Secular Monte-Carlo method

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I propose a new approach to simulations of long-term (secular) evolution of stellar systems that follow a series of quasi-equilibrium configurations.
It can be regarded as a bridge between self-consistent field (SCF) method for collisionless dynamics, and Monte-Carlo method for globular cluster evolution.

General potential-density representation and the Schwarzschild method

The potential and density profile of a stellar system with a well-defined center and without clumpy substructure may be well represented as a finite sum of basis functions with certain coefficients A_{nlm}:

$$\begin{aligned}\Phi(r,\theta,\phi) &= \sum_{n=0}^{n_{max}} \sum_{l=0}^{l_{max}} \sum_{m=-l}^{l} A_{nlm}\,\Phi_{nlm}(r,\theta,\phi) \\ \rho(r,\theta,\phi) &= \sum_{n=0}^{n_{max}} \sum_{l=0}^{l_{max}} \sum_{m=-l}^{l} A_{nlm}\,\rho_{nlm}(r,\theta,\phi)\end{aligned} \qquad (1)$$

Typically, the basis functions are factorized as $\Phi_{nlm} = \Phi_{nl}(r)\,Y_l^m(\theta,\phi)$, where the angular part of basis functions is represented in spherical harmonic functions. For the radial basis, several choices exist, most widely used is the Hernquist-Ostriker(1992) basis set, for which the zeroth-order term corresponds to Hernquist profile, $\rho(r) = r^{-1}(1+r)^{-3}$. Another option is a direct representation of radial part of expansion as smooth functions of radius:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{l_{max}} \sum_{m=-l}^{l} A_{lm}(r)\,Y_l^m(\theta,\phi) \qquad (2)$$

In practice, A_{lm} are evaluated at a finite number of grid points in radius, and spline-interpolated between them.

The coefficients of expansion in either case may be computed from a smooth analytical model, or directly from a set of point masses. In the latter case, integration of orbits in a smooth potential initialized from an N-body snapshot may be used to study the orbital structure of a given system, and gives a better approximation to the underlying density model than a superposition of potentials of discrete "frozen" particles (Vasiliev, 2013).

The potential-density expansions presented above may be used to construct self-consistent equilibrium models using the Schwarzschild (1979) method. One computes a large library of N_{orb} orbits by numerically integrating equations of motion in the given potential, evaluates the contribution of each orbit to the density of the model, parametrized by coefficients $A_{nlm,i}, i = 1..N_{orb}$ of density expansion for each orbit, and solves for weights w_i of orbits which result in their superposition satisfying the original density profile:

$$\sum_{i=1}^{N_{orb}} w_i\,A_{nlm,i} = A_{nlm} \quad \text{for } \forall n,l,m \qquad (3)$$

Time-smoothed SCF method

We can go one step further and drop the requirement that the superposition of orbits described above remains stationary in time. Namely, we may let the coefficients of potential expansion depend on t, while remaining consistent with the density generated by the set of orbits. It may be regarded as a time-smoothed generalization of Hernquist & Ostriker (1992) self-consistent field (SCF) method, in which the basic building block is changed from a single particle to the entire orbit (represented by $N_{samp} \gg 1$ of sampling points from its trajectory), and the coefficients of expansion are averaged over a time interval of many dynamical times. The obvious advantages are:

- Reduced discreteness noise: instead of N_{body} we have $N_{orb} \times N_{samp}$ point masses to compute the potential, while we need to follow only N_{orb} trajectories.
- Time averaging over timescales longer than the dynamical time ensures adiabatic conservation of actions; two-body relaxation is suppressed.
- May impose a particular form of symmetry (e.g. axisymmetry) by keeping only relevant terms in potential expansion.

In addition, one may add random perturbations to trajectories, mimicking the effect of two-body relaxation with adjustable relaxation rate (i.e. it does not needs to correspond to the actual number of orbits in the system, but to a much larger $N_\star$ which is a free parameter). In this approach, it generalizes the well-known Monte-Carlo method for studying dynamics of globular clusters (in the Spitzer's, not Hénon's formulation) to arbitrary geometry.

A range of problems may be addressed with this method, including:

- Interplay between collisional and collisionless relaxation in galactic nuclei containing supermassive black holes.
- Evolution of galactic shape due to diffusion of chaotic orbits.
- Non-spherical star clusters in external tidal field.

SMILE – a new software for orbit analysis and Schwarzschild modelling

While the approach outlined above remains to be developed in details, the tool for orbit analysis and construction of self-consistent models by Schwarzschild's method already exists and is publicly available. The key features of *SMILE* are:

- A number of standard potential models (e.g. triaxial Dehnen), and several general-purpose potential approximations which may represent arbitrary non-spherical density profile.
- Orbit analysis methods: classification of orbit shapes (boxes, tubes, resonant and chaotic orbits); frequency analysis, detection of chaotic orbits using Lyapunov exponent and frequency diffusion rate. May analyse orbits from external simulations as well as from internal orbit integrator.
- Tools for studying global dynamics of a given potential: Poincaré surfaces of section, frequency map plots.
- Schwarzschild modelling for a theorist's usage (creating models with arbitrary density profile but without fitting observations).
- An interactive graphical interface and scripting support.
- Written in C++, portable to Linux, Mac and Windows. Modular structure allows to use parts of the code in external programs, and easily include additional features.

References

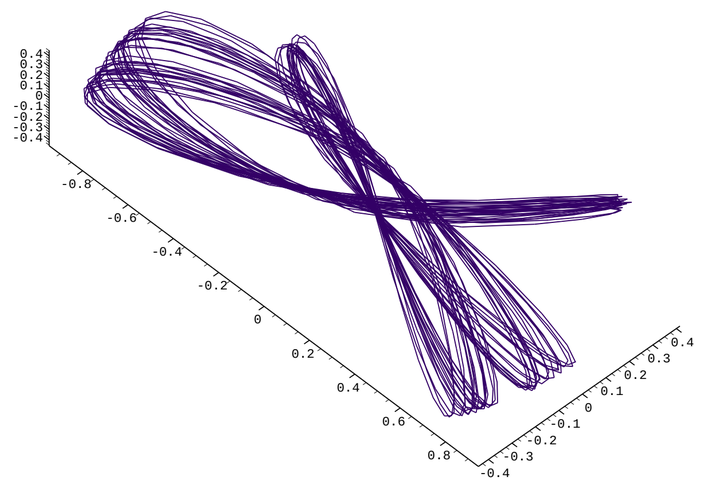
L. Hernquist, J. Ostriker, 1992, ApJ, 386, 375

Y. Papaphilippou, J. Laskar, 1998, A&A, 329, 451

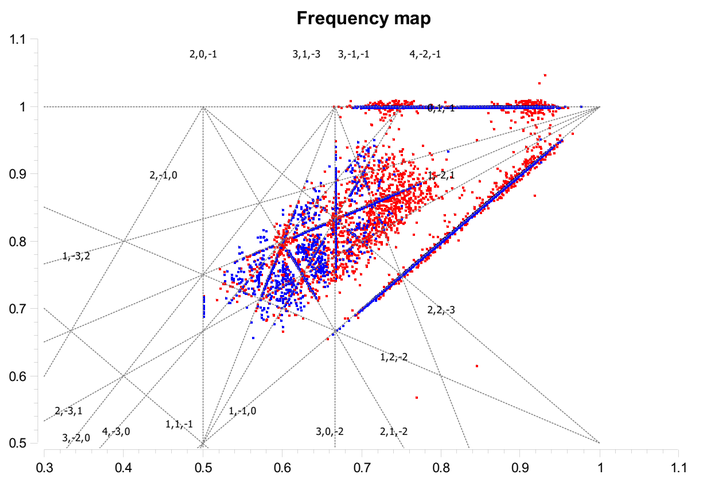
M. Schwarzschild, 1979, ApJ, 232, 236

E. Vasiliev, 2013, MNRAS, 434, 3174

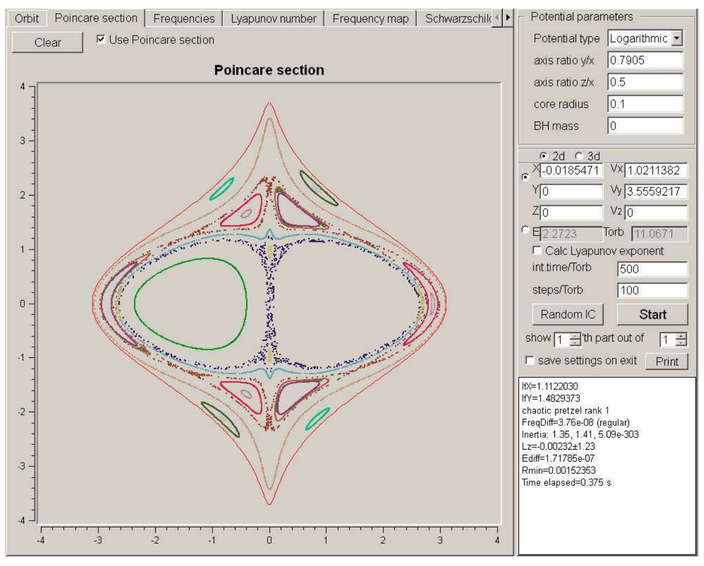
Screenshots



3d rendering of a 3:4:5 resonant orbit.



Frequency map plot (e.g. Papaphilippou & Laskar, 1998): each point corresponds to an orbit with the ratio of leading frequencies ω_y/ω_x and ω_z/ω_x as coordinates. Families of resonant orbits are visible as concentration of points along lines; color denotes the regular(blue) or chaotic(red) character of an orbit.



Poincaré surface of section for a 2d logarithmic potential

The software is available for download at
`http://td.lpi.ru/~eugvas/smile/`