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Generalized compound matrix method

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Abstract

In this work we demonstrate how the extension of the Evans function method using the compound matrix approach can be implemented to undertake the stability analysis (normally done through numerical means) of nonlinear travelling waves. The main advantage of this approach is that it can easily overcome the stiffness which is normally associated with these kinds of problems. We present a general approach which allows this method to be used for a general class of nonlinear travelling wave problems.

Keywords: Linear stability; Evans function; Compound matrix method; Stiff equations

1. Introduction

The problem of the stability of a travelling wave is considered. Linearization about the travelling wave often results in the following linear stability problem:

$$\mathbf{z}_\xi = \mathbf{A}(\xi, \lambda)\mathbf{z}, \tag{1}$$

where $\mathbf{z}$ is an n dimensional vector representing the linear perturbation, $\mathbf{A}$ is an $n \times n$ matrix whose entries are functions of the coordinate ξ and parameters λ. The physical background of the problems

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usually requires the coordinates of vector $\mathbf{z}$ to decay exponentially as $\xi \to \pm\infty$. Examples of such behaviour can be found in many physical, chemical and biological phenomena, modelled by nonlinear Partial Differential Equations (PDEs) exhibiting travelling wave solutions, such as combustion waves, optical solitons, chemical reactions, propagation of dominant genes and nerve pulses [1–3]. For this reason it is useful to define the limit problems as

$$\mathbf{z}_\xi = \mathbf{A}^{\pm}(\lambda)\mathbf{z}, \tag{2}$$

where

$$\mathbf{A}^{\pm}(\lambda) = \lim_{\xi\to\pm\infty} \mathbf{A}(\xi, \lambda). \tag{3}$$

The properties of Eq. (2) are used in order to define the boundary conditions for (1) explicitly. Suppose that n is an even number and matrices $\mathbf{A}^{\pm}$ have $n/2$ eigenvalues with positive real parts and $n/2$ eigenvalues with negative real parts so that

$$\begin{aligned} &\text{Re}\ \mu_i^{\pm} > 0 \quad \text{for } 1 < i \le n/2, \\ &\text{Re}\ \mu_i^{\pm} < 0 \quad \text{for } n/2 < i \le n, \end{aligned} \tag{4}$$

and

$$\text{Re}\ \mu_i^{\pm} \le \text{Re}\ \mu_{i+1}^{\pm} \quad \text{for} \quad 1 \le i < n. \tag{5}$$

Let us define the corresponding eigenvectors as $\mathbf{k}_i^{\pm}$. In this case (1) has $n/2$ solutions bounded for $\xi \to -\infty$ and satisfying

$$\mathbf{k}_i^{-} = \lim_{\xi\to-\infty} \mathrm{e}^{-\mu_i^{-}\xi} z_i(\xi), \quad 1 < i \le n/2, \tag{6}$$

and $n/2$ solutions bounded for $\xi \to +\infty$ and satisfying

$$\mathbf{k}_i^{+} = \lim_{\xi\to+\infty} \mathrm{e}^{-\mu_i^{+}\xi} z(\xi), \quad n/2 < i \le n. \tag{7}$$

The assumption that the limit matrices $\mathbf{A}^{+}$ and $\mathbf{A}^{-}$ defined in Eqs. (2) and (3) have dimensions n which are even numbers and have exactly $p = n/2$ eigenvalues with positive real parts is taken for the sake of simplicity. Although for many problems described by a system of PDEs of the second order such as the propagation of combustion waves in [4] this is certainly the case (n is even and $p = n/2$), for other problems like the KdV–Burgers equation [10] (which is a PDE of the third order, so $n = 3$) the above assumption does not hold. It will be seen later on that the algorithm proposed in this work applies for the general case of arbitrary n and p.

In order to investigate the linear stability of the travelling waves the solutions have to be found, which can often only be done numerically. However direct numerical integration of (1) finds only solutions $\mathbf{z}_1$ and $\mathbf{z}_n$ corresponding to the maximum rate of exponential growth μ_n^{-} and μ_1^{+} as we integrate forwards from $-\infty$ to 0 or backwards from ∞ to 0 (strictly speaking we integrate from $-L_1$ to 0 and from L_2 to 0 where $L_{1,2}$ are large positive numbers). This is due to the stiffness of (1) (see [4] and references therein for the detailed description). In order to find other solutions ($\mathbf{z}_i$ for $i \ne 1, n$) some orthogonalization procedure has to be introduced. One way to determine these solutions is to use the compound matrix method [5–7].

The compound matrix method was used in [5,6] to integrate the Orr–Sommerfeld equation which is a fourth-order differential equation (therefore it can be represented just as a system of four differential

equations of first order). The Orr–Sommerfeld equation is an important equation in the area of hydrodynamics [7]. It is also an example of a stiff equation which requires orthogonalization to be undertaken to maintain the linear independence of the solutions. At present the compound matrix method is widely used to calculate the Evans function [4,8], which is now a standard tool in spectral theory for calculating the unstable eigenvalues of the linear differential operators. There are explicit algorithms for implementing the compound matrix method for the system (1). However this approach is restricted by the number of governing equations; that is, the number of equations must be less than six [5,6]. Here we generalize this approach for systems containing an arbitrary number of equations. We describe our generalized approach in the next section.

2. Compound matrix method

Following [8] we use the exterior algebra formalism [9] which allows us to present the results in a more compact form compared to that of [5,6].

According to the compound matrix method, in order to find $\mathbf{z}_i \in \mathbf{C}^n$ instead of integrating the Eq. (1) subject to the boundary conditions (6) and (7) n times, the linear stability problem (1) is considered in the space of p-vectors $\mathbf{z}_1 \wedge \mathbf{z}_2 \wedge \cdots \wedge \mathbf{z}_p \in \bigwedge^p(\mathbf{C}^n)$, where $\wedge$ represents the wedge product, $\bigwedge^p(\mathbf{C}^n)$ is the p-th exterior power of the vector space $\mathbf{C}^n$ (where $p = n/2$ in our case). The properties of $\bigwedge^p(\mathbf{C}^n)$ are described in [9]. In particular, it is shown that if $\{\mathbf{e}_i\}$, $i = 1, \ldots, n$, is the orthonormal basis in $\mathbf{C}^n$, then

$$\mathbf{g}_j = \mathbf{e}_{i(1)} \wedge \mathbf{e}_{i(2)} \wedge \cdots \wedge \mathbf{e}_{i(p)}, \quad 1 \le i(1) < i(2) < \cdots < i(k) \le n \tag{8}$$

form an orthonormal basis in the space of p-vectors of a dimension given as

$$\dim \bigwedge^p(\mathbf{C}^n) = \frac{n!}{(n-p)!p!}. \tag{9}$$

It can be shown [8] that the p-vectors $\mathbf{v} = \mathbf{z}_1 \wedge \mathbf{z}_2 \wedge \cdots \wedge \mathbf{z}_p$ satisfy the equations

$$\mathbf{v}_\xi = \mathbf{B}(\xi, \lambda)\mathbf{v}, \tag{10}$$

where $\mathbf{B}$ is the induced matrix. The induced boundary conditions are given as

$$\mathbf{k}_1^- \wedge \cdots \wedge \mathbf{k}_p^- = \lim_{\xi \to -\infty} \mathrm{e}^{-s^-\xi}\mathbf{v}(\xi), \tag{11}$$

instead of (6) and

$$\mathbf{k}_{p+1}^+ \wedge \cdots \wedge \mathbf{k}_n^+ = \lim_{\xi \to +\infty} \mathrm{e}^{-s^+\xi}\mathbf{v}(\xi), \tag{12}$$

instead of (7). Here $s^- = \mu_1^- + \cdots + \mu_p^- > 0$ and $s^+ = \mu_{p+1}^+ + \cdots + \mu_n^+ < 0$. The p-vector $\mathbf{v}$ satisfying the Eq. (10) and boundary conditions (11) represents the subspace of solutions $\mathbf{z}_i$ $(i = 1, \ldots, p)$ of the linear stability problem (1) bounded for $\xi \to -\infty$ and satisfying (6). Similarly, the p-vector $\mathbf{v}$ satisfying Eq. (10) and boundary conditions (12) represents the subspace of solutions $\mathbf{z}_i$ $(i = p+1, \ldots, n)$ of the linear stability problem (1) bounded for $\xi \to \infty$ and satisfying (7). Therefore to find the subspace of (1) solutions bounded for $\xi \to -\infty$ we integrate (10) subject to (11) from $-\infty$ to 0. In other words we have to find a single solution of (10) which corresponds to the largest rate of exponential growth, s^-, when integrating forward from $-\infty$ as shown in [4]. This can always be done numerically by using the

Table 1
Structure of sets $\{i(1),\ldots,i(p)\}$ and $\{j(1),\ldots,j(p)\}$ in the case of a single different number $i(k)\neq j(k)$

$k-1$	k	$k+1$
$i(k-1)$	$i(k)$	$i(k+1)$
$i(k-1)$	$j(k)$	$i(k+1)$

technique introduced in [10]. Similarly, to determine the space of solutions to (10) bounded for $\xi\to\infty$ and satisfying (7), Eq. (10) are integrated backward from ∞ to 0 subject to (12).

In order to investigate the stability of the solution it is sufficient to find two solutions of (10) at $\xi=0$, the first solution subject to initial condition (11) and the second subject to initial condition (12). Then the Evans function is constructed and the linear stability problem is solved by employing the technique described in [4]. Therefore to carry out the stability analysis, induced matrix $\mathbf{B}$ has to be constructed. In [5,6] the induced matrix is given explicitly in two cases: $n=4$, $p=2$ and $n=6$, $p=3$. In this work we present the algorithm for constructing this induced matrix in the general case.

We consider matrix $\mathbf{B}$ in the basis $\{\mathbf{g}_i\}$. In [8] the elements of $\mathbf{B}$ are shown to be

$$\begin{aligned}\mathbf{B}_{nm}&=\langle\mathbf{g}_n|\mathbf{B}\mathbf{g}_m\rangle=\langle\mathbf{e}_{i(1)}\wedge\mathbf{e}_{i(2)}\wedge\cdots\wedge\mathbf{e}_{i(p)}|\mathbf{A}\mathbf{e}_{j(1)}\wedge\mathbf{e}_{j(2)}\wedge\cdots\wedge\mathbf{e}_{j(p)}\rangle\\&\quad+\langle\mathbf{e}_{i(1)}\wedge\mathbf{e}_{i(2)}\wedge\cdots\wedge\mathbf{e}_{i(p)}|\mathbf{e}_{j(1)}\wedge\mathbf{A}\mathbf{e}_{j(2)}\wedge\cdots\wedge\mathbf{e}_{j(p)}\rangle+\cdots\\&\quad+\langle\mathbf{e}_{i(1)}\wedge\mathbf{e}_{i(2)}\wedge\cdots\wedge\mathbf{e}_{i(p)}|\mathbf{e}_{j(1)}\wedge\mathbf{e}_{j(2)}\wedge\cdots\wedge\mathbf{A}\mathbf{e}_{j(p)}\rangle\\&=\sum_{k=1}^{p}\langle\mathbf{e}_{i(1)}\wedge\cdots\wedge\mathbf{e}_{i(p)}|\mathbf{e}_{j(1)}\wedge\cdots\wedge\mathbf{A}\mathbf{e}_{j(k)}\wedge\cdots\wedge\mathbf{e}_{j(p)}\rangle,\end{aligned}\tag{13}$$

where $\langle\cdot|\cdot\rangle$ denotes the scalar product, $\mathbf{g}_n=\mathbf{e}_{i(1)}\wedge\mathbf{e}_{i(2)}\wedge\cdots\wedge\mathbf{e}_{i(p)}$ and $\mathbf{g}_m=\mathbf{e}_{j(1)}\wedge\mathbf{e}_{j(2)}\wedge\cdots\wedge\mathbf{e}_{j(p)}$. The latter expression in (13) can be rewritten as

$$\mathbf{B}_{nm}=\sum_{k=1}^{p}\sum_{s=1}^{n}a_{s,j(k)}\langle\mathbf{e}_{i(1)}\wedge\cdots\wedge\mathbf{e}_{i(p)}|\mathbf{e}_{j(1)}\wedge\cdots\wedge\mathbf{e}_{s}\wedge\cdots\wedge\mathbf{e}_{j(p)}\rangle,\tag{14}$$

where $\mathbf{A}=\|a_{i,k}\|$. According to [9], terms under the double sum in (14) do not vanish if and only if: (i) all numbers $j(1),j(2),\ldots,s,\ldots,j(k)$ are different; (ii) the sets of numbers $\{i(1),\ldots,i(p)\}$ and $\{j(1),\ldots,s,\ldots,j(p)\}$ are identical (which would represent the same vector $\mathbf{g}_n$). This is possible only in several cases which we will consider next.

Firstly, this occurs when $n=m$ ($\{i(1),\ldots,i(p)\}$ and $\{j(1),\ldots,j(p)\}$ are identical). In this case instead of (14) we have

$$\mathbf{B}_{nn}=\sum_{k=1}^{p}a_{i(k),i(k)}.\tag{15}$$

The second option is when $n\neq m$ so that all numbers $\{i(1),\ldots,i(p)\}$ are the same as $\{j(1),\ldots,j(p)\}$ except one and we let it be the k-th number, i.e. $i(k)\neq j(k)$. This situation is illustrated in Table 1, where we show a part of set $\{i(1),\ldots,i(p)\}$ (second row) and a part of set $\{j(1),\ldots,j(p)\}$ (third row) which are different. In the third row we have replaced all $j(s)$ with $i(s)$ in cases where they are equal.

In this case the corresponding matrix element is given as

$$\mathbf{B}_{nm}=a_{i(k),j(k)}.\tag{16}$$

Table 2
Structure of sets $\{i(1), \ldots, i(p)\}$ and $\{j(1), \ldots, j(p)\}$ in the case of a single different number $i(k) \neq j(k)$

$k-1$	k	$k+1$	$k-1$	k	$k+1$
$i(k-1)$	$i(k)$	$i(k+1)$	$i(k-1)$	$i(k)$	$i(k+1)$
$i(k)$	$j(k)$	$i(k+1)$	$i(k-1)$	$j(k)$	$i(k)$

Besides this there are also two other configurations such that all numbers $\{i(1), \ldots, i(p)\}$ are the same as $\{j(1), \ldots, j(p)\}$ except one. They are illustrated schematically in Table 2. The first configuration results in the following matrix elements:

$$\mathbf{B}_{nm} = -a_{i(k-1),\,j(k)}, \tag{17}$$

whereas the latter configuration yields

$$\mathbf{B}_{nm} = -a_{i(k+1),\,j(k)}. \tag{18}$$

In all other cases, i.e. when sets $\{i(1), \ldots, i(p)\}$ and $\{j(1), \ldots, j(p)\}$ differ by two or more numbers, the resulting matrix elements vanish.

System (1) is usually solved numerically since the original nonlinear problem yielding (1) after the linearization process can be integrated only numerically. In other words, the nonlinear travelling wave solution is found numerically and the matrix $\mathbf{A}(\xi_i, \lambda)$ is defined on some grid points $\{\xi_i\}$ for a given set of parameter values. In this case matrix elements $\mathbf{B}_{n,m}(\xi_i, \lambda)$ can be determined on each of the grid points by using the following scheme: (i) we list all possible elements of both set $\{i(1), \ldots, i(p)\}$ corresponding to m and $\{j(1), \ldots, j(p)\}$ corresponding to n; (ii) by comparing these elements and using formulas (15)–(18) we determine all nonzero elements of $\mathbf{B}$.

As an example let us consider how matrix $\mathbf{B}_{ij}$ is constructed in the case $n = 4$ (i.e. vector $\mathbf{z}$ in (1) is four dimensional) and $p = 2$ (i.e. system (1) has a two dimensional subspace of solutions bounded for $\xi \to -\infty$ (and $+\infty$)). Firstly, we list all possible elements of set $\{i(1), i(2)\}$ corresponding to m, where m is an index of the basis vector $\mathbf{g}_m$ in (8) and $i(1, 2)$ is index of the vectors $\mathbf{e}_{i(1,2)}$ in (8). These are: $\{1, 2\}$ for $m = 1$, $\{1, 3\}$ for $m = 2$, $\{1, 4\}$ for $m = 3$, $\{2, 3\}$ for $m = 4$, $\{2, 4\}$ for $m = 5$, $\{3, 4\}$ for $m = 6$. Now we are ready to determine elements of $\mathbf{B}_{mn}$. For brevity let us consider only the elements of the first row, i.e. $\mathbf{B}_{1n}$. According to (15) $\mathbf{B}_{11} = a_{11} + a_{22}$. To find $\mathbf{B}_{12}$ we compare $\{1, 2\}$ and $\{1, 3\}$ corresponding to $n = 2$. They differ in one index $i(2) = 2 \neq j(2) = 3$ and it follows from (16) that $\mathbf{B}_{12} = a_{23}$. Similarly, comparing $\{1, 2\}$ and $\{1, 4\}$ ($n = 3$) we obtain $\mathbf{B}_{13} = a_{24}$. Next, $n = 4$ corresponds to $\{2, 3\}$ and it differs from $\{1, 2\}$ in one index $i(1) = 1 \neq j(2) = 3$. Therefore, according to (17), $\mathbf{B}_{14} = -a_{13}$. Similarly, for $n = 5$ corresponding to $\{2, 4\}$ we have $\mathbf{B}_{15} = -a_{14}$. For the last element in this row $n = 6$ and $\{3, 4\}$ differs from $\{1, 2\}$ in two indexes; therefore, $\mathbf{B}_{16} = 0$. Elements of the other rows can be constructed in exactly the same manner to give

$$\mathbf{B} = \begin{pmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{pmatrix}. \tag{19}$$

Such an algorithm can be easily implemented in standard packages like Matlab, Mathematica, Maple, or programmed in C++, Fortran etc. After the elements of **B** have been determined for all grid points ξ_i, we can find the solutions of (10) numerically using standard integrators.

3. Conclusions

In this work we have considered the linear stability problem which arises from the linearization of the PDEs exhibiting travelling wave solutions. This problem is formulated as a system of n first-order ordinary differential equations. This type of system is usually stiff [5,7] which makes the numerical calculation very difficult. Consequently orthogonalization routines such as the compound matrix method are required to overcome this stiffness. Having achieved this, the Evans function can be subsequently calculated and the stability of the travelling solutions determined using the Evans function approach [4]. Previously the compound matrix method has been used for the cases where the number of equations, n, was less than or equal to six [5,6]. In this work we have generalized the method for the case of an arbitrary even number of equations. Extension to the general case when n is arbitrary and there is a set of k solutions bounded for $\xi \to -\infty$ and $n - k$ solutions bounded for $\xi \to \infty$ is a straightforward procedure.

The Evans function method is now becoming one of the standard approaches for determining the stability of the solutions from PDEs. We believe that by adopting the generalized approach outlined in this work, the Evans function method can not only be utilized to investigate the stability of a wider class of PDEs but also be implemented in automated packages for numerical bifurcation analysis such as AUTO [11].

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