Self-consistent models of our Galaxy in the Gaia era

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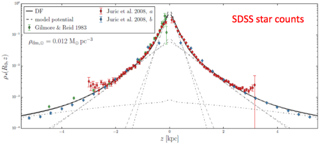
Strasbourg, March 2018

Data

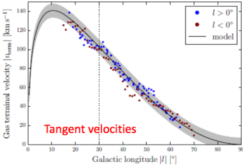
- star counts
- gas rotation curve
- tidal streams
- stellar kinematics
- chemical tags, ages

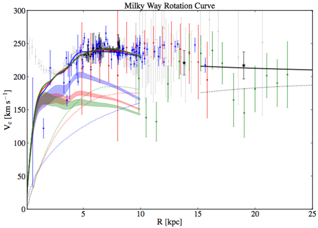
Model

- stellar density profile
- total gravitational potential; halo density profile and shape
- velocity distribution functions
- population synthesis

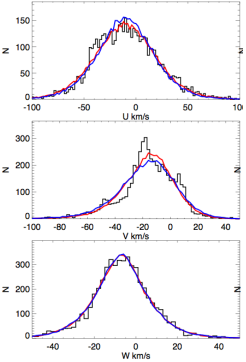


[Juric+ 2008]





[Bland-Hawthorn & Gerhard 2016]



[Sharma+ 2014]

Self-consistent models

- Stars are described by a distribution function f, which must depend only on the integrals of motion (Jeans theorem):

$$f = f\big(\mathcal{I}(\mathbf{x}, \mathbf{v})\big) \ , \quad \mathcal{I} = \{E, L, \dots\}.$$

depend on the potential Φ

- The density of stars is just the 0th moment of the distribution function:

$$\rho(\mathbf{x}) = \iiint d^3v \, f(\mathbf{x}, \mathbf{v}).$$

- The potential is related to the *total* density (stars + dark matter) through the Poisson equation:

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi \, G \, \rho(\mathbf{x}).$$

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Iterative approach

1. Assume a particular distribution function $f\left(\mathcal{I}\right)$;
2. Adopt an initial guess for $\Phi(\mathbf{x})$;
3. Establish the integrals of motion $\mathcal{I}(\mathbf{x},\mathbf{v})$ in this potential;
4. Compute the density $\rho(\mathbf{x}) = \iiint d^3v\ f\left(\mathcal{I}(\mathbf{x},\mathbf{v})\right)$;
5. Solve the Poisson equation to find the new potential $\Phi(\mathbf{x})$;
6. Repeat until convergence.

Origin: Prendergast & Tomer 1970;

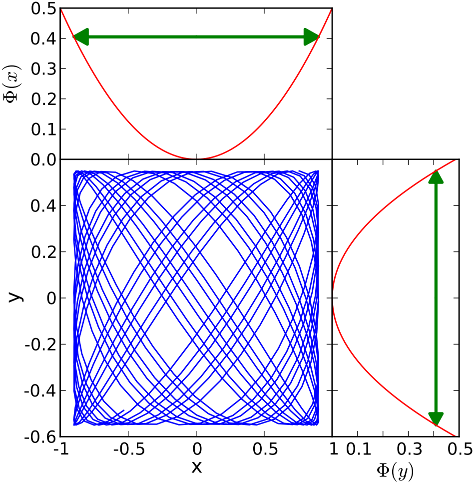
used in Kuijken & Dubinski 1995, Widrow+ 2008, Taranu+ 2017 (GalactICs),

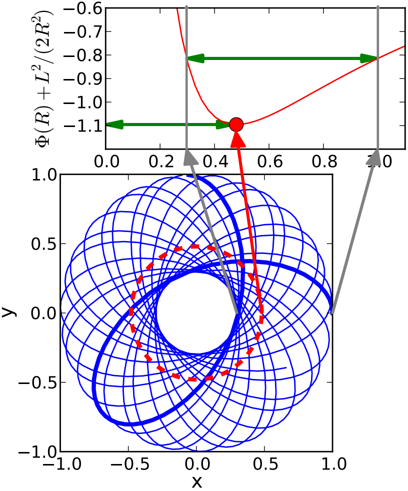
Piffl+ 2014, Cole & Binney 2016, Sanders & Evans 2016 (action-based formalism).

Actions as integrals of motion

One may use any set of integrals of motion, **but** actions are special:

$J = \frac{1}{2\pi} \oint \mathbf{p} \; d\mathbf{x}$, where **p** are canonically conjugate momenta for **x**





Advantages of action/angle variables

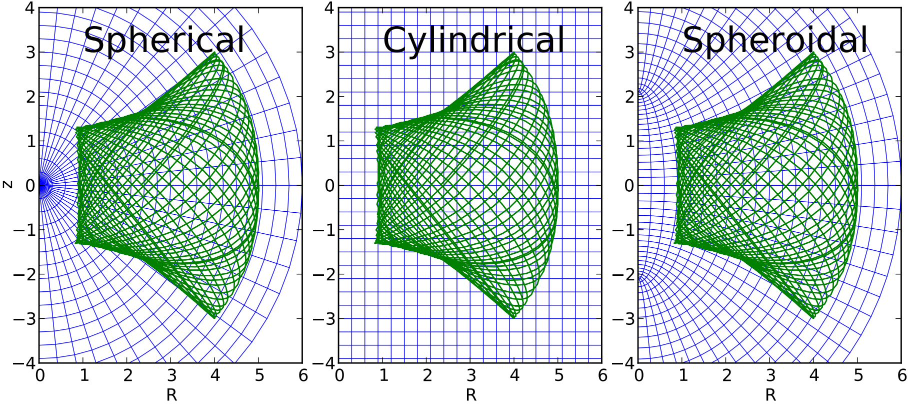
- Clear physical meaning
  (describe the extent of oscillations in each dimension).
- Most natural description of motion (angles change linearly with time).
- Possible range for each action variable is $[0..\infty)$ or $(-\infty..\infty)$, independently of the other ones (unlike E and L, say).
- Canonical coordinates $\Rightarrow$ total mass is computed trivially
  $M = \int f\big(\mathbf{x},\mathbf{v}\big)\; d^3x\; d^3v = \int f\big(\mathbf{J}\big)\; d^3J\; d^3\theta = \int f\big(\mathbf{J}\big)\; d^3J\; (2\pi)^3$,
  does not depend on Φ, does not change between iterations.
- Actions are adiabatic invariants (are conserved under slow variation of potential) $\Rightarrow$ easy to construct multicomponent models.
- Serve as a good starting point in perturbation theory.
- Efficient methods for conversion between $\{\mathbf{x},\mathbf{v}\}$ and $\{\mathbf{J},\boldsymbol{\theta}\}$ exist
  (e.g., Stäckel fudge, Binney 2012, or Torus machine, Binney & McMillan 2016).

"Classical" methods

- Spherical systems:
  two of the actions can be taken to be the *azimuthal action* $J_\phi \equiv L_z$ and the *latitudinal action* $J_\vartheta \equiv L - |L_z|$;
  the third one (the *radial action*) is given by a 1d quadrature:
  $$J_r = \frac{1}{\pi} \int_{r_\mathrm{min}}^{r_\mathrm{max}} dr \, \sqrt{2[E - \Phi(r)] - L^2/r^2},$$
  where $r_\mathrm{min}, r_\mathrm{max}$ are the peri- and apocentre radii.
  Angles are given by 1d quadratures. For special cases (the isochrone potential, and its limiting cases – Kepler and harmonic potentials), these integrals are computed analytically.
  Note: a related concept in celestial mechanics are the Delaunay variables.
- Flattened axisymmetric systems – the **epicyclic approximation**: motion close to the disk plane is nearly separable into the in-plane motion (J_ϕ and J_r as in the spherical case) and the vertical oscillation with a fixed energy E_z in a nearly harmonic potential (J_z).

State of the art: Stäckel fudge

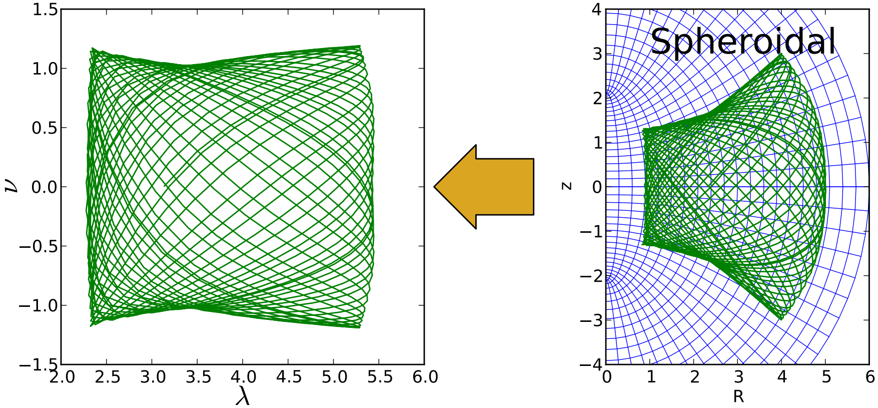
Fact: orbits in realistic axisymmetric galactic potentials are much better aligned with prolate spheroidal coordinates.



State of the art: Stäckel fudge

Fact: orbits in realistic axisymmetric galactic potentials are much better aligned with prolate spheroidal coordinates.

One may explore the assumption that the motion is separable in these coordinates (λ, ν).



Stäckel fudge (Binney 2012)

The most general form of potential that satisfies the separability condition is the Stäckel potential[1]: $\Phi(\lambda, \nu) = -\dfrac{f_1(\lambda) - f_2(\nu)}{\lambda - \nu}$.

The motion in λ and ν directions, with canonical momenta p_λ, p_ν, is governed by two separate equations:

$$2(\lambda - \Delta^2)\,\lambda\, p_\lambda^2 = \left[E - \frac{L_z^2}{2(\lambda - \Delta^2)}\right]\lambda - [I_3 + (\lambda - \nu)\Phi(\lambda, \nu)],$$
$$2(\nu - \Delta^2)\,\nu\, p_\nu^2 = \left[E - \frac{L_z^2}{2(\nu - \Delta^2)}\right]\nu - [I_3 + (\nu - \lambda)\Phi(\lambda, \nu)].$$

Under the approximation that the separation constant I_3 is indeed conserved along the orbit, this allows to compute the actions:

$$J_\lambda = \frac{1}{\pi}\int_{\lambda_{\min}}^{\lambda_{\max}} p_\lambda\, d\lambda, \quad J_\nu = \frac{1}{\pi}\int_{\nu_{\min}}^{\nu_{\max}} p_\nu\, d\nu.$$

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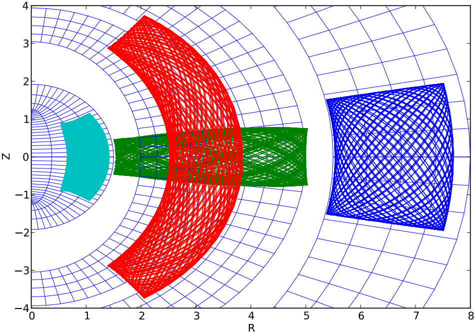
[1] Note that the potential of the Perfect Ellipsoid (de Zeeuw 1985) is of the Stäckel form, but it is only one example of a much wider class of potentials.

Stäckel fudge in practice

A rather flexible approximation: for each orbit, we have the freedom of using two functions $f_1(\lambda)$, $f_2(\nu)$ (directly evaluated from the actual potential $\Phi(R, z)$) to describe the motion in two independent directions.

These functions are different for each orbit (implicitly depend on E, L_z, I_3).

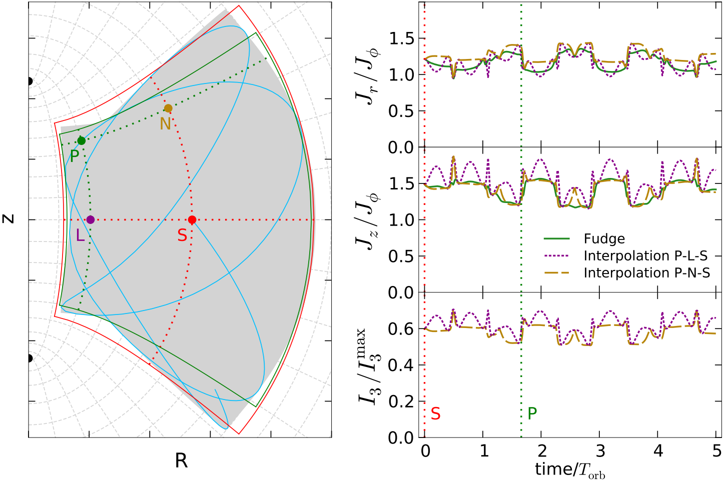
Moreover, we may choose the interfocal distance Δ of the auxiliary prolate spheroidal coordinate system for each orbit independently.



Accuracy of Stäckel fudge

Accuracy of action conservation using the Stäckel fudge:
$\lesssim 1\%$ for most disk orbits, $\lesssim 10\%$ even for high-eccentricity orbits.

Interpolation of J_r, J_z on a 3d grid of E, L_z, I_3:
10x speed-up at the expense of a moderate decrease in accuracy.



How to compute the potential

1. Direct integration:

$$\Phi(\mathbf{x}) = -\iiint d^3x'\,\rho(\mathbf{x}') \times \frac{G}{|\mathbf{x}-\mathbf{x}'|}.$$

2. Azimuthal harmonic expansion:

$$\Phi(R,z,\phi) = \sum_{m=-\infty}^{\infty} \Phi_m(R,z)\,\mathrm{e}^{im\phi}.$$

interpolated functions

3. Spherical harmonic expansion:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \Phi_{lm}(r)\,Y_l^m(\theta,\phi).$$

4. Basis-set expansion:

$$\Phi(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \Phi_{nlm}\,A_{nl}(r)\,Y_l^m(\theta,\phi).$$

(example: self-consistent field method of Hernquist&Ostriker 1992)

How to compute the potential of a spheroical system

3. Spherical-harmonic expansion:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi_{lm}(r)\, Y_l^m(\theta,\phi),$$

$$\Phi_{lm}(r) = -\frac{4\pi G}{2l+1} \times$$
$$\times \left[r^{-1-l} \int_0^r dr'\, \rho_{lm}(r')\, r'^{\,l+2} + r^l \int_r^\infty dr'\, \rho_{lm}(r')\, r'^{\,1-l} \right],$$

$$\rho_{lm}(r) = \int_0^\pi d\theta \int_0^{2\pi} d\phi\, \rho(r,\theta,\phi)\, Y_l^{m*}(\theta,\phi).$$

How to compute the potential of a flattened system

2. Azimuthal-harmonic (Fourier) expansion:

$$\Phi(R,z,\phi) = \sum_{m=-\infty}^{\infty} \Phi_m(R,z)\, \mathrm{e}^{im\phi},$$

$$\rho_m(R,z) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\; \rho(R,z,\phi)\mathrm{e}^{-im\phi},$$

$$\Phi_m(R,z) = -\iint dR'\, dz'\, \rho_m(R',z') \times \Xi_m(R,z,R',z'),$$

analytic expr. for Green's function:

$$\Xi_m(R,z,R',z') \equiv \int_0^{\infty} dk\; 2\pi G\; J_m(kR)\, J_m(kR')\, \exp(-k|z-z'|) =$$

$$= \frac{2\sqrt{\pi}\,\Gamma\left(m+\frac{1}{2}\right)\; {}_2F_1\left(\frac{3}{4}+\frac{m}{2},\; \frac{1}{4}+\frac{m}{2};\; m+1;\; \xi^{-2}\right)}{\sqrt{RR'}\,(2\xi)^{m+1/2}\,\Gamma(m+1)}$$

$$\text{where } \xi \equiv \frac{R^2+R'^2+(z-z')^2}{2RR'}.$$

Gravitational potential extracted from N-body models

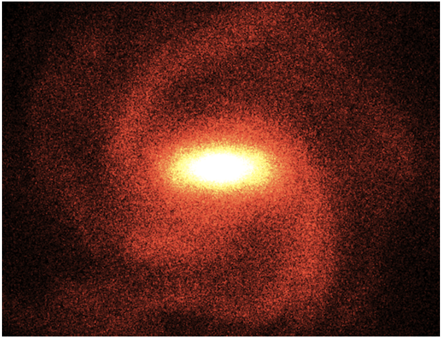
The spherical-harmonic and azimuthal-harmonic potential approximations can also be constructed from N-body models.

Advantages:

fast evaluation, smooth forces, suitable for orbit analysis.

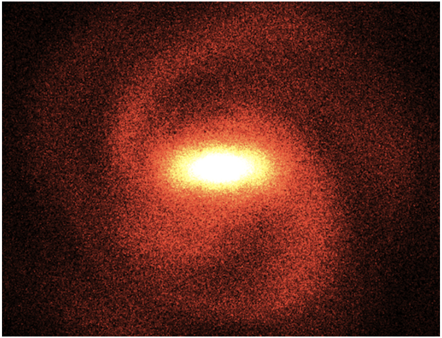
Real N-body model

(from Roca-Fabrega et al. 2013, 2014)



Potential approximation

(suitable for test-particle integrations, e.g. Romero-Gomez et al. 2011)



Distribution functions in action space

- Spheroidal components (halo, bulge): double-power-law DF
  [Binney 2014, Posti+ 2015, Williams & Evans 2015]

$$f(\mathbf{J}) = \frac{M}{(2\pi\, J_0)^3} \left(\frac{h(\mathbf{J})}{J_0}\right)^{-\Gamma} \left[1 + \left(\frac{g(\mathbf{J})}{J_0}\right)^{\eta}\right]^{\frac{\Gamma-B}{\eta}} \exp\left[-\left(\frac{g(\mathbf{J})}{J_\mathrm{cut}}\right)^{\zeta}\right] \left(1 + \varkappa \tanh \frac{J_\phi}{J_{\phi,0}}\right),$$

$$g(\mathbf{J}) \equiv g_r J_r + g_z J_z + g_\phi |J_\phi|, \quad h(\mathbf{J}) \equiv h_r J_r + h_z J_z + h_\phi |J_\phi|$$

- Disk components: quasi-isothermal DF [Binney & McMillan 2011]

$$f(\mathbf{J}) = \frac{\tilde\Sigma\,\Omega}{2\pi^2\,\kappa^2} \times \frac{\kappa}{\tilde\sigma_r^2} \exp\left(-\frac{\kappa\, J_r}{\tilde\sigma_r^2}\right) \times \frac{\nu}{\tilde\sigma_z^2} \exp\left(-\frac{\nu\, J_z}{\tilde\sigma_z^2}\right) \times \begin{cases} 1 & \text{if } J_\phi \ge 0, \\ \exp\left(\frac{2\Omega\, J_\phi}{\tilde\sigma_r^2}\right) & \text{if } J_\phi < 0, \end{cases}$$

$$\tilde\Sigma(R_c) \equiv \Sigma_0 \exp\left(-\tfrac{R_c}{R_\mathrm{disk}}\right), \quad \tilde\sigma_r^2(R_c) \equiv \sigma_{r,0}^2 \exp\left(-\tfrac{2R_c}{R_{\sigma,r}}\right), \quad \tilde\sigma_z^2(R_c) \equiv 2\, h_\mathrm{disk}^2\, \nu^2(R_c).$$

- Alternative disk DF (exponential):

$$f(\mathbf{J}) = \tfrac{M}{(2\pi)^3}\, \tfrac{J}{J_{\phi,0}^2}\, \exp\left(-\tfrac{J}{J_{\phi,0}}\right) \times \tfrac{J}{J_{r,0}^2} \exp\left(-\tfrac{J\,J_r}{J_{r,0}^2}\right) \times \tfrac{J}{J_{z,0}^2} \exp\left(-\tfrac{J\,J_z}{J_{z,0}^2}\right) \times \begin{cases} 1 & \text{if } J_\phi \ge 0 \\ \exp\left(\tfrac{J\,J_\phi}{J_{r,0}^2}\right) \end{cases}$$

Construction of self-consistent models

Modelling procedure:

- Assume the parameters for the stellar and dark matter DFs
- Iteratively find the self-consistent potential/density corresponding to this DF:
  - Assume an initial guess for the potential
  - Initialize the action mapper for this potential
  - Recompute the density by integrating the DFs over velocity
  - Recompute the potential
- Compute the likelihood of the model given the data (compare the velocity distributions, microlensing depth, rotation curve)
- Adjust the parameters of the DFs

5 – 10 iter.

The result: ~ 15 parameters of DFs (mass, scale lengths and heights, velocity dispersions, etc.) and the final self-consistent potential.

Self-consistent models for the Milky Way

Observational constraints:

- gas terminal velocities [Malhorta 1995]
- masers with 6d phase-space coords [Reid+ 2014]
- proper motion of SgrA* [Reid & Brunthaler 2004]
- vertical density profile in the Solar neighborhood [Jurić+ 2008]
- kinematics of local stars from RAVE [Kordopatis+ 2013] and Gaia
- microlensing depth towards Galactic bulge [Sumi & Penny 2016]

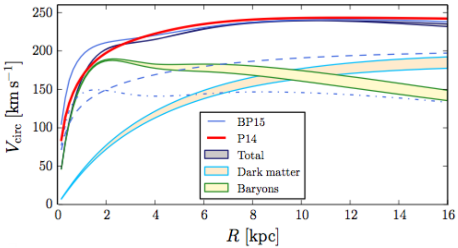
Likelihood analysis of discrete kinematic data:

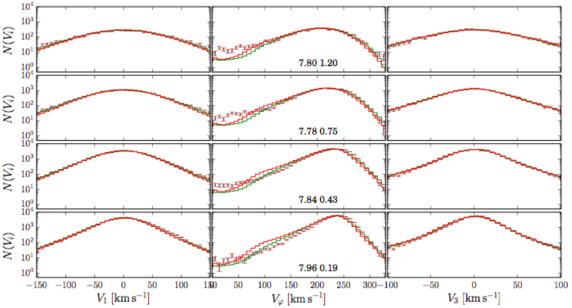
$$\mathcal{L} = \prod_{\text{stars}} \frac{\int \mathrm{d}\underbrace{w'}_{\text{true coordinates}}\ \underbrace{E}_{\text{error distribution}}(\underbrace{\mathbf{w}_s}_{\text{observed coordinates}}|\mathbf{w}')\ \underbrace{f}_{\text{distribution function}}\big(\mathbf{J}[\mathbf{w}']\big)\ \underbrace{S}_{\text{selection function}}(\mathbf{w})}{\underbrace{\int \mathrm{d}w'\ f\big(\mathbf{J}[\mathbf{w}']\big)\ S(\mathbf{w}')}_{\text{normalization factor}}}$$

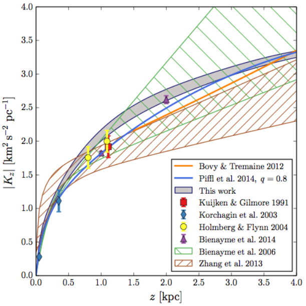
Self-consistent models for the Milky Way

[Cole & Binney 2016]

using the previous implementation







Advantages of models based on distribution function

- Clear physical meaning
  (localized structures in the space of integrals of motion);
- Easy to compare different models
  (how to compare two N-body or N-orbit models?);
- Easy to compare models to discrete observational data;
- Easy to sample particles from the distribution function
  (convert to an N-body model);
- Stability analysis
  (perturbation theory most naturally formulated in terms of actions);

Caveats:

- Implicitly rely on the integrability of the potential,
  ignore the presence of resonant orbit families (but see Binney 2017);
- So far implemented only for axisymmetric models
  (not a fundamental limitation).

AGAMA library – All-purpose galaxy modeling architecture

- Extensive collection of gravitational potential models
  (analytic profiles, azimuthal- and spherical-harmonic expansions)
  constructed from smooth density profiles or N-body snapshots;
- Conversion to/from action/angle variables;
- Self-consistent multicomponent models with action-based DFs;
- Schwarzschild orbit-superposition models;
- Generation of initial conditions for N-body simulations;
- Various math tools: 1d,2d,3d spline interpolation, penalized spline fitting and density estimation, multidimensional sampling;
- Efficient and carefully designed C++ implementation, examples, Python and Fortran interfaces, plugins for Galpy, NEMO, AMUSE.

`https://github.com/GalacticDynamics-Oxford/Agama`

Outlook

- Wealth of observational data calls for adequate modelling approaches
- State-of-the-art self-consistent models based on distribution functions in action space
- Work in progress on incorporating data from other surveys such as APOGEE, LAMOST, and eventually Gaia DR2
- Software available for the community

`arXiv:1802.08239, 1802.08255`

`https://github.com/GalacticDynamics-Oxford/Agama`