{\pi}\int_{R_-}^{R_+} p_R(R;\, E, p_\phi)\, dR$$

$$= \frac{1}{\pi}\int_{R_-}^{R_+} \sqrt{2\big[(E - \Phi_{\text{eff}}(R)\big]}\, dR$$



Action–angle variables for a 2d planar axisymmetric potential

Motion in ϕ: $\dot{p}_\phi = 0 \quad \Rightarrow \quad p_\phi = \text{const}$,

hence the azimuthal action is

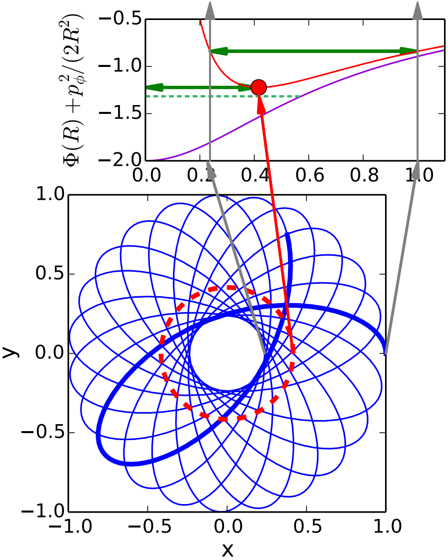
$$J_\phi = \frac{1}{2\pi} \int_0^{2\pi} p_\phi \, d\phi = p_\phi.$$

The actions J_R, J_ϕ describe the extent of the orbit in two complementary dimensions:

J_ϕ corresponds to the "guiding radius" (the radius of a circular orbit with the given angular momentum J_ϕ),

J_R gives the extent of radial oscillation about this guiding radius.

They can be varied independently, and any possible choice (provided that $J_R \geq 0$) corresponds to some trajectory.



Angles and frequencies

Note that $\dot{\phi} = p_\phi/R^2(t) \neq$ const, so ϕ is not a canonically conjugate angle variable to p_ϕ!

Such variable is θ_ϕ defined to increase linearly with time,
and similarly the radial phase angle θ_R also increases linearly with time:

$\theta_R = \Omega_R\, t, \quad \theta_\phi = \Omega_\phi\, t$, where

$\Omega_R \equiv \dfrac{\partial H(J_R, J_\phi)}{\partial J_R}, \quad \Omega_\phi \equiv \dfrac{\partial H(J_R, J_\phi)}{\partial J_\phi}$ are orbital frequencies.

$$\theta_R(R;\, E, p_\phi) = \Omega_R \int_{R_-}^{R} \frac{dt}{dR}\, dR = \Omega_R \int_{R_-}^{R} \frac{dR}{p_R(R;\, E, p_\phi)}$$

Radial orbital period $T_R \equiv \dfrac{2\pi}{\Omega_R} = 2\displaystyle\int_{R_-}^{R_+} \frac{dR}{p_R} = 2\int_{R_-}^{R_+} \frac{dR}{\sqrt{2\big[E - \Phi(R)\big] - \frac{p_\phi^2}{R^2}}}$

Azimuthal period $T_\phi \equiv \dfrac{2\pi}{\Omega_\phi} = \dfrac{2\pi\, \int_{R_-}^{R_+} dR/p_R}{p_\phi\, \int_{R_-}^{R_+} dR/(R^2\, p_R)}$

Action–angle variables for a 3d spherical potential

Spherical coordinates: $r, \theta, \phi, p_r, p_\theta, p_\phi$

Hamiltonian: $H = \Phi(r) + \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\,\sin^2\theta}\right)$

Integrals of motion: $E, L_x, L_y, L_z \left[, L \equiv \sqrt{L_x^2 + L_y^2 + L_z^2}\right]$

Radial action: $J_r = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{2\left[E - \Phi(r)\right] - \frac{L^2}{r^2}}\, dr \quad \geq 0$

Azimuthal action: $J_\phi = L_z$ (any sign)

Vertical action: $J_\theta \equiv J_z = L - |L_z| \quad \geq 0$

In general, actions, angles, frequencies, or $H(\mathbf{J})$ do not have analytic expressions. One exception is the isochrone potential [Hénon 1959]:

$$\Phi(r) = -\frac{G\,M}{b + \sqrt{b^2 + r^2}}$$ (includes Kepler and harmonic oscliiator as limiting cases)

$$H(\mathbf{J}) = -\frac{2\,(G\,M)^2}{\left(2J_r + L + \sqrt{L^2 + 4\,G\,M\,b}\right)^2}$$

Action–angle variables for a 3d axisymmetric potential

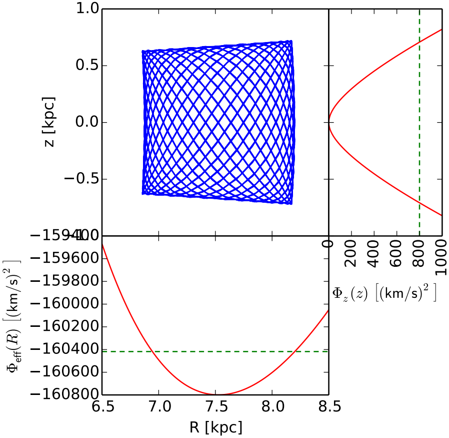
Epicyclic approximation for nearly-circular orbits close to the equatorial plane:

$\Phi(R,z) \approx \Phi_R(R) + \Phi_z(z)$,

motion in R,ϕ as in the planar axisymmetric problem with an effective potential

$\Phi_{\rm eff} = \Phi_R(R) + \frac{1}{2} L^2/r^2$,

and independent, nearly harmonic motion in z.



Epicyclic frequencies:

azimuthal: $\Omega = \dfrac{v_\circ(R)}{R} = \sqrt{\dfrac{1}{R}\dfrac{\partial\Phi}{\partial R}}$,

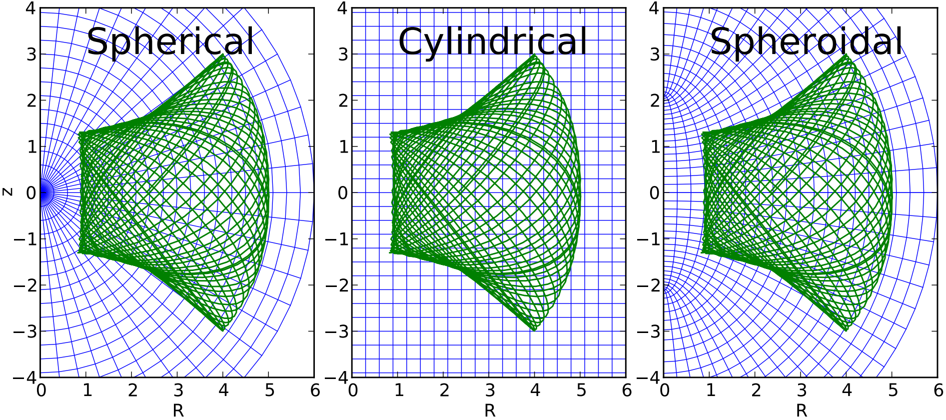
radial: $\kappa = \sqrt{\dfrac{\partial^2\Phi_{\rm eff}}{\partial R^2}} = \sqrt{\dfrac{\partial^2\Phi}{\partial R^2} + \dfrac{3}{R}\dfrac{\partial\Phi}{\partial R}}$

vertical: $\nu = \sqrt{\dfrac{\partial^2\Phi}{\partial z^2}}$.

However, it becomes increasingly inaccurate for orbits with high eccentricity and/or inclination.

Stäckel fudge

Fact: orbits in realistic axisymmetric galactic potentials are much better aligned with prolate spheroidal coordinates.



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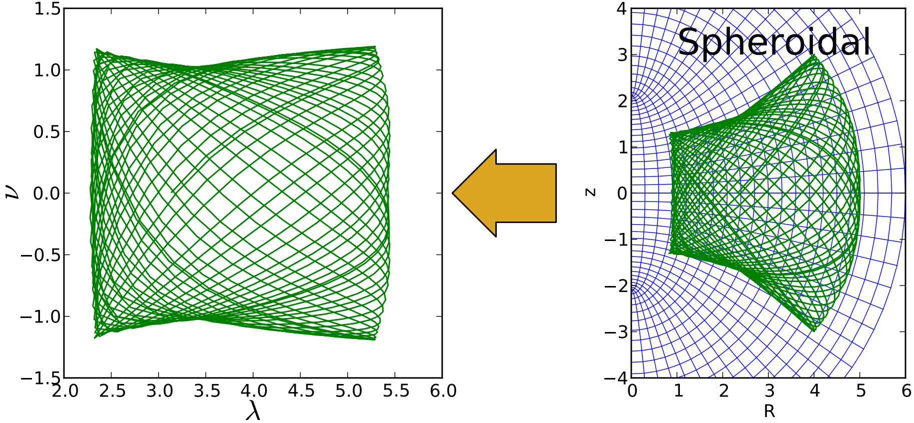
Fact: orbits in realistic axisymmetric galactic potentials are much better aligned with prolate spheroidal coordinates.

One may explore the assumption that the motion is separable in these coordinates (λ, ν).



Stäckel fudge

The most general form of potential that satisfies the separability condition is the Stäckel potential[1]: $\Phi(\lambda,\nu) = -\dfrac{f_1(\lambda) - f_2(\nu)}{\lambda - \nu}$.

The motion in λ and ν directions, with canonical momenta p_λ, p_ν, is governed by two separate equations:

$$2(\lambda - \Delta^2)\,\lambda\,p_\lambda^2 = \left[E - \frac{L_z^2}{2(\lambda - \Delta^2)}\right]\lambda - [I_3 + (\lambda - \nu)\Phi(\lambda,\nu)],$$
$$2(\nu - \Delta^2)\,\nu\,p_\nu^2 = \left[E - \frac{L_z^2}{2(\nu - \Delta^2)}\right]\nu - [I_3 + (\nu - \lambda)\Phi(\lambda,\nu)].$$

Under the approximation that the separation constant I_3 is indeed conserved along the orbit, actions are computed as

$$J_\lambda = \frac{1}{\pi}\int_{\lambda_{\mathrm{min}}}^{\lambda_{\mathrm{max}}} p_\lambda\,d\lambda, \quad J_\nu = \frac{1}{\pi}\int_{\nu_{\mathrm{min}}}^{\nu_{\mathrm{max}}} p_\nu\,d\nu.$$

[1] Note that the potential of the Perfect Ellipsoid [de Zeeuw 1985] is of the Stäckel form, but it is only one example of a much wider class of potentials.

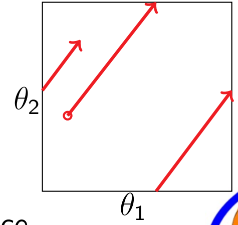
Stäckel fudge

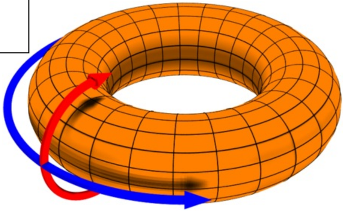
If one *pretends* that the actual galactic potential is of the Stäckel form, these expressions provide a good approximation to the true actions [Binney 2012], with a typical accuracy $\sim 1 - 10\%$.

This approach works reasonably well for most realistic axisymmetric galactic potentials, except when the orbit is a resonant one.

An alternative approach is based on the concept of canonical transformation, approximating the real actions with a convergent Fourier series [Sanders&Binney 2014; Bovy 2014], also known as *torus mapping* [McGill&Binney 1990; Kaasalainen 1994; McMillan&Binney 2008].
See Sanders&Binney 2016 for a general review.

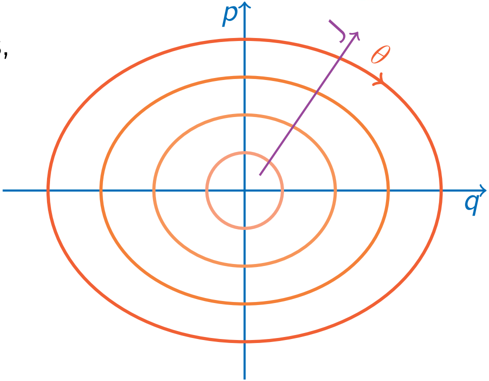
Invariant tori





In an integrable potential, the motion is multiperiodic in angles: $\theta_i + 2\pi \cong \theta_i$, restricted to a D-dimensional hypersurface of the $2D$-dimensional phase space.

Arnold–Liouville theorem:
this hypersurface is diffeomorphic to (i.e., could be smoothly deformed into) a D-torus, parametrized by D periodic variables $\boldsymbol{\theta} \in [0..2\pi)$.



The entire 6d phase space is foliated into non-intersecting 3d orbital tori.

Actions tell you which orbit the star is on, angles – where it is located on this orbit.

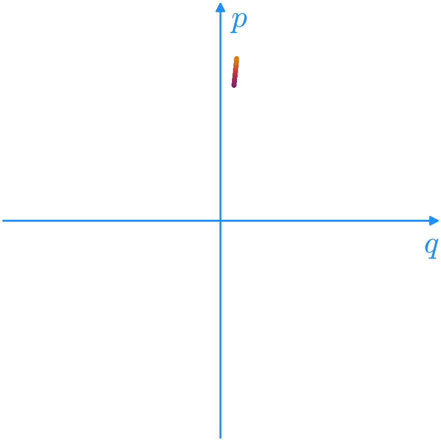
Invariant tori and phase mixing

Since the frequencies $\boldsymbol{\Omega} \equiv \dfrac{\partial H}{\partial \mathbf{J}} \neq \text{const}$, an initially localized ensemble of points eventually spreads out in angles (mixes in orbital phase) and fills the entire torus.

Thus in the time-averaged sense, only actions are important, and the distribution in angles is assumed to be uniform.

For chaotic orbits, the mixing is even more efficient.

However, at early stages of evolution (e.g., of a recently disrupted star cluster), the angle distribution is not yet mixed.



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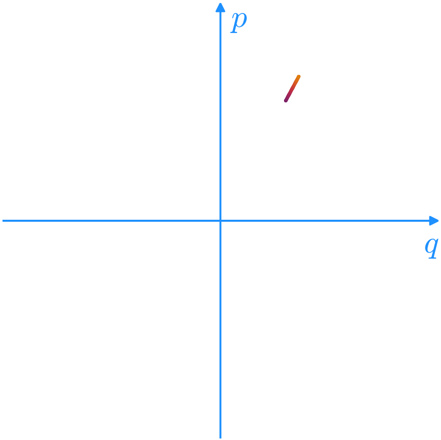

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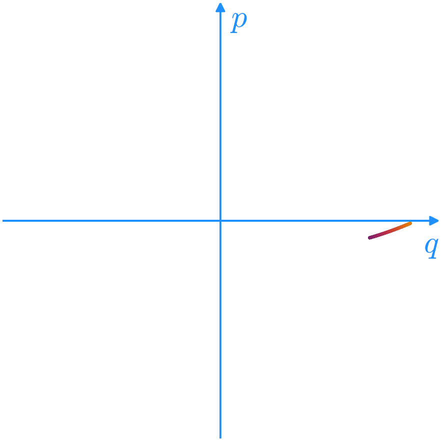

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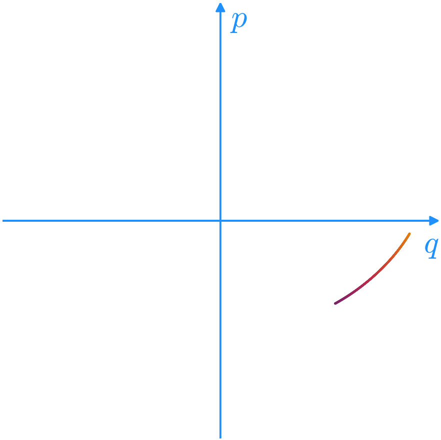

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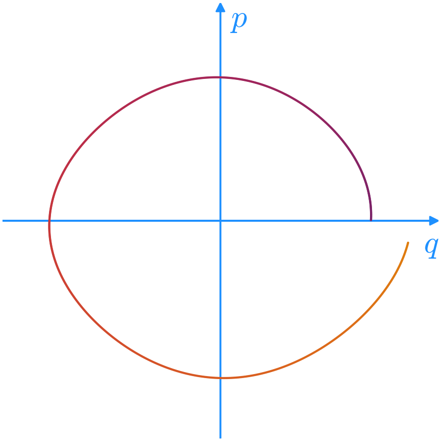

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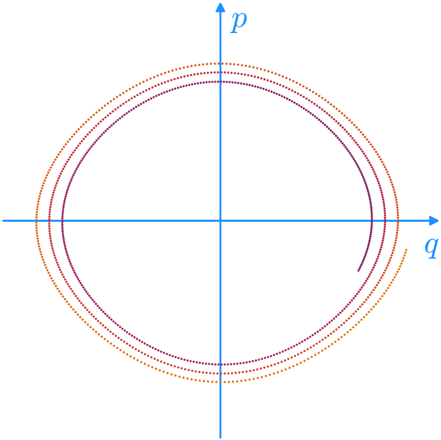

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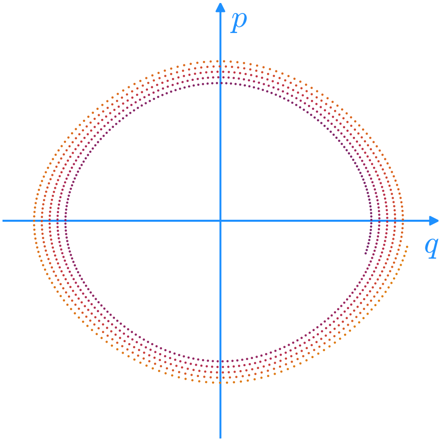

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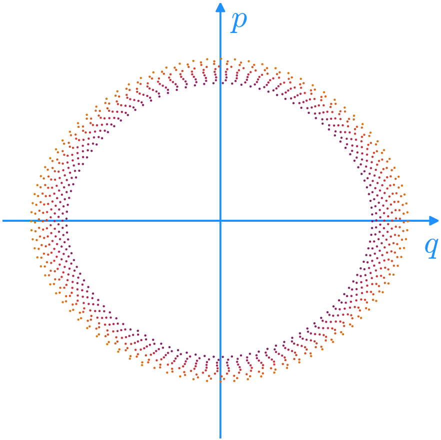
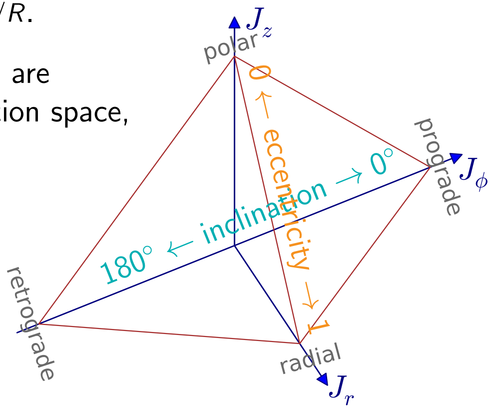


Fun facts / rules of thumb about actions

- Dimension of actions is length×velocity:
  if a star at a galactocentric distance r travels with velocity v, then [at least one of the actions] $J \sim r\,v$.
- Frequencies: $\Omega_i(\mathbf{J}) = \partial H/\partial J_i$
  characteristic velocity $v_i \sim \sqrt{\Omega_i\,J_i}$
  e.g., for a circular orbit $J_\phi = R\,v_\phi$, $\Omega_\phi = v_\phi/R$.
- Surfaces of constant energy $H(\mathbf{J}) = E$ are approximately tetrahedra in the 3d action space, with $E \approx E(\Omega_r\,J_r + \Omega_z\,J_z + \Omega_\phi\,J_\phi)$.



Advantages of action/angle variables

- Clear physical meaning (describe the extent of oscillations in each dimension).
- Most natural description of motion (angles change linearly with time).
- Possible range for each action variable is $[0..\infty)$ or $(-\infty..\infty)$, independently of the other ones (unlike E and L, say).
- Canonical coordinates $\Rightarrow$ the 6d phase-space volume element is $d^3x\; d^3v = d^3J\; d^3\theta$.
- Actions are adiabatic invariants (are conserved under slow variation of potential).
- Perturbation theory most naturally formulated in terms of actions.
- Efficient methods for conversion between $\{\mathbf{x}, \mathbf{v}\}$ and $\{\mathbf{J}, \boldsymbol{\theta}\}$ now exist.